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Extremal mild solutions for impulsive fractional evolution equations with nonlocal initial conditions

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Abstract

In this article, the theory of positive semigroup of operators and the monotone iterative technique are extended for the impulsive fractional evolution equations with nonlocal initial conditions. The existence results of extremal mild solutions are obtained. As an application that illustrates the abstract results, an example is given.

Keywords: impulsive fractional evolution equations; nonlocal initial conditions; extremal mild solutions; monotone iterative technique

1 Introduction

In this article, we use the monotone iterative technique to investigate the existence of extremal mild solutions of the impulsive fractional evolution equation with nonlocal initial conditions in an ordered Banach space X

$$\begin{cases} D^\alpha u(t) + Au(t) = f(t, u(t)), & t \in I, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \dots, m, \\ u(0) + g(u) = x_0 \in X, \end{cases} \tag{1.1}$$

where D^α is the Caputo fractional derivative of order $0 < \alpha < 1$, $A : D(A) \subset X \to X$ is a linear closed densely defined operator, $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $T(t)$ $(t \geq 0)$, $I = [0, T]$, $T > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $f : I \times X \to X$ is continuous, $g : PC(I, X) \to X$ is continuous ($PC(I, X)$ will be defined in Section 2), the impulsive function $I_k : X \to X$ is continuous, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary real or complex order. The subject is as old as differential calculus, and goes back to the time when Leibnitz and Newton invented differential calculus. Fractional derivatives have been extensively applied in many fields which have been seen an overwhelming growth in the last three decades. Examples abound: models admitting backgrounds of heat transfer, viscoelasticity, electrical circuits, electro-chemistry, economics, polymer physics, and even biology are always concerned with fractional derivative [1–6]. Fractional evolution equations have attracted many researchers in recent years, for example, see [7–14].

A strong motivation for investigating the problem (1.1) comes form physics. For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$, namely

$$\partial_t^\alpha u(y,t) = Au(y,t), \quad t \geq 0, y \in R, \tag{1.2}$$

we can take $A = \partial_y^{\beta_1}$, for $\beta_1 \in (0,1]$, or $A = \partial_y + \partial_y^{\beta_2}$ for $\beta_2 \in (1,2]$, where ∂_t^α, $\partial_y^{\beta_1}$, $\partial_y^{\beta_2}$ are the fractional derivatives of order α, β_1, β_2, respectively.

The existence results to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [15, 16]. Deng [17] indicated that, using the nonlocal condition $u(0) + g(u) = x_0$ to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem $u(0) = x_0$. For example, $g(u)$ can be given by

$$g(u) = \sum_{i=1}^{n} c_i u(\tau_i), \tag{1.3}$$

where c_i $(i = 1, 2, \ldots, n)$ are given constants and $0 < \tau_1 < \tau_2 < \cdots < \tau_n < T$. On the other hand, the differential equations involving impulsive effects appear as a natural description of observed evolution phenomena introduction of the basic theory of impulsive differential equations, we refer the reader to [18] and the references therein. The study of impulsive evolution equations with nonlocal initial conditions has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works [12–14]. They use the contraction mapping principle, the Krasnoselskii fixed point theorem and the Schaefer fixed point theorem.

To the authors' knowledge, there are no studies on the existence of solutions for the impulsive fractional evolution equations with nonlocal initial conditions by using the monotone iterative technique in the presence of lower and upper solutions. Nevertheless, the monotone iterative technique concerning upper and lower solutions is a powerful tool to solve the differential equations with various kinds of boundary conditions, see [19–21]. This technique is that, for the considered problem, starting from a pair ordered lower and upper, one constructs two monotone sequences such that them uniformly converge to the extremal solutions between the lower and upper solutions. In this article, based on Mu [8], we obtained the existence of extremal mild solutions of the problem (1.1) by using the monotone iterative technique.

