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Convergence rate of solutions toward stationary solutions to the bipolar Navier-Stokes-Poisson equations in a half line

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Abstract
In this paper, we show the convergence rate of a solution toward the stationary solution to the initial boundary value problem for the one-dimensional bipolar compressible Navier-Stokes-Poisson equations. For the supersonic flow at spatial infinity, if an initial perturbation decays with the algebraic or the exponential rate in the spatial asymptotic point, the solution converges to the corresponding stationary solution with the same rate in time as time tends to infinity. For the transonic flow at spatial infinity, the solution converges to the stationary solution in time with the lower rate than that of the initial perturbation in the spatial. These results are proved by the weighted energy method.
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1 Introduction

In this paper, we are concerned with the following bipolar Navier-Stokes-Poisson equations:

$$\begin{cases} \partial_t\rho_1 + \partial_x(\rho_1 u_1) = 0, \\ \partial_t(\rho_1 u_1) + \partial_x(\rho_1 u_1^2 + P_1(\rho_1)) = \mu_1 u_{1xx} + \rho_1 E, \\ \partial_t\rho_2 + \partial_x(\rho_2 u_2) = 0, \\ \partial_t(\rho_2 u_2) + \partial_x(\rho_2 u_2^2 + P_2(\rho_2)) = \mu_2 u_{2xx} - \rho_2 E, \\ E_x = \rho_1 - \rho_2, \end{cases} \tag{1.1}$$

in a one-dimensional half space $\mathbb{R}_+ := (0,\infty)$. Here the unknown functions are the densities ρ_i $(i = 1,2) > 0$, the velocities u_i $(i = 1,2)$, and the electron field E. $P_i(\rho_i)$ $(i = 1,2)$ is the pressure depending only on the density. μ_i $(i = 1,2)$ is viscosity coefficient. Throughout this paper, we assume that two fluids of electrons and ions have the same equation of state $P_1(\cdot) = P_2(\cdot) = P(\cdot)$ with $P(\rho) = K\rho^\gamma$ for $K > 0$ and $\gamma \ge 1$, and also they have the same viscosity coefficients $\mu_1 = \mu_2 = 1$. The bipolar Navier-Stokes-Poisson system is used to simulate the transport of charged particles (*e.g.*, electrons and ions). It consists of the compressible Navier-Stokes equation of two-fluid under the influence of the electro-static potential force governed by the self-consisted Poisson equation. Note that when we only consider one particle in the fluids, we also have the unipolar Navier-Stokes-Poisson equations. For more details, we can refer to [1–4].

Recently, some important progress was made for the compressible unipolar Navier-Stokes-Poisson system. The local and/or global existence of a renormalized weak solution for the Cauchy problem of the multi-dimensional compressible Navier-Stokes-Poisson system were proved in [5–7]. Chan [8] also considered the nonexistence of global weak solutions to the Navier-Stokes-Poisson equations in $\mathbb{R}^N$. Hao and Li [9] established the global strong solutions of the initial value problem for the multi-dimensional compressible Navier-Stokes-Poisson system in a Besov space. The global existence and L^2-decay rate of the smooth solution of the initial value problem for the compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$ were achieved by Li and his collaborators in [10, 11]. The pointwise estimates of the smooth solutions for the three-dimensional isentropic compressible Navier-Stokes-Poisson equation were obtained in [12]. The quasineutral limit of the compressible Navier-Stokes-Poisson system was studied in [13–15]. However, the results about the bipolar Navier-Stokes-Poisson equations are very few. Lastly, Li *et al.* [16] showed the global existence and asymptotic behavior of smooth solutions for the initial value problem of the bipolar Navier-Stokes-Poisson equations. Duan and Yang [17] studied the unique existence and asymptotic stability of a stationary solution for the initial boundary value problem, and they showed that the large-time behavior of solutions for the bipolar Navier-Stokes-Poisson equations coincided with the one for the single Navier-Stokes system in the absence of the electric field. The consistency is also observed and proved between the bipolar Euler-Poisson system and the single damped Euler equation; for example, see [18–20] and the references therein.

