Luo et al. *Boundary Value Problems* 2012, **2012**:76
http://www.boundaryvalueproblems.com/content/2012/1/76

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Existence of solutions to strongly damped plate or beam equations

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Abstract
In this paper, we study a strongly damped plate or beam equation. By using spatial sequence techniques and energy estimate methods, we obtain an existence theorem of the solution to abstract strongly damped plate or beam equation and to a nonlinear plate or beam equation.
MSC: 35L05; 35L20; 35D30; 35D35

Keywords: existence; solution; plate; beam; strongly damped

1 Introduction

We consider the following nonlinear strongly damped plate or beam equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - k\dfrac{\partial \triangle u}{\partial t} = f(x, \triangle^2 u) + g(x, u, Du, D^2 u, D^3 u), & k > 0, \\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0, \\ u(x,0) = \varphi, \qquad u_t(x,0) = \psi, \end{cases} \tag{1.1}$$

where $\triangle$ is the Laplacian operator, Ω denotes an open bounded set of R^N $(N = 1, 2)$ with a smooth boundary $\partial\Omega$ and u denotes a vertical displacement at (x, t).

It is well known that flexible structures like suspension bridges or overhead power transmission lines can be subjected to oscillations due to various causes. Simple models for such oscillations are described with second- and fourth-order partial differential equations as can be seen for example in [1–8]. The problem (1.1) can be applied in the mechanics of elastic constructions for the study of equilibrium forms of the plate and beam, which has a long history. The abstract theory of Eq. (1.1) was investigated by several authors [9–14].

The main objective of this article is to find proper conditions on f and g to ensure the existence of solutions of Eq. (1.1). This article uses the spatial sequence techniques, each side of the equation to be treated in different spaces, which is an important way to get more extensive and wonderful results.

The outline of the paper is as follows. In Section 2 we provide an essential definition and lemma of solutions to abstract equations from [15–18]. In Section 3, we give an existence theorem of solutions to abstract strongly damped plate or beam equations. In Section 4, we present the main result and its proof.

2 Preliminaries

We introduce two spatial sequences:

$$\begin{cases} X \subset H_3 \subset X_2 \subset X_1 \subset H, \\ X_2 \subset H_2 \subset H_1 \subset H, \end{cases} \tag{2.1}$$

where H, H_1, H_2, H_3 are Hilbert spaces, X is a linear space, and X_1, X_2 are Banach spaces. All embeddings of (2.1) are dense. Let

$$\begin{cases} L : X \to X_1 \quad \text{be one-one dense linear operator}, \\ \langle Lu, v\rangle_H = \langle u, v\rangle_{H_1}, \quad \forall u, v \in X. \end{cases} \tag{2.2}$$

Furthermore, L has eigenvectors $\{e_k\}$ satisfying

$$Le_k = \lambda_k e_k \quad (k = 1, 2, \ldots), \tag{2.3}$$

and $\{e_k\}$ constitutes a common orthogonal basis of H and H_3.

We consider the following abstract equation:

$$\begin{cases} \dfrac{d^2u}{dt^2} + k\dfrac{d}{dt}\mathcal{L}u = G(u), \quad k > 0, \\ u(0) = \varphi, \qquad u_t(0) = \psi, \end{cases} \tag{2.4}$$

where $G : X_2 \times R^+ \to X_1^*$ is a mapping, $R^+ = [0, \infty)$ and $\mathcal{L} : X_2 \to X_1$ is a bounded linear operator satisfying

$$\langle \mathcal{L}u, Lv\rangle_H = \langle u, v\rangle_{H_2}, \quad \forall u, v \in X_2. \tag{2.5}$$

Definition 2.1 [15] We say $u \in W^{1,\infty}_{\mathrm{loc}}((0,\infty), H_1) \cap L^\infty_{\mathrm{loc}}((0,\infty), X_2)$ is a global weak solution of Eq. (2.4) provided that $(\varphi, \psi) \in X_2 \times H_1$

$$\langle u_t, v\rangle_H + k\langle \mathcal{L}u, v\rangle_H = \int_0^t \bigl\langle G(u), v\bigr\rangle\, dt + \langle \psi, v\rangle_H + k\langle \mathcal{L}\varphi, v\rangle_H, \tag{2.6}$$

for all $v \in X_1$ and $0 \le t < \infty$.