In following section, we introduce some preliminaries which are used throughout this article. In Section 3, by combining the theory of positive semigroup of linear operators and the monotone iterative technique coupled with the method of upper and lower solutions, we construct two groups of monotone iterative sequences, and then prove these sequences monotonically converge to the maximal and minimal mild solutions of the problem (1.1), respectively, under some monotone conditions and noncompactness measure conditions of f, g, and I_k. In Section 4, in order to illustrate our results, an impulsive fractional partial differential equation with nonlocal initial condition is also considered.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this article.

Definition 2.1 [22] The fractional integral of order α with the lower limit zero for a function $f \in AC[0,\infty)$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0, 0 < \alpha < 1, \tag{2.1}$$

provided the right side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 [22] The Riemann-Liouville derivative of order α with the lower limit zero for a function $f \in AC[0,\infty)$ can be written as

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}}\, ds, \quad t > 0, 0 < \alpha < 1. \tag{2.2}$$

Definition 2.3 [22] The Caputo fractional derivative of order α for a function $f \in AC[0,\infty)$ can be written as

$$D^{\alpha} f(t) = {}^{L}D^{\alpha}\big(f(t) - f(0)\big), \quad t > 0, 0 < \alpha < 1. \tag{2.3}$$

Remark 2.4

(i) If $f \in C^1[0,\infty)$, then

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}}\, ds, \quad t > 0, 0 < \alpha < 1. \tag{2.4}$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then the integrals and derivatives which appear in Definitions 2.1-2.3 are taken in Bochner's sense.

Let X be an ordered Banach space with norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{y \in X | y \geq \theta\}$ (θ is the zero element of X) is normal with normal constant N. Let $C(I,X)$ be the Banach space of all continuous X-value functions on interval I with norm $\|u\|_C = \max_{t\in I} \|u(t)\|$. Let $PC(I,X) = \{u : I \to X | u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. Evidently, $PC(I,X)$ is an ordered Banach space with norm $\|u\|_{PC} = \sup_{t\in I} \|u(t)\|$ and the partial order $\leq$ reduced by the positive cone $K_{PC} = \{u \in PC(I,X) | u(t) \geq \theta, t \in I\}$. K_{PC} is also normal with the same normal constant N. For $u, v \in PC(I,X)$, $u \leq v$ if $u(t) \leq v(t)$ for all $t \in I$. For $v, w \in PC(I,X)$ with $v \leq w$, denote the ordered interval $[v,w] = \{u \in PC(I,X) | v \leq u \leq w\}$ in $PC(I,X)$, and $[v(t), w(t)] = \{y \in X | v(t) \leq y \leq w(t)\}$ ($t \in I$) in X. Set $C^{\alpha}(I,X) = \{u \in C(I,X) | D^{\alpha}u$ exists and $D^{\alpha}u \in C(I,X)\}$. Let $I' = I \setminus \{t_1, t_2, \ldots, t_m\}$. By X_1 we denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. An abstract function $u \in PC(I,X) \cap C^{\alpha}(I',X) \cap C(I',X_1)$ is called a solution of (1.1) if $u(t)$ satisfies all the equalities of (1.1). We note that $-A$ is the infinitesimal generator of a uniformly bounded analytic semigroup $T(t)$ $(t \geq 0)$. This means there exists $M \geq 1$ such that

$$\|T(t)\| \leq M, \quad t \geq 0. \tag{2.5}$$

Definition 2.5 If $v_0 \in PC(I,X) \cap C^\alpha(I',X) \cap C(I',X_1)$ and satisfies inequalities

$$\begin{cases} D^\alpha v_0(t) + Av_0(t) \leq f(t, v_0(t)), & t \in I, t \neq t_k, \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k)), & k = 1,2,\ldots,m, \\ v_0(0) + g(v_0) \leq x_0, \end{cases} \tag{2.6}$$

then v_0 is called a lower solution of problem (1.1); if all inequalities of (2.6) are inverse, we call it an upper solution of the problem (1.1).

