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Existence and uniqueness of solutions for periodic-integrable boundary value problem of second order differential equation

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Abstract
In this paper we deal with one kind of second order periodic-integrable boundary value problem. Using the lemma on bilinear form and Schauder's fixed point theorem, we give the existence and uniqueness of solutions for the problem under Lazer type nonresonant condition.
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1 Introduction and main results

In this paper, we consider the solutions to the following periodic-integrable boundary value problem (for short, PIBVP):

$$\begin{aligned} &\big(p(t)x'\big)' + f(t,x) = 0, \\ &x(0) = x(T), \qquad \int_0^T x(s)\,ds = 0, \end{aligned} \tag{1.1}$$

where $p(t) \in C^1(R,R)$ is a given T-periodic function in $t \in R$, and $p(t) > 0$; $f \in C^1(R \times R, R)$ is T-periodic in t.

Throughout this paper, we assume

(A1) there exist two constants m and M such that

$$m \le \frac{f_x(t,x)}{p(t)} \le M$$

for all $t \in R$ and $x \in R$;

(A2) there exists $N \in Z^+$ such that

$$\frac{4\pi^2}{T^2}N^2 < m \le M < \frac{4\pi^2}{T^2}(N+1)^2.$$

Recently, boundary value problems with integral conditions have been studied extensively [6–10]. As we all know Lazer type conditions are essential for the existence and uniqueness of periodic solutions of equations [1–4]. In [5] the existence of periodic solutions has been considered for the following second order equation:

$$\big(p(t)x'\big)' + f(x,t) = 0.$$

Motivated by the above works, we will consider periodic-integrable boundary value problem (1.1). The main result obtained by us is the following theorem.

Theorem 1 *Assume that (A1) and (A2) are satisfied. Then PIBVP (1.1) has a unique solution.*

This paper is organized as follows. Section 2 deals with a linear problem. There, using the bilinear lemma developed by Lazer, one proves the uniqueness of solutions for linear equations. In Section 3, applying the result in Section 2 and Schauder's fixed point theorem, we complete the proof of Theorem 1.

2 Linear equation

Consider the following linear PIBVP:

$$\begin{aligned} &\big(p(t)x'\big)' + q(t)x = 0, \\ &x(0) = x(T), \qquad \int_0^T x(s)\,\mathrm{d}s = 0, \end{aligned} \tag{2.1}$$

here $p(t) \in C^1(R,R)$ is a given T-periodic function in $t \in R$, and $p(t) > 0$; $q(t) \in C(R,R)$ is a T-periodic function. Assume that

(L1) there exist two constants m and M such that

$$m \le \frac{q(t)}{p(t)} \le M$$

for all $t \in R$. Moreover, m and M suit (A2).

Theorem 2 *Assume that (L1) and (A2) are satisfied, then PIBVP (2.1) has only a trivial solution.*

In order to prove Theorem 2, let us give some following concepts.

First, for any interval $[\alpha,\beta] \subset [0,T]$, define

$$\begin{aligned} \mathcal{O}_{\alpha,\beta} = \big\{ &u(t) \in L^2(0,T) : u'(t) \text{ is absolutely continuous on } [\alpha,\beta], \\ &\text{and } u(t) = 0 \text{ for any } t \in [0,\alpha] \cup [\beta,T] \big\}. \end{aligned}$$

It is clear that $\mathcal{O}_{\alpha,\beta}$ is a linear space with the norm as follows:

$$\|u\| = \max_{t\in[0,T]} \big|u(t)\big| + \max_{t\in[0,T]} \big|u'(t)\big|.$$

Define a bilinear form on $\mathcal{O}_{\alpha,\beta}$ as follows:

$$H_{\alpha,\beta}(u,v) = \int_0^T \big[p(t)u'(t)v'(t) - q(t)u(t)v(t)\big]\,\mathrm{d}t$$

for any $u(t) \in \mathcal{O}_{\alpha,\beta}$ and $v(t) \in \mathcal{O}_{\alpha,\beta}$. Let