Lemma 2.2 [18] *Let $u \in L^p_{\mathrm{loc}}((-\infty,\infty), X)$, X be a Banach space. If $u_h = \frac{1}{h}\int_h^{t+h} u(s)\,ds$ $(0 < |h| < 1)$, then $\{u_h\} \in L^p_{\mathrm{loc}}((-\infty,\infty), X)$, satisfying*

$$\lim_{h\to 0} \bigl\| u_h(t) - u(t) \bigr\|_X = 0, \quad \text{a.e. } t \in (-\infty, \infty),$$

$$\lim_{h\to 0} \int_{-T}^{T} \bigl\| u_h(t) - u(t) \bigr\|_X^p\, dt = 0, \quad 0 < T < \infty.$$

3 Existence theorem of abstract equation

Let $G = A + B : X_2 \times R^+ \to X_1^*$. Assume:

(A1) There is a C^1 functional $F : X_2 \to R^1$ such that

$$\langle Au, Lv\rangle = \langle -DF(u), v\rangle, \quad \forall u, v \in X. \tag{3.1}$$

(A2) Functional $F : X_2 \to R^1$ is coercive, *i.e.*,

$$F(u) \to \infty, \quad \Leftrightarrow \quad \|u\|_{X_2} \to \infty. \tag{3.2}$$

(A3) B satisfies

$$\left|\langle Bu, Lv\rangle\right| \le CF(u) + \frac{k}{2}\|v\|_{H_2}^2 + g(t), \quad \forall u, v \in X, \tag{3.3}$$

for $g \in L^1_{\mathrm{loc}}(0,\infty)$.

Theorem 3.1 *If $G : X_2 \times R^+ \to X_1^*$ is bounded and continuous, and DF is monotone, i.e.,*

$$\langle DF(u_1) - DF(u_2), u_1 - u_2\rangle \ge 0, \quad \forall u_1, u_2 \in X_2, \tag{3.4}$$

then, for all $(\varphi, \psi) \in X_2 \times H_1$, the following assertions hold.

(1) If $G = A$ satisfies (A1) and (A2), then Eq. (2.4) has a global weak solution

$$u \in W^{1,\infty}\big((0,\infty), H_1\big) \cap W^{1,2}\big((0,\infty), H_2\big) \cap L^\infty\big((0,\infty), X_2\big). \tag{3.5}$$

(2) If $G = A + B$ satisfies (A1)-(A3), and $u_n \rightharpoonup^* u_0$ in $L^\infty((0,T), X_2)$ such that

$$\lim_{n\to\infty} \int_0^T \langle Bu_n, v\rangle\, dt = \int_0^T \langle Bu_0, v\rangle\, dt, \quad \forall v \in L^\infty\big((0,T), X_2\big), \tag{3.6}$$

$$\lim_{n\to\infty} \int_0^T \langle Bu_n, Lu_n\rangle\, dt = \int_0^T \langle Bu_0, Lu_0\rangle\, dt, \tag{3.7}$$

then Eq. (2.4) has a global weak solution

$$u \in W^{1,\infty}_{\mathrm{loc}}\big((0,\infty), H_1\big) \cap W^{1,2}_{\mathrm{loc}}\big((0,\infty), H_2\big) \cap L^\infty_{\mathrm{loc}}\big((0,\infty), X_2\big). \tag{3.8}$$

(3) Furthermore, if $G = A + B$ satisfies

$$\left|\langle Gu, v\rangle\right| \le CF(u) + \frac{1}{2}\|v\|_H^2 + g(t), \tag{3.9}$$

for $g \in L^1(0,T)$, then $u \in W^{2,2}_{\mathrm{loc}}((0,\infty); H)$.

Proof Let $\{e_k\} \subset X$ be a common orthogonal basis of H and H_3, satisfying (2.3). Set

$$\begin{cases} X_n = \left\{ \sum_{i=1}^n \alpha_i e_i \mid \alpha_i \in R^1 \right\}, \\ \tilde{X}_n = \left\{ \sum_{j=1}^n \beta_j(t) e_j \mid \beta_j \in C^2[0,\infty) \right\}. \end{cases} \tag{3.10}$$

Clearly, $LX_n = X_n$, $L\tilde{X}_n = \tilde{X}_n$.

By using Galerkin method, there exists $u_n \in C^2([0,\infty), X_n)$ satisfying

$$\begin{cases} \left\langle \frac{du_n}{dt}, v \right\rangle_H + k\langle \mathcal{L}u_n, v\rangle_H = \int_0^t \langle G(u_n), v\rangle\,dt + \langle \psi_n, v\rangle_H + k\langle \mathcal{L}\varphi_n, v\rangle_H, \\ u_n(0) = \varphi_n, \qquad u_n'(0) = \psi_n, \end{cases} \tag{3.11}$$

for $\forall v \in X_n$, and

$$\int_0^t \left[\left\langle \frac{d^2u_n}{d^2t}, v \right\rangle_H + k\left\langle \mathcal{L}\frac{du_n}{dt}, v \right\rangle_H \right] dt = \int_0^t \langle Gu_n, v\rangle\,dt \tag{3.12}$$

for $\forall v \in \tilde{X}_n$.