Lemma 2.6 [7] *If h satisfies a uniform Hölder condition, with exponent $\beta \in (0,1]$, then the unique solution of the linear initial value problem (LIVP) for the fractional evolution equation*

$$\begin{cases} D^\alpha u(t) + Au(t) = h(t), & t \in I, \\ u(0) = x_0 \in X, \end{cases} \tag{2.7}$$

is given by

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s)\,ds, \tag{2.8}$$

where

$$U(t) = \int_0^\infty \zeta_\alpha(\theta) T\big(t^\alpha\theta\big)\,d\theta, \qquad V(t) = \alpha \int_0^\infty \theta\zeta_\alpha(\theta) T\big(t^\alpha\theta\big)\,d\theta, \tag{2.9}$$

$\zeta_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$.

Remark 2.7 [9, 10]

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\rho_\alpha\big(\theta^{-\frac{1}{\alpha}}\big), \tag{2.10}$$

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{n=0}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty). \tag{2.11}$$

Remark 2.8 [10] $\zeta_\alpha(\theta) \geq 0$, $\theta \in (0,\infty)$, $\int_0^\infty \zeta_\alpha(\theta)\,d\theta = 1$, $\int_0^\infty \theta\zeta_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(1+\alpha)}$.

Definition 2.9 If $h \in C(I,X)$, by the mild solution of IVP (2.7), we mean that the function $u \in C(I,X)$ satisfying the integral Equation (2.8).

Form Definition 2.9, we can easily obtain the following result.

Lemma 2.10 *For any $h \in PC(I,X)$, $y_k \in X$, $k = 1,2,\ldots,m$, the LIVP for the linear impulsive fractional evolution equation*

$$\begin{cases} D^\alpha u(t) + Au(t) = h(t), & t \in I, t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, & k = 1,2,\ldots,m, \\ u(0) = x_0 \in X, \end{cases} \tag{2.12}$$

has the unique mild solution $u \in PC(I,X)$ given by

$$u(t) = \begin{cases} U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s)\,ds, & t \in [0,t_1], \\ U(t)[u(t_1) + y_1] + \int_{t_1}^t (t-s)^{\alpha-1} V(t-s)h(s)\,ds, & t \in (t_1,t_2], \\ \vdots \\ U(t)[u(t_m) + y_m] + \int_{t_m}^t (t-s)^{\alpha-1} V(t-s)h(s)\,ds, & t \in (t_m,T], \end{cases} \tag{2.13}$$

where $U(t)$ and $V(t)$ are given by (2.9).

Remark 2.11 We note that $U(t)$ and $V(t)$ do not possess the semigroup properties. The mild solution of (2.12) can be expressed only by using piecewise functions.

Definition 2.12 A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup, if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Definition 2.13 A bounded linear operator K on X is called to be positive, if $Kx \geq \theta$ for all $x \geq \theta$.

Remark 2.14 By (2.9) and Remark 2.8, $U(t)$ and $V(t)$ are positive, if $\{T(t)\}_{t\geq 0}$ is a positive semigroup.

Remark 2.15 From Remark 2.14, if $T(t)$ $(t \geq 0)$ is a positive semigroup generated by $-A$, $h \geq \theta$, $x_0 \geq \theta$ and $y_k \geq \theta$, $k = 1,2,\ldots,m$, then the mild solution $u \in PC(I,X)$ of (2.12) satisfies $u \geq \theta$. For the applications of positive operators semigroup, one can refer to [23–25].

Now, we recall some properties of the measure of noncompactness will be used later. Let $\mu(\cdot)$ denotes the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [26]. For any $B \subset C(I,X)$ and $t \in I$, set $B(t) = \{u(t) | u \in B\}$. If B is bounded in $C(I,X)$, then $B(t)$ is bounded in X, and $\mu(B(t)) \leq \mu(B)$. If E is a precompact set in X, then $\mu(E) = 0$.