$$\mathcal{X}_{\alpha,\beta} = \left\{ x \in \mathcal{O}_{\alpha,\beta} : x = \sum_{i=N+1}^{\infty} \left(a_i \cos \frac{2\pi i}{T} t + b_i \sin \frac{2\pi i}{T} t \right) \right\},$$
$$\mathcal{Y}_{\alpha,\beta} = \left\{ y \in \mathcal{O}_{\alpha,\beta} : y = c_0 + \sum_{k=1}^{N} \left(c_k \cos \frac{2\pi i}{T} t + d_k \sin \frac{2\pi i}{T} t \right) \right\},$$

where N suits assumption (L1), and a_m, b_m, c_0, c_k and d_k are some constants. Then $\mathcal{O}_{\alpha,\beta} = \mathcal{X}_{\alpha,\beta} \oplus \mathcal{Y}_{\alpha,\beta}$.

From $p(t) \in C^1(R, R)$ and $p(t) > 0$, we can obtain that there exist two constants M_1 and M_2 such that

$$0 \le M_1 \le p(t) \le M_2$$

for all $t \in R$. Then from assumptions (L1) and (A2), we have

$$\begin{aligned} H_{\alpha,\beta}(x,x) &= \int_0^T p(t) \left(x'^2(t) - \frac{q(t)}{p(t)} x^2(t) \right) \mathrm{d}t \\ &\ge \int_0^T p(t) \left(x'^2(t) - Mx^2(t) \right) \mathrm{d}t \\ &\ge \frac{2\pi^2 M_1}{T} \sum_{i=N+1}^{\infty} \left(i^2 - (N+1)^2 \right) \left(a_i^2 + b_i^2 \right) \\ &\ge 0 \end{aligned}$$

for all $x \in \mathcal{X}_{\alpha,\beta}$, and

$$\begin{aligned} H_{\alpha,\beta}(y,y) &= \int_0^T p(t) \left(y'^2(t) - \frac{q(t)}{p(t)} y^2(t) \right) \mathrm{d}t \\ &\le \int_0^T p(t) \left(y'^2(t) - my^2(t) \right) \mathrm{d}t \\ &\le \frac{2\pi^2 M_2}{T} \sum_{k=1}^{N} \left(k^2 - N^2 \right) \left(c_k^2 + d_k^2 \right) - mM_2 T c_0^2 \\ &\le 0 \end{aligned}$$

for all $y \in \mathcal{Y}_{\alpha,\beta}$. Thus, $H_{\alpha,\beta}$ is positive definite on $\mathcal{X}_{\alpha,\beta}$ and negative definite on $\mathcal{Y}_{\alpha,\beta}$. By the lemma in [1], we assert that if $H_{\alpha,\beta}(u,v) = 0$ for all $u \in \mathcal{O}_{\alpha,\beta}$, then $v \equiv 0$.

For every x on $[\alpha, \beta]$ with $x(\alpha) = x(\beta) = 0$, we introduce an auxiliary function

$$x^{\alpha,\beta}(t) = \begin{cases} x(t), & t \in [\alpha, \beta], \\ 0, & t \in [0, \alpha] \cup [\beta, T]. \end{cases}$$

The following lemma is very useful in our proofs.

Lemma 1 *If $p(t)$, $q(t)$ are continuous and satisfy (L1) and (A2), then the following two points boundary value problem*

$$\begin{aligned} &(p(t)x')' + q(t)x = 0, \\ &x(\alpha) = x(\beta) = 0 \end{aligned} \tag{2.2}$$

has only a trivial solution.