In this paper, we are going to discuss the initial-boundary value problem for the one-dimensional bipolar Navier-Stokes-Poisson equations. Now we give the initial condition

$$(\rho_1, u_1, \rho_2, u_2)(x,0) = (\rho_{10}, u_{10}, \rho_{20}, u_{20})(x) \to (\rho_+, u_+, \rho_+, u_+) \quad \text{as } x \to \infty, \tag{1.2}$$

and the boundary date

$$u_1(0,t) = u_2(0,t) = u_b < 0. \tag{1.3}$$

Here, we suppose $\inf_{x\in\mathbb{R}_+} \rho_{i0}$ $(i = 1,2) > 0$ and further the compatibility condition $u_b = u_{10}(0) = u_{20}(0)$. Moreover, for the unique existence, we also assume

$$E(+\infty, t) = 0. \tag{1.4}$$

In [17], the authors showed that the solution to (1.1)-(1.4) converges to the corresponding stationary solution of the single Navier-Stokes system in the absence of the electric field

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u + P(\rho)) = u_{xx}, \end{cases} \tag{1.5}$$

as time tends to infinity. Then, let $(\tilde{\rho}, \tilde{u})(x)$ be the stationary solution to the system (1.5). We know that the stationary solution $(\tilde{\rho}, \tilde{u})$ satisfies

$$\begin{cases} (\tilde{\rho}\tilde{u})_x = 0, \\ (\tilde{\rho}\tilde{u}^2 + P(\tilde{\rho}))_x = \mu \tilde{\rho}_{xx}, \end{cases} \tag{1.6}$$

and the boundary and spatial asymptotic conditions

$$\tilde{u}(0) = u_b, \qquad \lim_{x\to\infty}(\tilde{\rho}, \tilde{u}) = (\rho_+, u_+), \qquad \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x) > 0. \tag{1.7}$$

In this paper, we are mainly concerned with the decay rate of solutions to (1.1)-(1.4) toward the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\rho}, \tilde{u}, 0)$. Now we state the main result in the following theorem.

Theorem 1.1 *Suppose that $M_+ \geq 1$ and $u_b < u_*$ hold. The initial data $(\rho_{10}, u_{10}, \rho_{20}, u_{20}, E_0)(x)$ is supposed to satisfy*

$$(\rho_{i0}, u_{i0})\ (i = 1,2) \in H^1(\mathbb{R}_+), \qquad E_0(x) \in L^2(\mathbb{R}_+), \qquad \inf_{x\in\mathbb{R}_+}(\rho_{10}, \rho_{20}) > 0, \tag{1.8}$$

and there exists a positive constant ε_0 such that

$$\left\|(\rho_{10} - \tilde{\rho}, u_{10} - \tilde{u}, \rho_{20} - \tilde{\rho}, u_{20} - \tilde{u})\right\|_1 + \|E_0\| + \delta < \varepsilon_0. \tag{1.9}$$

(i) *When $M_+ > 1$, in addition, the initial data also satisfies $(1+x)^{\frac{\alpha}{2}}(\rho_{10} - \tilde{\rho})$, $(1+x)^{\frac{\alpha}{2}}(u_{10} - \tilde{u})$, $(1+x)^{\frac{\alpha}{2}}(\rho_{20} - \tilde{\rho})$, $(1+x)^{\frac{\alpha}{2}}(u_{20} - \tilde{u})$, $(1+x)^{\frac{\alpha}{2}}E_0 \in L^2(\mathbb{R}_+)$ for a certain positive constant α, then the solution $(\rho_1, u_1, \rho_2, u_2, E)$ to* (1.1)-(1.3) *satisfies the decay estimate*

$$\left\|(\rho_1 - \tilde{\rho}, u_1 - \tilde{u}, \rho_2 - \tilde{\rho}, u_2 - \tilde{u}, E)\right\|_{L^\infty} \leq C(1+x)^{-\frac{\alpha}{2}}. \tag{1.10}$$

On the other hand, if the initial data satisfies $e^{\frac{\zeta}{2}x}(\rho_{10} - \tilde{\rho})$, $e^{\frac{\zeta}{2}x}(u_{10} - \tilde{u})$, $e^{\frac{\zeta}{2}x}(\rho_{20} - \tilde{\rho})$, $e^{\frac{\zeta}{2}x}(u_{20} - \tilde{u})$, $e^{\frac{\zeta}{2}x}E_0 \in L^2(\mathbb{R}_+)$ for a certain positive constant ζ, then there exists a positive constant α such that the solution $(\rho_1, u_1, \rho_2, u_2, E)$ to (1.1)-(1.3) *satisfies*

$$\left\|(\rho_1 - \tilde{\rho}, u_1 - \tilde{u}, \rho_2 - \tilde{\rho}, u_2 - \tilde{u}, E)\right\|_{L^\infty} \leq Ce^{-\alpha t}. \tag{1.11}$$