Lemma 2.16 [27] *Let $B = \{u_n\} \subset C(I,X)$ $(n = 1,2,\ldots)$ be a bounded and countable set. Then $\mu(B(t))$ is Lebesgue integral on I, and*

$$\mu\left(\left\{\int_I u_n(t)\,dt \,\middle|\, n = 1,2,\ldots\right\}\right) \leq 2\int_I \mu\bigl(B(t)\bigr)\,dt. \tag{2.14}$$

In order to prove our results, we also need a generalized Gronwall inequality for fractional differential equation.

Lemma 2.17 [28] *Suppose $b \geq 0$, $\beta > 0$, and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s)\, ds \tag{2.15}$$

on this interval; then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T. \tag{2.16}$$

3 Main results

Theorem 3.1 *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)$ ($t \geq 0$) is positive, the Cauchy problem (1.1) has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \leq w_0$, and the following conditions are satisfied:*

(H_1) There exists a constant $C \geq 0$ such that

$$f(t,x_2) - f(t,x_1) \geq -C(x_2 - x_1), \tag{3.1}$$

for any $t \in I$, and $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$. That is, $f(t,x) + Cx$ is increasing in x for $x \in [v_0(t), w_0(t)]$.

(H_2) $g(u)$ is decreasing in u for $u \in [v_0, w_0]$.

(H_3) $I_k(x)$ is increasing in x for $x \in [v_0(t), w_0(t)]$ ($t \in I$).

(H_4) There exists a constant $L \geq 0$ such that

$$\mu\big(\{f(t,x_n)\}\big) \leq L\mu\big(\{x_n\}\big), \tag{3.2}$$

for any $t \in I$, and increasing or decreasing monotonic sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.

(H_5) $\{g(u_n)\}$ is precompact in X, for any increasing or decreasing monotonic sequence $\{u_n\} \subset [v_0, w_0]$. That is, $\mu(\{g(u_n)\}) = 0$.

Then the Cauchy problem (1.1) has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.

Proof It is easy to see that $-(A + CI)$ generates an positive analytic semigroup $S(t) = e^{-Ct}T(t)$. Let $\Phi(t) = \int_0^\infty \zeta_\alpha(\theta)S(t^\alpha\theta)\, d\theta$, $\Psi(t) = \alpha \int_0^\infty \theta\zeta_\alpha(\theta)S(t^\alpha\theta)\, d\theta$. By Remark 2.14, $\Phi(t)$ ($t \geq 0$) and $\Psi(t)$ ($t \geq 0$) are positive. By (2.5) and Remark 2.8, we have that

$$\|\Phi(t)\| \leq M, \qquad \|\Psi(t)\| \leq \frac{\alpha}{\Gamma(\alpha+1)} M \triangleq M_1, \quad t \geq 0. \tag{3.3}$$

Let $D = [v_0, w_0]$, $J_1' = [t_0, t_1] = [0, t_1]$, $J_k' = (t_{k-1}, t_k]$, $k = 2, 3, \dots, m+1$. We define a mapping $Q : D \to PC(I, X)$ by

$$Qu(t) = \begin{cases} \Phi(t)[x_0 - g(u)] + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,u(s)) + Cu(s)]\,ds, & t \in J_1', \\ \Phi(t)[u(t_1) + I_1(u(t_1))] & \\ \quad + \int_{t_1}^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,u(s)) + Cu(s)]\,ds, & t \in J_2', \\ \vdots & \\ \Phi(t)[u(t_m) + I_m(u(t_m))] & \\ \quad + \int_{t_m}^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,u(s)) + Cu(s)]\,ds, & t \in J_{m+1}'. \end{cases} \tag{3.4}$$