Proof It is clear that 0 is a solution of two points boundary value problem (2.2). If $v(t)$ is a solution of problem (2.2), then $v^{\alpha,\beta}(t) \in \mathcal{O}_{\alpha,\beta}$. For any $u \in \mathcal{O}_{\alpha,\beta}$, we have

$$\int_\alpha^\beta \left[(p(t)v'(t))' u(t) + q(t)u(t)v(t) \right] \mathrm{d}t = 0$$

by using (2.2). Integrating the first terms by parts, we derive

$$-H_{\alpha,\beta}(u, v^{\alpha,\beta}) = -\int_0^T \left[p(t)u'(t)v^{\alpha,\beta\prime}(t) - q(t)u(t)v^{\alpha,\beta}(t) \right] \mathrm{d}t = 0.$$

By assumption (L1), $H_{\alpha,\beta}$ is positive definite on $\mathcal{X}_{\alpha,\beta}$ and negative definite on $\mathcal{Y}_{\alpha,\beta}$. These show $v^{\alpha,\beta}(t) \equiv 0$ for $t \in [0, T]$, *that is*, $v(t) \equiv 0$ for $t \in [\alpha, \beta]$. The proof of Lemma 1 is ended. □

Proof of Theorem 2 It is clear that PIBVP (2.1) has at least one solution, for example, $x_* \equiv 0$. Assume that PIBVP (2.1) possesses a nontrivial solution $x^* \not\equiv 0$. The proof is divided into three parts.

Case 1: $x^*(0) = x^*(T) = 0$. By Lemma 1 ($\alpha = 0$ and $\beta = T$), PIBVP (2.1) has only a trivial solution. This contradicts $x^* \not\equiv 0$.

Case 2: $x^*(0) = x^*(T) = \eta > 0$. Denote

$$S = \{ t \in [0, T] : x^*(t) = 0 \}.$$

Take

$$a = \inf_{t\in S} t \quad \text{and} \quad b = \sup_{t\in S} t.$$

From $\int_0^{2\pi} x^*(s)\,\mathrm{d}s = 0$, there are at least two points in the set S, which implies that $0 < a < b < 2\pi$ and $x^*(a) = x^*(b) = 0$. By Lemma 1 ($\alpha = a$ and $\beta = b$) the two points boundary value problem

$$\begin{aligned} &(p(t)x')' + q(t)x = 0, \\ &x(a) = x(b) = 0 \end{aligned} \tag{2.3}$$

only has a trivial solution. Hence we obtain $x^*(t) \equiv 0$, $t \in [a, b]$. By the definitions of a and b, one has

$$x^*(t) > 0 \quad \text{for all } t \in [0, a) \cup (b, T].$$

From $\int_0^T x^*(s)\,\mathrm{d}s = 0$, we get

$$\int_0^a x^*(s)\,\mathrm{d}s + \int_b^T x^*(s)\,\mathrm{d}s = 0.$$

This contradicts $\int_0^a x^*(s)\,\mathrm{d}s > 0$ and $\int_b^T x^*(s)\,\mathrm{d}s > 0$.

Case 3: $x^*(0) = x^*(T) = \eta < 0$. This case is similar to Case 2.

Thus, we complete the proof of Theorem 2. □

Theorem 3 *If $p(t)$, $q(t)$ are continuous and satisfy (L1) and (A2), then the following PIBVP*

$$\begin{aligned} &\bigl(p(t)x'\bigr)' + q(t)x = h(t), \\ &x(0) = x(T), \qquad \int_0^T x(s)\,\mathrm{d}s = 0 \end{aligned} \tag{2.4}$$

has a unique solution.

Proof Let $x_{*1}(t)$ and $x_{*2}(t)$ be two linear independent solutions of the following linear equation:

$$\bigl(p(t)x'\bigr)' + q(t)x = 0.$$

Assume that $x_*(t) = c_1x_{*1}(t) + c_2x_{*2}(t)$ is a solution of PIBVP (2.1), where c_1 and c_2 are constants. Then by the boundary value conditions of (2.1),

$$\begin{cases} (x_{*1}(0) - x_{*1}(T))c_1 + (x_{*2}(0) - x_{*2}(T))c_2 = 0, \\ \int_0^T x_{*1}(s)\,\mathrm{d}sc_1 + \int_0^T x_{*2}(s)\,\mathrm{d}sc_2 = 0. \end{cases}$$

By Theorem 3, PIBVP (2.1) has only a trivial solution, which shows

$$\begin{vmatrix} x_{*1}(0) - x_{*1}(T) & x_{*2}(0) - x_{*2}(T) \\ \int_0^T x_{*1}(s)\,\mathrm{d}s & \int_0^T x_{*2}(s)\,\mathrm{d}s \end{vmatrix} \neq 0. \tag{2.5}$$