Firstly, we consider $G = A$. Let $v = \frac{d}{dt}Lu_n$ in (3.12). Taking into account (2.2)and (3.1), it follows that

$$\begin{aligned} 0 &= \int_0^t \left[\frac{1}{2}\frac{d}{dt}\left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_1} + k\left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_2} + \left\langle DF(u_n), \frac{du_n}{dt} \right\rangle \right] dt \\ &= \frac{1}{2}\left\| \frac{du_n}{dt} \right\|_{H_1}^2 - \frac{1}{2}\|\psi_n\|_{H_1}^2 + k\int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 dt + F(u_n) - F(\varphi_n). \end{aligned}$$

We get

$$\frac{1}{2}\left\| \frac{du_n}{dt} \right\|_{H_1}^2 + k\int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 dt + F(u_n) = F(\varphi_n) + \frac{1}{2}\|\psi_n\|_{H_1}^2. \tag{3.13}$$

Let $\varphi \in H_3$. From (2.1) and (2.2), it is known that $\{e_n\}$ is an orthogonal basis of H_1. We find that $\varphi_n \to \varphi$ in H_3, and $\psi_n \to \psi$ in H_1. From that $H_3 \subset X_2$ is an imbedding, it follows that

$$\begin{cases} \varphi_n \to \varphi & \text{in } X_2, \\ \psi_n \to \psi & \text{in } H_1. \end{cases} \tag{3.14}$$

From (3.2), (3.13) and (3.14), we obtain

$$\{u_n\} \subset W^{1,\infty}\big((0,\infty), H_1\big) \cap W^{1,2}\big((0,\infty), H_2\big) \cap L^{\infty}\big((0,\infty), X_2\big) \text{ is bounded.}$$

Let

$$\begin{cases} u_n \rightharpoonup^* u_0 & \text{in } W^{1,\infty}\big((0,\infty), H_1\big) \cap L^{\infty}\big((0,\infty), X_2\big), \\ u_n \rightharpoonup u_0 & \text{in } W^{1,2}\big((0,\infty), H_2\big), \end{cases} \tag{3.15}$$

which implies that $u_n \to u_0$ in $W^{1,2}((0,\infty), H)$ is uniformly weakly convergent from that $H_2 \subset H$ is a compact imbedding.

According to (2.2), (2.4), (2.5) and (3.4), we obtain that

$$\begin{aligned} 0 &\geq \int_0^t \langle DF(v) - DF(u_n), u_n - v\rangle\,d\tau \\ &= \int_0^t \langle Av, Lv - Lu_n\rangle\,d\tau + \int_0^t \langle Au_n, Lu_n - Lv\rangle\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \langle Av, Lv - Lu_n\rangle \, d\tau + \int_0^t \left\langle \frac{d^2u_n}{dt^2} + k\frac{d}{dt}\mathcal{L}u_n, Lu_n - Lv\right\rangle_H d\tau \\
&= \int_0^t \langle Av, Lv - Lu_n\rangle \, d\tau + \int_0^t \left\langle \frac{d^2u_n}{dt^2}, Lu_n\right\rangle_H d\tau + k\int_0^t \left\langle \frac{d}{dt}\mathcal{L}u_n, Lu_n\right\rangle_H d\tau \\
&\quad - \int_0^t \left\langle \frac{d^2u_n}{dt^2}, Lv\right\rangle_H d\tau - k\int_0^t \left\langle \frac{d}{dt}\mathcal{L}u_n, Lv\right\rangle_H d\tau \\
&= \int_0^t \langle Av, Lv - Lu_n\rangle \, d\tau + \int_0^t \left\langle \frac{d^2u_n}{dt^2}, u_n\right\rangle_{H_1} d\tau + k\int_0^t \left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_2} d\tau \\
&\quad - \int_0^t \left\langle \frac{d^2u_n}{dt^2}, v\right\rangle_{H_1} d\tau - k\int_0^t \left\langle \frac{du_n}{dt}, v\right\rangle_{H_2} d\tau \\
&= \int_0^t \langle Av, Lv - Lu_n\rangle \, d\tau + \left\langle u_n, \frac{du_n}{dt}\right\rangle_{H_1} - \langle \varphi_n, \psi_n\rangle_{H_1} - \int_0^t \left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_1} d\tau \\
&\quad + \frac{k}{2}\langle u_n, u_n\rangle_{H_2} - \frac{k}{2}\langle \varphi_n, \varphi_n\rangle_{H_2} - \left\langle \frac{du_n}{dt}, v\right\rangle_{H_1} + \langle \psi_n, v(0)\rangle_{H_1} + \int_0^t \left\langle \frac{du_n}{dt}, \frac{dv}{dt}\right\rangle_{H_1} d\tau \\
&\quad - k\langle u_n, v\rangle_{H_2} + k\langle \varphi_n, v(0)\rangle_{H_2} + k\int_0^t \left\langle u_n, \frac{dv}{dt}\right\rangle_{H_2} d\tau.
\end{aligned}
$$

Let $n \to \infty$. From (3.15), we get

$$
\begin{aligned}
&\int_0^t \langle Av, Lv - Lu_0\rangle \, d\tau + \left\langle u_0, \frac{du_0}{dt}\right\rangle_{H_1} - \langle \varphi, \psi\rangle_{H_1} - \lim_{n\to\infty}\int_0^t \left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_1} d\tau \\
&\quad + \frac{k}{2}\langle u_0, u_0\rangle_{H_2} - \frac{k}{2}\langle \varphi, \varphi\rangle_{H_2} - \left\langle \frac{du_0}{dt}, v\right\rangle_{H_1} + \langle \psi, v(0)\rangle_{H_1} + \int_0^t \left\langle \frac{du_0}{dt}, \frac{dv}{dt}\right\rangle_{H_1} d\tau \\
&\quad - k\langle u_0, v\rangle_{H_2} + k\langle \varphi, v(0)\rangle_{H_2} + k\int_0^t \left\langle u_0, \frac{dv}{dt}\right\rangle_{H_2} d\tau \le 0. \qquad (3.16)
\end{aligned}
$$