Clearly, $Q : D \to PC(I, X)$ is continuous. By Lemma 2.10, $u \in D$ is a mild solution of problem (1.1) if and only if

$$u = Qu. \tag{3.5}$$

For $u_1, u_2 \in D$ and $u_1 \le u_2$, from the positivity of operators $\Phi(t)$ and $\Psi(t)$, (H_1), (H_2), and (H_3), we have inequality

$$Qu_1 \le Qu_2. \tag{3.6}$$

Now, we show that $v_0 \le Qv_0$, $Qw_0 \le w_0$. Let $D^\alpha v_0(t) + Av_0(t) + Cv_0(t) \triangleq \sigma(t)$. By Definition 2.5, Lemma 2.10, the positivity of operators $\Phi(t)$ and $\Psi(t)$, for $t \in J_1'$, we have that

$$\begin{aligned} v_0(t) &= \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\sigma(s)\,ds \\ &\le \Phi(t)\big[x_0 - g(v_0)\big] + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\big[f\big(s, v_0(s)\big) + Cv_0(s)\big]\,ds. \end{aligned} \tag{3.7}$$

For $t \in J_2'$, we have that

$$\begin{aligned} v_0(t) &= \Phi(t)\big[v_0(t_1) + \Delta v_0|_{t=t_1}\big] + \int_{t_1}^t (t-s)^{\alpha-1}\Psi(t-s)\sigma(s)\,ds \\ &\le \Phi(t)\big[v_0(t_1) + I_1\big(v_0(t_1)\big)\big] + \int_{t_1}^t (t-s)^{\alpha-1}\Psi(t-s) \\ &\quad \times \big[f\big(s, v_0(s)\big) + Cv_0(s)\big]\,ds. \end{aligned} \tag{3.8}$$

Continuing such a process interval by interval to J_{m+1}', by (3.4), we obtain that $v_0 \le Qv_0$. Similarly, we can show that $Qw_0 \le w_0$. For $u \in D$, in view of (3.6), then $v_0 \le Qv_0 \le Qu \le Qw_0 \le w_0$. Thus, $Q : D \to D$ is a continuous increasing monotonic operator. We can now define the sequences

$$v_n = Qv_{n-1}, \qquad w_n = Qw_{n-1}, \quad n = 1, 2, \dots, \tag{3.9}$$

and it follows from (3.6) that

$$v_0 \le v_1 \le \cdots v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.10}$$

Let $B = \{v_n\}$ and $B_0 = \{v_{n-1}\}$, $n = 1, 2, \ldots$. By (3.10) and the normality of the positive cone P, then B and B_0 are bounded. It follows from $B_0 = B \cup \{v_0\}$ that $\mu(B(t)) = \mu(B_0(t))$ for $t \in I$. Let

$$\varphi(t) = \mu\big(B(t)\big) = \mu\big(B_0(t)\big), \quad t \in I. \tag{3.11}$$

From (H_4), (H_5), (3.3), (3.4), (3.9), (3.11), Lemma 2.16 and the positivity of operator $\Psi(t)$, for $t \in J_1'$, we have that

$$\begin{aligned}
\varphi(t) &= \mu\big(B(t)\big) = \mu\big(QB_0(t)\big) \\
&= \mu\bigg(\bigg\{\Phi(t)\big[x_0 - g(v_{n-1})\big] \\
&\quad + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\big[f\big(s, v_{n-1}(s)\big) + Cv_{n-1}(s)\big]\,ds \bigg| n = 1, 2, \ldots\bigg\}\bigg) \\
&\le M\mu\big(\{g(v_{n-1})\}\big) \\
&\quad + \mu\bigg(\bigg\{\int_0^t (t-s)^{\alpha-1}\Psi(t-s)\big[f\big(s, v_{n-1}(s)\big) + Cv_{n-1}(s)\big]\,ds \bigg| n = 1, 2, \ldots\bigg\}\bigg) \\
&\le 2\int_0^t \mu\big(\big\{(t-s)^{\alpha-1}\Psi(t-s)\big[f\big(s, v_{n-1}(s)\big) + Cv_{n-1}(s)\big] | n = 1, 2, \ldots\big\}\big)\,ds \\
&\le 2M_1 \int_0^t (t-s)^{\alpha-1}(L + C)\mu\big(B_0(s)\big)\,ds \\
&= 2M_1(L + C)\int_0^t (t-s)^{\alpha-1}\varphi(s)\,ds.
\end{aligned} \tag{3.12}$$