Let $x_\star(t) = c_3x_{*1}(t) + c_4x_{*2}(t) + x_{*0}(t)$ be a solution of PIBVP (2.4), where $x_{*0}(t)$ is a solution of the equation

$$\bigl(p(t)x'\bigr)' + q(t)x = h(t).$$

From the boundary value conditions, we have

$$\begin{cases} (x_{*1}(0) - x_{*1}(T))c_3 + (x_{*2}(0) - x_{*2}(T))c_4 = x_{*0}(0) - x_{*0}(T), \\ \int_0^T x_{*1}(s)\,\mathrm{d}sc_3 + \int_0^T x_{*2}(s)\,\mathrm{d}sc_4 = \int_0^T x_{*0}(s)\,\mathrm{d}s. \end{cases}$$

From (2.5) constants c_3, c_4 are unique. Thus, PIBVP (2.4) has only one solution. □

3 Nonlinear equations

Let us prove Theorem 1. Rewrite (1.1) as follows:

$$\begin{aligned} &\big(p(t)x'\big)' + h(t,x)x = -f(t,0), \\ &x(0) = x(T), \qquad \int_0^T x(s)\,\mathrm{d}s = 0, \end{aligned} \tag{3.1}$$

where

$$h(t,x) = \int_0^1 f_x(t,\theta x)\,\mathrm{d}\theta.$$

Define

$$\begin{aligned} \mathcal{O}^* = \bigg\{ &u(t) \in L^2(0,T) : u'(t) \text{ is absolutely continuous on[0,T]}, \\ &u(0) = u(T) \text{ and } \int_0^T u(s)\,\mathrm{d}s = 0 \bigg\}. \end{aligned}$$

Fix $y \in \mathcal{O}^*$, introduce an auxiliary PIBVP

$$\begin{aligned} &\big(p(t)x'\big)' + h(t,y)x = -f(t,0), \\ &x(0) = x(T), \qquad \int_0^T x(s)\,\mathrm{d}s = 0. \end{aligned} \tag{3.2}$$

To prove the main result, we need the following Lemma 2.

Lemma 2 *If f satisfies (A1) and (A2), then for any given $y \in \mathcal{O}^*$, PIBVP (3.2) has only one solution, denoted as $x_y(t)$ and $\|x_y\| \le M$.*

Proof From condition (A2), it follows that

$$m \le \frac{h(t,y)}{p(t)} \le M.$$

By Theorem 3, PIBVP (3.2) has only one solution $x_y(t)$. If $\|x_y\| \le M$ does not hold, there would exist a sequence $\{y_m(t)\}$ such that $\|x_{y_m}\| \to \infty$, $m \to \infty$. Choose a subsequence of $\{h(t,y_m)\}_{m=1}^{\infty}$, without loss of generality, express as itself, such that the sequences are weakly convergent in $L^2(0,T)$. Denote the limit as $h_0(t)$. It is obvious that $h_0(t) \in L^2(0,T)$.

Because the set

$$S = \bigg\{ q(t) \in L^2[0,T] : m \le \frac{q(t)}{p(t)} \le M \bigg\}$$

is bounded convex in $L^2(0,T)$, by the Mazur theorem, we have $h_0(t) \in S$. Hence,

$$m \le \frac{h_0(t)}{p(t)} \le M.$$

By the Arzela-Ascoli theorem, passing to a subsequence, we may assume that

$$x_m = \frac{x_{y_m}}{\|x_{y_m}\|} \to x_0,$$

and $x'_m \to z(t)$ in $C([0,T],R)$. Thus, $x_0(0) = x_0(T)$ and $\int_0^T x_0(s)\,\mathrm{d}s = 0$.

By

$$x_m(t) = x_m(0) + \int_0^t x'_m(s)\,\mathrm{d}s,$$

one has

$$x_0(t) = x_0(0) + \int_0^t z(s)\,\mathrm{d}s,$$

which implies $z(t) = x'_0(t)$, for any $t \in [0,T]$. Hence, $\|x_0\| = 1$.