(ii) *When $M_+ = 1$, and there exists a positive constant ε_0 such that if the initial data also satisfies $\|(1+x)^{\frac{\alpha}{2}}(\rho_{10} - \tilde{\rho}, u_{10} - \tilde{u}, \rho_{20} - \tilde{\rho}, u_{20} - \tilde{u})\|_1 + \|(1+x)^{\frac{\alpha}{2}}E_0\| < \varepsilon_0$ for a certain constant α satisfying $\alpha \in [2, \alpha^*)$, where α^* is a constant defined by*

$$\alpha^*\left(\alpha^* - 2\right) = \frac{4}{\gamma + 1} \quad \textit{and} \quad \alpha^* > 0,$$

then the solution $(\rho_1, u_1, \rho_2, u_2, E)$ to (1.1)-(1.3) *satisfies*

$$\left\|(\rho_1 - \tilde{\rho}, u_1 - \tilde{u}, \rho_2 - \tilde{\rho}, u_2 - \tilde{u}, E)\right\|_{L^\infty} \leq C(1+t)^{-\frac{\alpha}{4}}, \tag{1.12}$$

where M_+, u_* and δ are defined in Section 2, and $E_0(x) = -\int_x^\infty(\rho_{10} - \rho_{20})(y)\,dy$.

Notations Throughout this paper, $C > 0$ denotes the generic positive constant independent of time. $L^p(\mathbb{R})$ $(1 \leq p < \infty)$ denotes the space of measurable functions with the finite norm $\|\cdot\|_{L^p} = (\int_{\mathbb{R}} |\cdot|^p\,dx)^{\frac{1}{p}}$, and L^∞ is the space of bounded measurable functions on $\mathbb{R}$ with the norm $\|\cdot\|_{L^\infty} = \operatorname{ess\,sup}_x |\cdot|$. We use $\|\cdot\|$ to denote the L^2-norm. $H^k(\mathbb{R})$ $(k \geq 0)$ stands for the space of $L^2(\mathbb{R})$-functions f whose derivatives (in the sense of distribution)

$D_x^l f$ ($l \leq k$) are also $L^2(\mathbb{R})$-functions with the norm $\|\cdot\|_k = (\sum_{l=0}^{k} \|D^l \cdot\|^2)^{\frac{1}{2}}$. Moreover, $C^k([0,T];H^l(\mathbb{R}))$ ($k,l \geq 0$) denotes the space of the k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\mathbb{R})$.

The rest of the paper is organized as follows. In Section 2, we review the results of the stationary solution and the non-stationary solutions, then we reformulate our problem. Finally, we give the *a priori* estimates for the cases $M_+ > 1$ and $M_+ = 1$ in Section 3 and 4, respectively.

2 Stationary solution and global existence of non-stationary solution

In this section we mainly review the property of a stationary solution, and the unique existence and asymptotic behavior of non-stationary solutions for (1.1)-(1.3). To begin with, we recall the stationary equation

$$\begin{cases} (\tilde{\rho}\tilde{u})_x = 0, \\ (\tilde{\rho}\tilde{u}^2 + P(\tilde{\rho}))_x = \mu\tilde{\rho}_{xx} \end{cases} \tag{2.1}$$

with

$$\tilde{u}(0) = u_b < 0, \qquad \lim_{x\to\infty}(\tilde{\rho},\tilde{u}) = (\rho_+,u_+), \qquad \inf_{x\in\mathbb{R}_+}\tilde{\rho}(x) > 0. \tag{2.2}$$

Integrating $(2.1)_1$ over (x,∞) yields $\tilde{\rho}(x) = \rho_+u_+(\tilde{u}(x))^{-1}$, which implies by letting $x \to 0^+$, $\rho_b = \tilde{\rho}(0) = \rho_+u_+(u_b)^{-1}$. Namely, $\tilde{u} = \frac{u_+}{v_+}\tilde{v}$ ($\tilde{v} = \frac{1}{\tilde{\rho}}$, $v_+ = \frac{1}{\rho_+}$), which together with (2.1) implies

$$u_+ = \frac{v_+}{v(0)}u_b < 0. \tag{2.3}$$

Thus, the condition $u_+ < 0$ has to be assumed whenever the outflow problem, *i.e.*, the case $u_b < 0$, is consider. Moreover, let the strength of the boundary layer $(\tilde{\rho},\tilde{u})(x)$ be measured by $\delta := |u_+ - u_b|$. Finally, we also define (v^*,u^*) as follows:

$$u_+ = \frac{u_+}{v_+}v^* - \sqrt{(v_+ - v^*)\left[P\left(\frac{1}{v^*}\right) - P\left(\frac{1}{v_+}\right)\right]}, \qquad u^* = \frac{u_+}{v_+}v^*, \tag{2.4}$$

and denote the Mach number at infinity $M_+ =: \frac{|u_+|}{\sqrt{P'(\rho_+)}}$. Then one has the following lemma.