By (3.12) and Lemma 2.17, we obtain that $\varphi(t) \equiv 0$ on J_1'. In particular, $\mu(B(t_1)) = \mu(B_0(t_1)) = \varphi(t_1) = 0$. This means that $B(t_1)$ and $B_0(t_1)$ are precompact in X. Thus, $I_1(B_0(t_1))$ is precompact in X and $\mu(I_1(B_0(t_1))) = 0$. For $t \in J_2'$, using the same argument as above for $t \in J_1'$, we have that

$$\begin{aligned}
\varphi(t) &= \mu\big(B(t)\big) = \mu\big(QB_0(t)\big) \\
&= \mu\bigg(\bigg\{\Phi(t)\big[v_{n-1}(t_1) + I_1\big(v_{n-1}(t_1)\big)\big] \\
&\quad + \int_{t_1}^t (t-s)^{\alpha-1}\Psi(t-s)\big[f\big(s, v_{n-1}(s)\big) + Cv_{n-1}(s)\big]\,ds \bigg| n = 1, 2, \ldots\bigg\}\bigg) \\
&\le M\big[\mu\big(B_0(t_1)\big) + \mu\big(I_1\big(B_0(t_1)\big)\big)\big] + 2M_1(L + C)\int_{t_1}^t (t-s)^{\alpha-1}\varphi(s)\,ds \\
&= 2M_1(L + C)\int_{t_1}^t (t-s)^{\alpha-1}\varphi(s)\,ds.
\end{aligned} \tag{3.13}$$

By (3.13) and Lemma 2.17, $\varphi(t) \equiv 0$ on J_2'. Then, $\mu(B_0(t_2)) = \mu(I_1(B_0(t_2))) = 0$. Continuing such a process interval by interval to J_{m+1}', we can prove that $\varphi(t) \equiv 0$ on every J_k', $k = 1, 2, \ldots, m + 1$. This means $\{v_n(t)\}$ $(n = 1, 2, \ldots)$ is precompact in X for every $t \in I$. So, $\{v_n(t)\}$ has a convergent subsequence in X. In view of (3.10), we can easily prove that $\{v_n(t)\}$ itself is convergent in X. That is, there exist $\underline{u}(t) \in X$ such that $v_n(t) \to \underline{u}(t)$ as $n \to \infty$ for every

$t \in I$. By (3.4) and (3.9), we have that

$$v_n(t) = \begin{cases} \Phi(t)[x_0 - g(v_{n-1})] \\ \quad + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,v_{n-1}(s)) + Cv_{n-1}(s)]\,ds, & t \in J_1', \\ \Phi(t)[v_{n-1}(t_1) + I_1(v_{n-1}(t_1))] \\ \quad + \int_{t_1}^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,v_{n-1}(s)) + Cv_{n-1}(s)]\,ds, & t \in J_2', \\ \vdots \\ \Phi(t)[v_{n-1}(t_m) + I_m(v_{n-1}(t_m))] \\ \quad + \int_{t_m}^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,v_{n-1}(s)) + Cv_{n-1}(s)]\,ds, & t \in J_{m+1}'. \end{cases} \tag{3.14}$$