From PIBVP (3.2), we obtain

$$\begin{aligned} &\bigl(p(t)x'_m\bigr)' + h(t,y_m)x_m = \frac{-f(t,0)}{\|x_{y_m}\|}, \\ &x_m(0) = x_m(T), \qquad \int_0^T x_m(s)\,\mathrm{d}s = 0. \end{aligned} \tag{3.3}$$

This shows that $x_0(t)$ is a nontrivial solution of the following PIBVP:

$$\begin{aligned} &\bigl(p(t)x'\bigr)' + h_0(t)x = 0, \\ &x(0) = x(T), \qquad \int_0^T x(s)\,\mathrm{d}s = 0. \end{aligned} \tag{3.4}$$

On the other hand, by Theorem 2, PIBVP (3.4) has only zero, which leads to a contradiction. The proof of Lemma 2 is completed. □

Set

$$B_{M^*} = \bigl\{x \in \mathcal{O}^* : \|x\| \le M^*\bigr\}.$$

Define an operator $F : \mathcal{O}^* \to \mathcal{O}^*$ by $Fy = x_y$. Applying Lemma 2, $F : B_{M^*} \to B_{M^*}$.

Lemma 3 *Operator F is completely continuous on $\mathcal{O}^*$.*

Proof We first prove that F is continuous. Given any $\{y_m\} \subset \mathcal{O}^*$ such that $y_m \to y_0 \in \mathcal{O}^*$. Put $u_m = x_{y_m} - x_{y_0}$. From the definition

$$\begin{aligned} &\bigl(p(t)u'_m\bigr)' + h(t,y_m)u_m = \bigl[h(t,y_0) - h(t,y_m)\bigr]x_{y_0}, \\ &u_m(0) = u_m(T), \qquad \int_0^T u_m(s)\,\mathrm{d}s = 0. \end{aligned} \tag{3.5}$$

We would prove that $u_m \to 0$ in $C^1([0,T],R)$. If not, then there would be a $c > 0$ such that

$$\lim_{m\to\infty} \sup \|u_m\| \geq c.$$

Utilizing Lemma 2 and Arzela-Ascoli theorem, passing to a subsequence, we may assume that $u_m \to u_0$. Similar to the proof of Lemma 2, we have $u_m' \to u_0'$. Then

$$\begin{aligned} &\big(p(t)u_0'\big)' + h(t,y_0)u_0 = 0, \\ &u_0(0) = u_0(T), \qquad \int_0^T u_0(s)\,\mathrm{d}s = 0. \end{aligned} \tag{3.6}$$

Moreover,

$$m \leq \frac{h(t,y_0)}{p(t)} \leq M.$$

Hence, from Theorem 2, $u_0(t) \equiv 0$. This implies F is continuous. By Lemma 2, for any bounded subset $D \subset \mathcal{O}^*$, $F(D)$ is also bounded. Hence, applying the continuity of F and Arzela-Ascoli theorem, $F(D)$ is relatively compact. This shows F is completely continuous on $\mathcal{O}^*$. The proof of Lemma 3 is completed. □

Proof of Theorem 1 By Lemma 2, Lemma 3 and Schauder's fixed point theorem, F has a fixed point in $\mathcal{O}^*$, *that is*, PIBVP (1.1) has a solution $x(t)$.

The following is to prove uniqueness. Let $x_1(t)$ and $x_2(t)$ be any two solutions of equation (1.1). Then $x(t) = x_1(t) - x_2(t)$ is a solution of the equation

$$\big(p(t)x'\big)' + x\int_0^1 f_x(t, x_2 + \theta x)\,\mathrm{d}\theta = 0.$$

Employing (A2), we have

$$m \leq \frac{\int_0^1 f_x(t,x_2+\theta x)\,\mathrm{d}\theta}{p(t)} \leq M.$$

Hence by Theorem 3, $x(t) \equiv 0$. The uniqueness is proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, HH, FC and YC contributed to each part of this work equally and read and approved the final version of the manuscript.

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