Lemma 2.1 (see [21, 22]) *Assume that the condition* (2.3) *holds. The boundary problem* (2.1)-(2.2) *has a smooth solution* $(\tilde{\rho},\tilde{u})(x)$, *if and only if* $M_+ \geq 1$ *and* $u_b < u_*$. *Moreover, if* $M_+ > 1$, *there exist two positive constants* λ *and* C *such that the stationary solution* $(\tilde{\rho},\tilde{u})$ *satisfies the estimate*

$$\left|\partial_x^k\left(\tilde{\rho}(x) - \rho_+, \tilde{u}(x) - u_+\right)\right| \leq C\delta e^{-\lambda x} \quad \textit{for } k = 0,1,2,\ldots. \tag{2.5}$$

If $M_+ = 1$, *the stationary solution* $(\tilde{\rho},\tilde{u})$ *satisfies*

$$\left|\partial_x^k\left(\tilde{\rho}(x) - \rho_+, \tilde{u} - u_+\right)\right| \leq C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} \quad \textit{for } k = 0,1,2,\ldots. \tag{2.6}$$

As to the stability of the stationary solution of (1.1)-(1.4), Duan and Yang showed the following results in [17].

Lemma 2.2 (see [17]) *Suppose that $M_+ \geq 1$ and $u_b < u_*$ hold. In addition, the initial data $(\rho_{10}, u_{10}, \rho_{20}, u_{20}, E_0)$ is supposed to satisfy*

$$(\rho_{10} - \tilde{\rho}, u_{10} - \tilde{u}, \rho_{20} - \tilde{\rho}, u_{20} - \tilde{u}) \in H^1(\mathbb{R}_+), \qquad E_0 \in L^2(\mathbb{R}_+), \qquad \inf_{x\in\mathbb{R}_+} (\rho_{10}, \rho_{20}) > 0.$$

Then there exists a positive constant ε_0 such that if

$$\big\|(\rho_{10} - \tilde{\rho}, u_{10} - \tilde{u}, \rho_{20} - \tilde{\rho}, u_{20} - \tilde{u})\big\|_1 + \|E_0\| + \delta < \varepsilon_0,$$

the initial boundary value problem (1.1)-(1.3) *has a unique solution $(\rho_1, u_1, \rho_2, u_2, E) \in X(0,T)$ for arbitrary $T > 0$. Moreover, the solution $(\rho_1, u_1, \rho_2, u_2, E)$ converges to the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\rho}, \tilde{u}, 0)$ as time tends to infinity*:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} |\rho_1 - \tilde{\rho}, u_1 - \tilde{u}, \rho_2 - \tilde{\rho}, u_2 - \tilde{u}, E| = 0.$$

Here the solution space $X(0,T)$ is defined by

$$\begin{aligned} X(0,T) = \big\{&(\rho_1, u_1, \rho_2, u_2, E) : \rho_1 - \tilde{\rho}, u_1 - \tilde{u}, \rho_2 - \tilde{\rho}, u_2 - \tilde{u} \in C\big(0,T;H^1\big), \\ &(\rho_1 - \tilde{\rho})_x, (\rho_2 - \tilde{\rho})_x \in L^2\big(0,T;L^2\big), (u_1 - \tilde{u})_x, (u_2 - \tilde{u})_x \in L^2\big(0,T;H^1\big), \\ &E \in C\big(0,T;L^2\big), (u_1 - \tilde{u})(t,0) = (u_2 - \tilde{u})(t,0) = 0 \ (0 \leq t \leq T)\big\}. \end{aligned}$$

Finally, to enclose this section, we reformulate the original problem in terms of the perturbed variables. Set $(\varphi_1, \psi_1, \varphi_2, \psi_2)$ from the stationary solution as

$$\varphi_i = \rho_i - \tilde{\rho}, \qquad \psi_i = u_i - \tilde{u}, \quad i = 1,2.$$

Due to (1.1) and (1.6), we have the system of equations for $(\varphi_1, \psi_1, \varphi_2, \psi_2, E)$ as

$$\begin{cases} \partial_t\varphi_1 + u_1\partial_x\varphi_1 + \rho_1\partial_x\psi_1 = -f_1, \\ \rho_1\partial_t\psi_1 + P'(\rho_1)\partial_x\varphi_1 + \rho_1u_1\partial_x\psi_1 = \psi_{1xx} - g_1 + \rho_1 E, \\ \partial_t\varphi_2 + u_2\partial_x\varphi_2 + \rho_2\partial_x\psi_2 = -f_2, \\ \rho_2\partial_t\psi_2 + P'(\rho_2)\partial_x\varphi_2 + \rho_2u_2\partial_x\psi_2 = \psi_{2xx} - g_2 - \rho_2 E, \\ E = -\int_x^\infty (\varphi_1 - \varphi_2)(y,t)\,dy, \end{cases} \tag{2.7}$$

where the nonlinear terms f_i $(i = 1,2)$ and g_i $(i = 1,2)$ are given by

$$f_i = \tilde{u}_x\varphi_i + \tilde{\rho}_x\psi_i, \qquad g_i = \tilde{u}_x(\tilde{u}\varphi_i + \rho_i\psi_i) + \tilde{\rho}_x\big(P'(\rho_i) - P'(\tilde{\rho})\big).$$