Since $\bigcup_{n=1}^{\infty} \tilde{X}_n$ is dense in $W^{1,\infty}((0,\infty), H_1) \cap W^{1,2}((0,\infty), H_2) \cap L^{\infty}((0,\infty), X_2)$, the above equality (3.16) holds for $\forall v \in W^{1,\infty}((0,\infty), H_1) \cap W^{1,2}((0,\infty), H_2) \cap L^{\infty}((0,\infty), X_2)$.

We set v the following variable:

$$
\begin{aligned}
u_h + \lambda w &= \frac{1}{h}\int_t^{t+h} u_0 \, d\tau + \lambda w, \\
u_{-h} + \lambda w &= \frac{1}{h}\int_{t-h}^{t} \tilde{u}_0 \, d\tau + \lambda w,
\end{aligned}
$$

where $w \in X_2$, λ is a real, $\tilde{u}_0 = u_0$ if $t \ge 0$, and $\tilde{u}_0 = 0$ if $t < 0$. Thus the equality (3.16) is read as

$$
\begin{aligned}
&\int_0^t \langle A(u_h + \lambda w), L(u_h + \lambda w) - Lu_0\rangle \, d\tau + \left\langle u_0(t), \frac{du_0(t)}{dt}\right\rangle_{H_1} - \langle \varphi, \psi\rangle_{H_1} \\
&\quad - \lim_{n\to\infty}\int_0^t \left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_1} d\tau + \frac{k}{2}\langle u_0(t), u_0(t)\rangle_{H_2} - \frac{k}{2}\langle \varphi, \varphi\rangle_{H_2} \\
&\quad - \left\langle \frac{du_0(t)}{dt}, u_h(t) + \lambda w\right\rangle_{H_1} + \langle \psi, u_h(0) + \lambda w\rangle_{H_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \left\langle \frac{du_0}{dt}, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_1} d\tau \\
&- k\langle u_0(t), u_h(t)+\lambda w\rangle_{H_2} + k\langle \varphi, u_h(0)+\lambda w\rangle_{H_2} \\
&+ k\int_0^t \left\langle u_0, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_2} d\tau \le 0,
\end{aligned}
\tag{3.17}
$$

and,

$$
\begin{aligned}
&\int_0^{t+h} \langle A(u_{-h}+\lambda w), L(u_{-h}+\lambda w)-Lu_0\rangle\, d\tau + \left\langle u_0(t+h), \frac{du_0(t+h)}{dt} \right\rangle_{H_1} - \langle \varphi, \psi\rangle_{H_1} \\
&- \lim_{n\to\infty} \int_0^{t+h} \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_1} d\tau + \frac{k}{2}\langle u_0(t+h), u_0(t+h)\rangle_{H_2} - \frac{k}{2}\langle \varphi, \varphi\rangle_{H_2} \\
&- \left\langle \frac{du_0(t+h)}{dt}, u_{-h}(t+h)+\lambda w \right\rangle_{H_1} + \langle \psi, u_{-h}(0)+\lambda w\rangle_{H_1} \\
&+ \int_0^{t+h} \left\langle \frac{du_0}{dt}, \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_1} d\tau - k\langle u_0(t+h), u_{-h}(t+h)+\lambda w\rangle_{H_2} \\
&+ k\langle \varphi, u_{-h}(0)+\lambda w\rangle_{H_2} + k\int_0^{t+h} \left\langle u_0, \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_2} d\tau \le 0.
\end{aligned}
\tag{3.18}
$$