Let $n \to \infty$, then by Lebesgue-dominated convergence theorem, we have that

$$\underline{u}(t) = \begin{cases} \Phi(t)[x_0 - g(\underline{u})] + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,\underline{u}(s)) + C\underline{u}(s)]\,ds, & t \in J_1', \\ \Phi(t)[\underline{u}(t_1) + I_1(\underline{u}(t_1))] \\ \quad + \int_{t_1}^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,\underline{u}(s)) + C\underline{u}(s)]\,ds, & t \in J_2', \\ \vdots \\ \Phi(t)[\underline{u}(t_m) + I_m(\underline{u}(t_m))] \\ \quad + \int_{t_m}^t (t-s)^{\alpha-1}\Psi(t-s)[f(s,\underline{u}(s)) + C\underline{u}(s)]\,ds, & t \in J_{m+1}'. \end{cases} \tag{3.15}$$

Then $\underline{u} \in C(I,X)$ and $\underline{u} = Q\underline{u}$. Similarly, we can prove that there exists $\overline{u} \in C(I,X)$ such that $\overline{u} = Q\overline{u}$. By (3.6), if $u \in D$, and u is a fixed point of Q, then $v_1 = Qv_0 \leq Qu = u \leq Qw_0 = w_1$. By induction, $v_n \leq u \leq w_n$. By (3.10) and taking the limit as $n \to \infty$, we conclude that $v_0 \leq \underline{u} \leq u \leq \overline{u} \leq w_0$. That means that $\underline{u}, \overline{u}$ are the minimal and maximal fixed points of Q on $[v_0, w_0]$, respectively. By (3.5), they are the minimal and maximal mild solutions of the Cauchy problem (1.1) on $[v_0, w_0]$, respectively. □

Corollary 3.2 *Let X be an ordered Banach space, whose positive cone P is regular. Assume that $T(t)$ ($t \geq 0$) is positive, the Cauchy problem (1.1) has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \leq w_0$, (H_1), (H_2), and (H_3) hold. Then the Cauchy problem (1.1) has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof Since P is regular, any ordered-monotonic and ordered-bounded sequence in X is convergent. For $t \in I$, let $\{x_n\}$ be an increasing or decreasing sequence in $[v_0(t), w_0(t)]$. By (H_1), $\{f(t,x_n) + Cx_n\}$ is an ordered-monotonic and ordered-bounded sequence in X. Then, $\mu(\{f(t,x_n) + Cx_n\}) = \mu(\{x_n\}) = 0$. By the properties of the measure of noncompactness, we have

$$\mu\big(\{f(t,x_n)\}\big) \leq \mu\big(\{f(t,x_n) + Cx_n\}\big) + C\mu\big(\{x_n\}\big) = 0. \tag{3.16}$$

So, (H_4) holds. Let $\{u_n\}$ be an increasing or decreasing sequence in $[v_0, w_0]$. By (H_2), $\{g(u_n)\}$ is an ordered-monotonic and ordered-bounded sequence in X. Then $\{g(u_n)\}$ is precompact in X. Thus, (H_5) holds. By Theorem 3.1, the proof is then complete. □

Corollary 3.3 *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)$ $(t \geq 0)$ is positive, the Cauchy problem (1.1) has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \leq w_0$, (H_1), (H_2), and (H_3) hold. Then the Cauchy problem (1.1) has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof In an ordered and weakly sequentially complete Banach space, the normal cone P is regular. Then the proof is complete. □

Corollary 3.4 *Let X be an ordered and reflective Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)$ $(t \geq 0)$ is positive, the Cauchy problem (1.1) has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \leq w_0$, (H_1), (H_2), and (H_3) hold. Then the Cauchy problem (1.1) has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof In an ordered and reflective Banach space, the normal cone P is regular. Then the proof is complete. □

4 Examples

Example 4.1 In order to illustrate our results, we consider the following impulsive fractional partial differential equation with nonlocal initial condition

$$\begin{cases} \frac{\partial^{\frac{1}{2}} u(t,x)}{\partial t^{\frac{1}{2}}} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,u(t,x)), \quad x \in [0,\pi], t \in [0,T], t \neq t_k, \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,T], \\ \Delta u(t,x)|_{t=t_k} = I_k(u(t_k,x)), \quad k = 1,2,\ldots,m, \\ u(0,x) + \sum_{i=0}^{n} c_i u(\tau_i,x) = u_0(x), \quad x \in [0,\pi], \end{cases} \tag{4.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $0 < \tau_1 < \tau_2 < \cdots < \tau_n < T$, $c_i \leq 0$ $(i = 1,2,\ldots,n)$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, $I_k : \mathbb{R} \to \mathbb{R}$ $(k = 1,2,\ldots,m)$ is continuous, $u_0 \in L^2([0,\pi],\mathbb{R})$.