The initial and boundary condition to (2.7) are derived from (1.2), (1.3) and (1.4) as follows:

$$(\varphi_i, \psi_i)(x,0) = (\varphi_{i0}, \psi_{i0})(x) := (\rho_{i0} - \tilde{\rho}, u_{i0} - \tilde{u}), \quad i = 1,2, \tag{2.8}$$

$$(\varphi_1, \psi_1, \varphi_2, \psi_2)(t,0) = 0. \tag{2.9}$$

The uniform bound of the solutions in the weighted Sobolev space is derived later in Sections 3 and 4. For this purpose, we introduce the function spaces $X_\omega(0,T)$ and $X^1_\omega(0,T)$ defined by

$$X_\omega(0,T) = \left\{(\varphi_1,\psi_1,\varphi_2,\psi_2,E) : (\sqrt{\omega}\varphi_1,\sqrt{\omega}\psi_1,\sqrt{\omega}\varphi_2,\sqrt{\omega}\psi_2,\sqrt{\omega}E) \in C\big(0,T;L^2(\mathbb{R}_+)\big)\right\}$$

and

$$\begin{aligned}X^1_\omega(0,T) = \big\{&(\varphi_1,\psi_1,\varphi_2,\psi_2,E) : (\sqrt{\omega}\varphi_1,\sqrt{\omega}\psi_1,\sqrt{\omega}\varphi_2,\sqrt{\omega}\psi_2) \in C\big(0,T;H^1(\mathbb{R}_+)\big);\\ &\sqrt{\omega}E \in C\big(0,T;L^2(\mathbb{R}_+)\big)\big\}.\end{aligned}$$

Here the two types of weight functions are considered: $\omega(x) := (1+x)^\alpha$, or $\omega(x) = e^{\alpha x}$. Also, we use the norms $|\cdot|_{2,\omega}$, $|\cdot|_{\mathrm{a},\alpha}$, and $|\cdot|_{\mathrm{e},\alpha}$ defined by

$$|f|_{2,\omega} := \left(\int_0^\infty \omega(x) f(x)^2\,dx\right)^{\frac{1}{2}}, \qquad |f|_{\mathrm{a},\alpha} := |f|_{2,(1+x)^\alpha}, \qquad |f|_{\mathrm{e},\alpha} := |f|_{2,e^{\alpha x}}.$$

The following lemma, concerning the existence of the solution locally in time, is proved by the standard iteration method. Hence we omit the proof.

Lemma 2.3 *If the initial data satisfies* (1.8) *and* $\sqrt{\omega}\varphi_{10},\sqrt{\omega}\psi_{10},\sqrt{\omega}\varphi_{20},\sqrt{\omega}\psi_{20},\sqrt{\omega}E_0 \in L^2(\mathbb{R}_+)$, *there exists a positive constant T such that the initial boundary value problem* (2.7)-(2.9) *has a unique solution* $(\varphi_1,\psi_1,\varphi_2,\psi_2,E) \in X_\omega(0,T)$. *Moreover, if the initial data satisfies* (1.8), (1.9) *and* $\sqrt{\omega}\varphi_{10},\sqrt{\omega}\psi_{10},\sqrt{\omega}\varphi_{20},\sqrt{\omega}\psi_{20} \in H^1(\mathbb{R}_+)$ *and* $\sqrt{\omega}E_0 \in L^2(\mathbb{R}_+)$, *there exists a unique solution* $(\varphi_1,\psi_1,\varphi_2,\psi_2,E)$ *in* $X^1_\omega(0,T)$.