In view of (3.17) and (3.18), we have

$$
\begin{aligned}
&\int_0^t \langle A(u_h+\lambda w), L(u_h+\lambda w)-Lu_0\rangle\, d\tau + \int_h^{t+h} \langle A(u_{-h}+\lambda w), L(u_{-h}+\lambda w)-Lu_0\rangle\, d\tau \\
&+ \left\langle u_0(t), \frac{du_0(t)}{dt} \right\rangle_{H_1} + \left\langle u_0(t+h), \frac{du_0(t+h)}{dt} \right\rangle_{H_1} \\
&- 2\langle \varphi, \psi\rangle_{H_1} - \lim_{n\to\infty} \int_0^t \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_1} d\tau \\
&- \lim_{n\to\infty} \int_0^{t+h} \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_1} d\tau + \frac{k}{2}\langle u_0(t), u_0(t)\rangle_{H_2} + \frac{k}{2}\langle u_0(t+h), u_0(t+h)\rangle_{H_2} \\
&- k\langle \varphi, \varphi\rangle_{H_2} - \left\langle \frac{du_0(t)}{dt}, u_h(t)+\lambda w \right\rangle_{H_1} + \langle \psi, u_h(0)+\lambda w\rangle_{H_1} \\
&- \left\langle \frac{du_0(t+h)}{dt}, u_{-h}(t+h)+\lambda w \right\rangle_{H_1} + \langle \psi, u_{-h}(0)+\lambda w\rangle_{H_1} \\
&- k\langle u_0(t), u_h+\lambda w\rangle_{H_2} + k\langle \varphi, u_h(0)+\lambda w\rangle_{H_2} - k\langle u_0(t+h), u_{-h}+\lambda w\rangle_{H_2} \\
&+ k\langle \varphi, u_{-h}(0)+\lambda w\rangle_{H_2} + \int_0^t \left\langle \frac{du_0}{dt}, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_1} d\tau \\
&+ \int_0^{t+h} \left\langle \frac{du_0}{dt}, \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_1} d\tau + k\int_0^t \left\langle u_0, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_2} d\tau \\
&+ k\int_0^{t+h} \left\langle u_0, \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_2} d\tau \le 0.
\end{aligned}
\tag{3.19}
$$

We know that

$$\begin{aligned}
&\int_0^t \left\langle \frac{du_0}{dt}, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_1} d\tau + \int_0^{t+h} \left\langle \frac{du_0}{dt}, \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_1} d\tau \\
&\quad = \int_0^t \left\langle \frac{du_0}{dt}, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_1} d\tau + \int_h^{t+h} \left\langle \frac{du_0(\tau)}{dt}, \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_1} d\tau \\
&\qquad + \int_0^h \left\langle \frac{du_0(\tau)}{dt}, \frac{u_0(\tau)}{h} \right\rangle_{H_1} d\tau \\
&\quad = \int_0^t \left\langle \frac{du_0}{dt}, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_1} d\tau + \int_0^t \left\langle \frac{du_0(\tau+h)}{dt}, \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_1} d\tau \\
&\qquad + \frac{1}{h}\int_0^h \left\langle \frac{du_0(\tau)}{dt}, u_0(\tau) \right\rangle_{H_1} d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \left\langle u_0(\tau), \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_2} d\tau + \int_0^{t+h} \left\langle u_0(\tau), \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_2} d\tau \\
&\quad = \int_0^t \left\langle u_0(\tau), \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_2} d\tau + \int_h^{t+h} \left\langle u_0(\tau), \frac{u_0(\tau)-u_0(\tau-h)}{h} \right\rangle_{H_2} d\tau \\
&\qquad + \int_0^h \left\langle u_0(\tau), \frac{u_0(\tau)}{h} \right\rangle_{H_2} d\tau \\
&\quad = \int_0^t \left\langle u_0(\tau), \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_2} d\tau + \int_0^t \left\langle u_0(\tau+h), \frac{u_0(\tau+h)-u_0(\tau)}{h} \right\rangle_{H_2} d\tau \\
&\qquad + \frac{1}{h}\int_0^h \langle u_0(\tau), u_0(\tau)\rangle_{H_2} d\tau \\
&\quad = \frac{1}{h}\int_t^{t+h} \langle u_0(\tau), u_0(\tau)\rangle_{H_2} d\tau.
\end{aligned}$$

Let $h \to 0^+$. (3.19) can be read as

$$\int_0^t \langle A(u_0+\lambda w), \lambda Lw\rangle d\tau - \left\langle \frac{du_0}{dt}, \lambda w \right\rangle_{H_1} - k\langle u_0, \lambda w\rangle_{H_2} + \langle \psi, \lambda w\rangle_{H_1} + k\langle \varphi, \lambda w\rangle_{H_2} \le 0.$$

According to (2.2) and (2.5), we obtain that

$$\int_0^t \langle A(u_0+\lambda w), Lw\rangle d\tau - \left\langle \frac{du_0}{dt}, Lw \right\rangle_{H} - k\langle \mathcal{L}u_0, Lw\rangle_{H} + \langle \psi, Lw\rangle_{H_1} + k\langle \mathcal{L}\varphi, Lw\rangle_{H} \le 0.$$

Let $\lambda \to 0^+$. It follows that

$$\int_0^t \langle A(u_0), Lw\rangle d\tau - \left\langle \frac{du_0}{dt}, Lw \right\rangle_{H} - k\langle \mathcal{L}u_0, Lw\rangle_{H} + \langle \psi, Lw\rangle_{H_1} + k\langle \mathcal{L}\varphi, Lw\rangle_{H} \le 0.$$

Since $L: X_2 \to X_1$ is dense, the above inequality can be rewritten as

$$\left\langle \frac{du_0}{dt}, v \right\rangle_{H} + k\langle \mathcal{L}u_0, v\rangle_{H} = \int_0^t \langle A(u_0), v\rangle d\tau + \langle \psi, v\rangle_{H} + k\langle \mathcal{L}\varphi, v\rangle_{H},$$

which implies that $u_0 \in W^{1,\infty}((0,\infty),H_1) \cap W^{1,2}((0,\infty),H_2) \cap L^\infty((0,\infty),X_2)$ is a global weak solution of Eq. (2.4).