Let $X = L^2([0,\pi],\mathbb{R})$, $P = \{v | v \in X, v(y) \geq 0 \text{ a.e. } y \in [0,\pi]\}$. Then X is a Banach space, and P is a regular cone in X. Define the operator A as follows:

$$D(A) = \{v \in X | v, v' \text{ are absolutely continuous}, v'' \in X, v(0) = v(\pi) = 0\}, \qquad Av = -v'',$$

then $-A$ generate an analytic semigroup of uniformly bounded linear operators $T(t)$ $(t \geq 0)$ in X (see [11]). By the maximum principle, we can easily find that $T(t)$ $(t \geq 0)$ is a positive semigroup. Denote $u(t)(x) = u(t,x)$, $f(t,u(t))(x) = f(t,u(t,x))$, $I_k(u(t_k))(x) = I_k(u(t_k,x))$, $g(u)(x) = \sum_{i=0}^{n} c_i u(\tau_i,x)$, $x_0 = u_0(x)$, then the system (4.1) can be reformulated as the problem (1.1) in X. It is easy to find that (H_2) holds. Moreover, we assume that the following conditions hold:

(a) $f(t,0) \geq 0$ for $t \in [0,T]$, $I_k(0) \geq 0$, $u_0(x) \geq 0$ for $x \in [0,\pi]$.

(b) There exists w such that

$$\begin{cases} \frac{\partial^{\frac{1}{2}} w(t,x)}{\partial t^{\frac{1}{2}}} - \frac{\partial^2 w(t,x)}{\partial x^2} \geq f(t, w(t,x)), & x \in [0,\pi], t \in [0,T], t \neq t_k, \\ w(t,0) = w(t,\pi) = 0, & t \in [0,T], \\ \Delta w(t,x)|_{t=t_k} \geq I_k(w(t_k,x)), & k = 1,2,\ldots,m, \\ w(0,x) + \sum_{i=0}^{n} c_i w(\tau_i,x) \geq w_0(x), & x \in [0,\pi], \end{cases} \tag{4.2}$$

where $w = w(x,t)$ ($x \in [0,\pi]$, $t \in [0,T]$), w is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $w(t_k^+, x)$ exists, $k = 1,2,\ldots,m$, $\frac{\partial^{\frac{1}{2}} w(t,x)}{\partial t^{\frac{1}{2}}}$ and $\frac{\partial^2 w(t,x)}{\partial x^2}$ are continuous at $t \neq t_k$.

(c) $f'_u(t,u)$ is continuous on any bounded and ordered interval.

(d) For any u_1, u_2 on a bounded and ordered interval, and $u_1 \leq u_2$, we have

$$I_k\big(u_1(t_k,x)\big) \leq I_k\big(u_2(t_k,x)\big), \quad x \in [0,\pi], k = 1,2,\ldots,m. \tag{4.3}$$

Theorem 4.2 *If (a)-(d) are satisfied, then the system (4.1) has the minimal and maximal mild solutions between* 0 *and* w.

Proof By (a) and (b), we know 0 and w are the lower and upper solutions of the problem (1.1), respectively. (c) implies that (H_1) are satisfied. (d) implies that (H_3) are satisfied. Then by Corollary 3.2, the system (4.1) has the minimal and maximal mild solutions between 0 and w. □

Competing interests

The author declares that she has no competing interests.

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