3 *A priori* estimates for $M_+ > 1$

In this section, we derive the *a priori* estimates of the solution $(\varphi_1,\psi_1,\varphi_2,\psi_2,E)$ for the case that $M_+ > 1$ holds in some Sobolev space. To summarize the *a priori* estimate, we use the following notation (see [23]) for a weight function $W(x,t) = \chi(t)\omega(x)$:

$$\begin{aligned}N(t) &= \sup_{0\le\tau\le t}\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2)(\tau)\big\|_1,\\ M(t)^2 &= \int_0^t \chi(\tau)\big(\big\|(\varphi_{1x},\varphi_{2x},E_x)(\tau)\big\|^2 + \big\|(\psi_{1x},\psi_{2x})(\tau)\big\|_1^2\big)\,d\tau\\ &\quad + \int_0^t \chi(\tau)\big(\varphi_1(\tau,0)^2 + \varphi_2(\tau,0)^2 + E(\tau,0)^2\big)\,d\tau,\\ L(t)^2 &= \int_0^t \chi_t(\tau)\big(\big|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)(\tau)\big|_{2,\omega}^2 + \big\|(\varphi_{1x},\psi_{1x},\varphi_{2x},\psi_{2x})(\tau)\big\|^2\big)\\ &\quad + \chi(\tau)\big(\big|(\psi_1,\psi_2)(\tau)\big|_{2,\omega_{xx}}^2 + \big|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)(\tau)\big|_{2,|\tilde{u}_x|\omega}^2\big)\,d\tau.\end{aligned}$$

Proposition 3.1 *Suppose that the same assumptions as in Theorem* 1.1 *hold*.

(i) (*Algebraic decay*) *Suppose that* $(\varphi_1,\psi_1,\varphi_2,\psi_2,E) \in X_{(1+x)^\alpha}(0,T)$ *is a solution to* (2.7)-(2.9) *for certain positive constants α and T. Then there exist positive constants ε_0 and C*

such that if $N(T) + \delta \le \varepsilon_0$, *then the solution* $(\varphi_1, \psi_1, \varphi_2, \psi_2, E)$ *satisfies the estimate*

$$
\begin{aligned}
&(1+t)^{\alpha+\varepsilon}\big(\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2)(t)\big\|_1^2 + \big\|E(t)\big\|^2\big) + \int_0^t (1+\tau)^{\alpha+\varepsilon}\big(\big\|(\varphi_{1x},\varphi_{2x},E_x)(\tau)\big\|^2 \\
&\quad + \big\|(\psi_{1x},\psi_{2x})(\tau)\big\|_1^2\big)\,d\tau + \int_0^t (1+\tau)^{\alpha+\varepsilon}\big|(\varphi_1,\varphi_2,E,\varphi_{1x},\varphi_{2x})(\tau,0)\big|^2\,d\tau \\
&\le C\big(\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20})\big\|_1^2 + \big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\big\|_{\mathrm{a},\alpha}^2\big)(1+t)^{\varepsilon}
\end{aligned}
\tag{3.1}
$$

for arbitrary $t \in [0,T]$ *and* $\varepsilon > 0$.

(ii) (*Exponential decay*) *Suppose that* $(\varphi_1, \psi_1, \varphi_2, \psi_2, E) \in X_{e^{\zeta x}}(0,T)$ *is a solution to* (2.7)-(2.9) *for certain positive constants* ζ *and* T. *Then there exist positive constants* ε_0, C, β ($< \zeta$) *and* α *satisfying* $\alpha \ll \beta$ *such that if* $N(T) + \delta \le \varepsilon_0$, *then the solution* $(\varphi_1, \psi_1, \varphi_2, \psi_2, E)$ *satisfies the estimate*

$$
\begin{aligned}
&e^{\alpha t}\big(\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2)(t)\big\|_1^2 + \big\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)(t)\big\|_{\mathrm{e},\beta}^2\big) + \int_0^t e^{\alpha\tau}\big(\big\|(\varphi_{1x},\varphi_{2x},E_x)(\tau)\big\|^2 \\
&\quad + \big\|(\psi_{1x},\psi_{2x})(\tau)\big\|_1^2\big)\,d\tau + \int_0^t e^{\alpha\tau}\big|(\varphi_1,\varphi_2,E,\varphi_{1x},\varphi_{2x})(\tau,0)\big|^2\,d\tau \\
&\quad + \int_0^t e^{\alpha\tau}\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E,\psi_{1x},\psi_{2x})(\tau)\big\|_{\mathrm{e},\beta}^2\,d\tau \\
&\le C\big(\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20})\big\|_1^2 + \big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\big\|_{\mathrm{e},\beta}^2\big).
\end{aligned}
\tag{3.2}
$$

For the sake of clarity, we divide the proof of Proposition 4.1 into the following lemmas. We first derive the basic energy estimate.