Secondly, we consider $G = A + B$. Let $v = \frac{d}{dt}Lu_n$ in (3.12). In view of (2.2) and (3.1), it follows that

$$\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2 + k\int_0^t \left\|\frac{du_n}{dt}\right\|_{H_2}^2 dt + F(u_n) = \int_0^t \left\langle B(u_n), \frac{d}{dt}Lu_n\right\rangle dt + F(\varphi_n) + \frac{1}{2}\|\psi_n\|_{H_1}^2.$$

From (3.3), we have

$$\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2 + F(u_n) + k\int_0^t \left\|\frac{du_n}{dt}\right\|_{H_2}^2 dt \le C\int_0^t \left[F(u_n) + \frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2\right] dt + f(t), \tag{3.20}$$

where $f(t) = \int_0^t g(\tau)\,d\tau + \frac{1}{2}\|\psi\|_{H_1}^2 + \sup_n F(\varphi_n)$.

By using the Gronwall inequality, it follows that

$$\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2 + F(u_n) \le f(0)e^{Ct} + \int_0^t f(\tau)e^{C(t-\tau)}\,d\tau, \tag{3.21}$$

which implies that for all $0 < T < \infty$,

$$\{u_n\} \subset W^{1,\infty}\big((0,T),H_1\big) \cap L^\infty\big((0,T),X_2\big) \text{ is bounded.}$$

From (3.20) and (3.21), it follows that

$$\{u_n\} \subset W^{1,2}\big((0,T),H_2\big) \text{ is bounded.}$$

Let

$$\begin{cases} u_n \rightharpoonup^* u_0 & \text{in } W^{1,\infty}\big((0,T),H_1\big) \cap L^\infty\big((0,T),X_2\big), \\ u_n \rightharpoonup u_0 & \text{in } W^{1,2}\big((0,T),H_2\big), \end{cases} \tag{3.22}$$

which implies that $u_n \to u_0$ in $W^{1,2}((0,T),H)$ is uniformly weakly convergent from that $H_2 \subset H$ is a compact imbedding.

The remaining part of the proof is same as assertion (1).

Lastly, assume (3.9) holds. Let $v = \frac{d^2u_n}{dt^2}$ in (3.12). It follows that

$$\begin{aligned} &\int_0^t \left\langle \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2}\right\rangle_H dt + \frac{k}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2 \\ &\quad\le \frac{k}{2}\|\psi_n\|_H^2 + \int_0^t \left[\frac{1}{2}\left\|\frac{d^2u_n}{dt^2}\right\|_H^2 + CF(u_n) + g(\tau)\right] d\tau. \end{aligned}$$

From (3.21), the above inequality implies

$$\int_0^t \left\|\frac{d^2u_n}{dt^2}\right\|_H^2 d\tau \le C \quad (C > 0 \text{ is constant}). \tag{3.23}$$

We see that for all $0 < T < \infty$, $\{u_n\} \subset W^{2,2}((0,T),H)$ is bounded. Thus $u \in W^{2,2}((0,T),H)$.
□

4 Main result

Now, we consider the nonlinear strongly damped plate or beam equation (1.1). Set

$$F(x,y) = \int_0^y f(x,z)\,dz. \tag{4.1}$$

We assume

$$\begin{cases} F(x,y) \geq C_1|y|^p - C_2, & p \geq 2, \\ |f(x,y)| \leq C\big(|y|^{p-1}+1\big), \end{cases} \tag{4.2}$$

$$\big(f(x,y_1) - f(x,y_2)\big)(y_1 - y_2) \geq 0, \tag{4.3}$$

$$g\big(x,u,Du,D^2u,D^3u\big)|_{\partial\Omega} = 0, \quad \forall u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0, \tag{4.4}$$

$$\big|g(x,\xi)\big| + \big|D_x g(x,\xi)\big| \leq C\bigg(\sum_{|\beta|\leq 3} |\xi_\beta|^{\frac{p}{2}} + 1\bigg), \tag{4.5}$$

$$\bigg|\sum_{|\alpha|\leq 3} D_{\xi_\alpha}\bigg| \leq C\bigg(\sum_{|\beta|\leq 3} |\xi_\beta|^{\frac{p}{2}-1} + 1\bigg), \tag{4.6}$$

where $\xi = \{\xi_\alpha \,|\, |\alpha| \leq 3\}$, ξ_α corresponds to $D^\alpha u$.