Lemma 3.2 *Suppose that the same assumptions as in Theorem* 1.1 *hold*. *Then there exists a positive constant* ε_0 *such that* $N(T) + \delta < \varepsilon_0$, *it holds that*

$$
\begin{aligned}
&\chi(t)\big|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\big|_{2,w}^2 + \int_0^t \chi(\tau)\big(\big|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)(\tau)\big|_{2,w_x}^2 + \big|(\psi_{1x},\psi_{2x})(\tau)\big|_{2,w}^2\big)\,d\tau \\
&\quad + \int_0^t \chi(\tau)\left(\sum_{i=1}^2 \varphi_i(\tau,0)^2 + E(\tau,0)^2\right)d\tau \\
&\le C\big|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\big|_{2,w}^2 + CL(t)^2.
\end{aligned}
\tag{3.3}
$$

Proof From (2.7), a direct computation yields

$$
\left(\rho_1\mathcal{E}_1 + \rho_2\mathcal{E}_2 + \frac{1}{2}E^2\right)_t - G_{1x} + \psi_{1x}^2 + \psi_{2x}^2 = (\psi_1\psi_{1x} + \psi_2\psi_{2x})_x + R_1,
\tag{3.4}
$$

here

$$
\begin{aligned}
&\mathcal{E}_i := \mathcal{E}(\rho_i,u_i) \quad (i=1,2), \\
&\mathcal{E}(\rho,u) := \Phi(\rho,\tilde{\rho}) + \frac{1}{2}|u-\tilde{u}|^2, \\
&\Phi(\rho,\tilde{\rho}) := \int_{\tilde{\rho}}^{\rho} \frac{P(s)-P(\tilde{\rho})}{s^2}\,ds,
\end{aligned}
$$

$$G_1 = -\rho_1 u_1 \mathcal{E}_1 - \rho_2 u_2 \mathcal{E}_2 - \big(P(\rho_1) - P(\tilde{\rho})\big)\psi_1 - \big(P(\rho_2) - P(\tilde{\rho})\big)\psi_2 - \frac{\tilde{u}}{2}E^2,$$

$$\begin{aligned} R_1 = &- \tilde{u}_x \bigg[P(\rho_1) - P(\tilde{\rho}) - P'(\tilde{\rho})\varphi_1 + (\rho_1 u_1 - \tilde{\rho}\tilde{u})\psi_1 \\ &+ P(\rho_2) - P(\tilde{\rho}) - P'(\tilde{\rho})\varphi_2 + (\rho_2 u_2 - \tilde{\rho}\tilde{u})\psi_2 - \frac{1}{2}E^2 \bigg] \\ &- \frac{1}{\tilde{\rho}} P(\tilde{\rho})_x \varphi_1 \psi_1 - \frac{1}{\tilde{\rho}} P(\tilde{\rho})_x \varphi_2 \psi_2. \end{aligned}$$

Owing to Lemmas 2.1 and 2.2, we see that the energy form $\mathcal{E}(\rho, u)$ is equivalent to $|(\rho - \tilde{\rho}, u - \tilde{u})|^2$. That is, there exist positive constants c and C such that

$$c\big(\varphi_i^2 + \psi_i^2\big) \le \mathcal{E}_i \le C\big(\varphi_i^2 + \psi_i^2\big), \quad i = 1, 2. \tag{3.5}$$

We also have positive bounds of ρ_i $(i = 1, 2)$ as

$$0 < c \le \rho_i \ (i = 1, 2) \le C, \qquad (t, x) \in [0, T] \times \mathbb{R}_+. \tag{3.6}$$

Further, multiplying (3.4) by a weight function $W(t, x) = \chi(t)\omega(x)$, we have

$$\begin{aligned} &\bigg(W\rho_1 \mathcal{E}_1 + W\rho_2 \mathcal{E}_2 + \frac{1}{2}WE^2 \bigg)_t - (WG_1)_x + W_x G_1 + W\psi_{1x}^2 + W\psi_{2x}^2 \\ &\quad = W_t \bigg(\rho_1 \mathcal{E}_1 + \rho_2 \mathcal{E}_2 + \frac{1}{2}E^2 \bigg) + \bigg(\sum_{i=1}^{2} W\psi_i \psi_{ix} - \frac{1}{2} W_x \psi_i^2 \bigg)_x \\ &\quad\quad + \frac{1}{2} W_{xx}\big(\psi_1^2 + \psi_2^2\big) + WR_1. \end{aligned} \tag{3.7}$$