Theorem 4.1 *Under the assumptions (4.1)-(4.6), if φ satisfies the bounded condition of Eq. (1.1), for $(\varphi,\psi) \in W^{4,p}(\Omega) \cap H^2 \times H_0^1(\Omega)$, then there exists a global strong solution for Eq. (1.1)*

$$\begin{aligned} &u \in L^\infty_{\mathrm{loc}}\big((0,\infty), W^{4,p}(\Omega)\big), \\ &u_t \in L^\infty_{\mathrm{loc}}\big((0,\infty), H^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big((0,\infty), H^3(\Omega)\big), \\ &u_{tt} \in L^{p'}\big((0,T)\times\Omega\big), \qquad p' = \frac{p}{p-1} \quad \forall 0 < T < \infty. \end{aligned}$$

Proof We introduce spatial sequences

$$\begin{aligned} &X = \big\{u \in C^\infty(\Omega) \big| \triangle^k u\big|_{\partial\Omega} = 0, k = 0,1,2,\ldots\big\}, \\ &X_1 = L^p(\Omega), \qquad X_2 = \big\{W^{4,p}(\Omega)|u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0\big\}, \\ &H = L^2(\Omega), \qquad H_1 = H^2(\Omega) \cap H_0^1(\Omega), \\ &H_2 = \big\{H^3(\Omega)|u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0\big\}, \\ &H_3 = \big\{u \in H^{4m}(\Omega)|u|_{\partial\Omega} = \cdots = \triangle^{2m-1}u|_{\partial\Omega}\big\}, \end{aligned}$$

where the inner products of H_1, H_2 and H_3 are defined by

$$\begin{aligned} &\langle u,v\rangle_{H_1} = \int_\Omega \triangle u \triangle v\,dx, \qquad \langle u,v\rangle_{H_2} = \int_\Omega \nabla\triangle u \nabla\triangle v\,dx, \\ &\langle u,v\rangle_{H_3} = \int_\Omega \triangle^{2m} u \triangle^{2m} v\,dx, \end{aligned}$$

where $m \geq 1$ such that $H_3 \subset X_2$ is an embedding.

Linear operator $\mathcal{L} : X_2 \to X_1$ and $L : X_2 \to X_1$ is defined by

$$\mathcal{L}u = -\triangle u, \qquad Lu = \triangle^2 u.$$

It is known that $\mathcal{L}$ and L satisfy (2.2), (2.3) and (2.5). Define $G = A + B : X_2 \to X_1^*$ by

$$\langle Au, v\rangle = \int_\Omega f\big(x, \triangle^2 u\big) v\, dx, \qquad \langle Bu, v\rangle = \int_\Omega g\big(x, u, Du, D^2u, D^3u\big) v\, dx, \quad \text{for } v \in X_1.$$

Let $F_1(u) = \int_\Omega F(x, \triangle u)\, dx$, where F is the same as in (4.2). We get

$$\begin{aligned}
&\langle Au, Lu\rangle = -\big\langle DF_1(u), v\big\rangle, \\
&F_1(u) \to \infty \quad \Leftrightarrow \quad \|u\|_{X_2} \to \infty,
\end{aligned}$$

which implies conditions (A1), (A2) of Theorem 3.1.

From (4.3), we have

$$\big\langle DF_1(u_1) - DF_1(u_2), u_1 - u_2\big\rangle \geq 0.$$

From (4.5) and (4.6), we obtain that $B : X_2 \to X_1^*$ is a compact operator. Then, B satisfies (3.6) and (3.7).

We will show (3.3) as follows. From (4.4) and (4.5), for $\forall u, v \in X$, it follows that

$$\begin{aligned}
\big|\langle Bu, Lv\rangle\big| &= \int_\Omega \big|g\big(x, u, Du, D^2u, D^3u\big)\triangle^2 v\big|\, dx \\
&= \int_\Omega \big|\nabla g\big(x, u, Du, D^2u, D^3u\big) \cdot \nabla\triangle v\big|\, dx \\
&\leq \frac{k}{2}\int_\Omega |\nabla\triangle v|^2\, dx + \frac{2}{k}\int_\Omega \big|\nabla g\big(x, u, Du, D^2u, D^3u\big)\big|^2\, dx \\
&\leq \frac{k}{2}\|v\|_{H_2}^2 + C\int_\Omega \left[|D_x g|^2 + \sum_{i=1}^n \sum_{|\alpha|\leq 3} |D_{\zeta_\alpha} g|^2 \big|D_i D^\alpha u\big|^2\right] dx \\
&\leq \frac{k}{2}\|v\|_{H_2}^2 + C\int_\Omega \left[\sum_{|\alpha|\leq 4} \big|D^\alpha u\big|^p + 1\right] dx,
\end{aligned}$$

which implies condition (A3) of Theorem 3.1. From Theorem 3.1, Eq. (1.1) has a solution

$$u \in L^\infty_{\mathrm{loc}}\big((0,\infty), W^{4,p}(\Omega)\big), \tag{4.7}$$

$$u_t \in L^\infty_{\mathrm{loc}}\big((0,\infty), H^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big((0,\infty), H^3(\Omega)\big). \tag{4.8}$$

Lastly, we show that $u'' \in L^{p'}(\Omega \times (0,T))$. By Definition 2.1, u satisfies

$$\begin{aligned}
\int_\Omega u_t(x,t) v\, dx + \int_\Omega u(x,t) v\, dx = {}& \int_0^t \int_\Omega \big[f(x, \triangle u) + g\big(x, u, Du, D^2u, D^3u\big)v\big]\, dx\, dt \\
&+ \int_\Omega \psi v\, dx + \int_\Omega \varphi v\, dx, \quad \forall v \in L^p(\Omega).
\end{aligned}$$

Then, for any $h > 0$, it follows that

$$\begin{aligned}&\int_\Omega \triangle_h^t u_t(x,t)v\,dx + \int_\Omega \triangle_h^t u(x,t)v\,dx\\&\quad = \frac{1}{h}\int_t^{t+h}\int_\Omega \bigl[f(x,\triangle u) + g\bigl(x,u,Du,D^2u,D^3u\bigr)v\bigr]\,dx\,dt,\end{aligned} \tag{4.9}$$

where $\triangle_h^t u = \frac{1}{h}(u(t+h) - u(t))$. Let $v = |\triangle_h^t u_t|^{p'-2}\triangle_h^t u_t$. From (4.9), we have

$$\begin{aligned}&\int_\Omega \bigl|\triangle_h^t u_t(x,t)\bigr|^{p'}\,dx + \int_\Omega \triangle_h^t u(x,t)\bigl|\triangle_h^t u_t\bigr|^{p'-2}\triangle_h^t u_t\,dx\\&\quad = \frac{1}{h}\int_t^{t+h}\int_\Omega \bigl[\bigl(f(x,\triangle u) + g\bigl(x,u,Du,D^2u,D^3u\bigr)\bigr)\bigl|\triangle_h^t u_t\bigr|^{p'-2}\triangle_h^t u_t\bigr]\,dx\,dt.\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\int_\Omega \bigl|\triangle_h^t u_t(x,t)\bigr|^{p'}\,dx\\&\quad \le \int_\Omega \bigl|\triangle_h^t u(x,t)\bigr|\bigl|\triangle_h^t u_t\bigr|^{p'-1}\,dx\\&\qquad + \frac{1}{h}\int_t^{t+h}\int_\Omega \bigl[\bigl(\bigl|f(x,\triangle u)\bigr| + \bigl|g\bigl(x,u,Du,D^2u,D^3u\bigr)\bigr|\bigr)\bigl|\triangle_h^t u_t\bigr|^{p'-1}\bigr]\,dx\,dt\\&\quad \le \frac{1}{2}\int_\Omega \bigl|\triangle_h^t u_t(x,t)\bigr|^{p'}\,dx + 4\int_\Omega \bigl|\triangle_h^t u(x,t)\bigr|^{p'}\,dx + \frac{4}{h^{p'}}\int_t^{t+h}\int_\Omega \bigl[|f|^{p'} + |g|^{p'}\bigr]\,dx\,dt.\end{aligned}$$

From (4.2) and (4.5), we have

$$\begin{aligned}&\int_0^T\int_\Omega \left|\frac{u_t(x,t+h) - u_t(x,t)}{h}\right|^{p'}\,dx\,dt\\&\quad \le C\int_0^T\int_\Omega \bigl|\triangle_h^t u(x,t)\bigr|^{p'}\,dx\,dt\\&\qquad + C\int_0^T\int_t^{t+h}\int_\Omega \bigl[|f|^{p'} + |g|^{p'}\bigr]\,dx\,d\tau\,dt\\&\quad \le C\int_0^T\int_\Omega \bigl|u_t(x,t)\bigr|^{\frac{p}{p-1}}\,dx\,dt\\&\qquad + C\int_0^T\int_\Omega \left[\bigl(|\triangle u|^{p-1} + 1\bigr)^{p'} + \left(\sum_{|\alpha|\le 3}\bigl|D^\alpha u\bigr|^{\frac{p}{2}} + 1\right)^{p'}\right]\,dx\,dt\\&\quad \le C\int_0^T\int_\Omega \bigl|u_t(x,t)\bigr|^{\frac{p}{p-1}}\,dx\,dt + C\int_0^T\int_\Omega \left[|\triangle u|^p + \sum_{|\alpha|\le 3}\bigl|D^\alpha u\bigr|^{\frac{p^2}{2(p-1)}} + 1\right]\,dx\,dt.\end{aligned}$$

By using the Sobolev embedding theorem, it follows that from (4.7) and (4.8) the right of the above inequality is bounded. Then, u_{tt} exists almost everywhere in $\Omega \times (0,T)$, and $u_{tt} \in L^{p'}(\Omega \times (0,T))$, $\forall 0 < T < \infty$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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Acknowledgement
The authors are very grateful to the anonymous referees whose careful reading of the manuscript and valuable comments enhanced the presentation of the manuscript. Supported by the National Natural Science Foundation of China (NO. 11071177), the NSF of Sichuan Science and Technology Department of China (NO. 2010JY0057) and the NSF of Sichuan Education Department of China (NO. 11ZA102).

Received: 11 April 2012 Accepted: 3 July 2012 Published: 20 July 2012

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doi:10.1186/1687-2770-2012-76
Cite this article as: Luo et al.: Existence of solutions to strongly damped plate or beam equations. *Boundary Value Problems* 2012 **2012**:76.