Due to the boundary conditions (1.3) and (2.9), the integration of the second term on the left-hand side of (3.7) over $\mathbb{R}_+$

$$\begin{aligned} &\int_{\mathbb{R}_+} \bigg(W \bigg[\rho_1 u_1 \mathcal{E}_1 + \rho_2 u_2 \mathcal{E}_2 + \big(P(\rho_1) - P(\tilde{\rho})\big)\psi_1 + \big(P(\rho_1) - P(\tilde{\rho})\big)\psi_2 + \frac{\tilde{u}}{2}E^2 \bigg] \bigg)_x \, dx \\ &\quad = -\chi(t)\rho_1(t,0)u_b \mathcal{E}_1(t,0) - \chi(t)\rho_2(t,0)u_b \mathcal{E}_2(t,0) - \frac{1}{2}u_b \chi(t)E(t,0)^2 \\ &\quad \ge C\chi(t)\big(\varphi_1(t,0)^2 + \varphi_2(t,0)^2 + E(t,0)^2\big), \end{aligned} \tag{3.8}$$

where we have used the estimates (3.5) and (3.6). Next, G_1 can be computed as

$$G_1 = G_{11} + G_{12} \tag{3.9}$$

with

$$\begin{aligned} G_{11} = &\frac{K\gamma \rho_+^{\gamma-2}|u_+|}{2}\big(\varphi_1^2 + \varphi_2^2\big) + \frac{\rho_+|u_+|}{2}\big(\psi_1^2 + \psi_2^2\big) \\ &- K\gamma \rho_+^{\gamma-1}(\varphi_1\psi_1 + \varphi_2\psi_2) - \frac{u_+}{2}E^2 \end{aligned}$$

and

$$
\begin{aligned}
G_{12} = &-\sum_{i=1}^{2}\biggl(\frac{K\gamma\rho_+u_+}{2\rho_i^2}\bigl(\tilde\rho^{\gamma-1}-\rho_+^{\gamma-3}\rho_i^2\bigr)\varphi_i^2+K\rho_+u_+\tilde\rho^{\gamma-1}\biggl[\Phi\biggl(\frac{\tilde\rho}{\rho_i}\biggr)-\frac{\gamma}{2}\biggl(\frac{\tilde\rho}{\rho_i}-1\biggr)^2\biggr]\\
&+K\gamma\bigl(\tilde\rho^{\gamma-1}-\rho_+^{\gamma-1}\bigr)\varphi_i\psi_i+K\tilde\rho^{\gamma}\biggl[\biggl(\frac{\rho_i}{\tilde\rho}\biggr)^{\gamma}-1-\gamma\biggl(\frac{\rho_i}{\tilde\rho}-1\biggr)\biggr]\psi_i+(\rho_iu_i-\rho_+u_+)\mathcal{E}_i\biggr)\\
&-\frac{1}{2}(\tilde u-u_+)E^2.
\end{aligned}
$$

The conditions $M_+ > 1$ and $u_+ < 0$ yield that the quadratic form G_{11} is positive definite since

$$
\begin{aligned}
G_{11} &= \sum_{i=1}^{2}\biggl[\biggl(\frac{P'(\rho_+)^{\frac{3}{2}}}{2\rho_+}\varphi_i^2+\frac{\rho_+\sqrt{\rho_+}}{2}\psi_i^2\biggr)(M_+-1)\\
&\quad+\frac{\sqrt{P'(\rho_+)}}{2\rho_+}\bigl(\sqrt{P'(\rho_+)}\varphi_i-\rho_+\psi_i\bigr)^2\biggr]-\frac{1}{2}u_+E^2\\
&\ge C\bigl(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2+E^2\bigr),
\end{aligned}
$$

which yields

$$
\int_0^t\int_{\mathbb{R}_+}W_xG_{11}\,dx\,d\tau \ge C\int_0^t\chi(\tau)\bigl|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\bigr|_{2,\omega_x}^2\,d\tau. \tag{3.10}
$$

Using (2.5), (3.5) and the inequalities $|\Phi(s)-\frac{r}{2}(s-1)^2| \le C|s-1|^2$, $|s^\gamma-1-\gamma(s-1)| \le C|s-1|^2$ for $|s-1| \ll 1$, we have the estimate for G_{12} as

$$
|G_{12}| \le C\bigl(N(t)+\delta\bigr)\bigl(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2+E^2\bigr),
$$

which implies

$$
\int_0^t\int_{\mathbb{R}_+}W_xG_{12}\,dx\,d\tau \le C\bigl(N(t)+\delta\bigr)\int_0^t\chi(\tau)\bigl|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\bigr|_{2,\omega_x}^2\,d\tau. \tag{3.11}
$$

Moreover, the positive bound of ρ_i ($i = 1,2$), (3.6) and the Schwarz inequality yield the estimate for R_1 as

$$
|R_1| \le C|\tilde u_x|\bigl(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2+E^2\bigr),
$$

then we have

$$
\int_0^t\int_{\mathbb{R}_+}WR_1\,dx\,d\tau \le C\int_0^t\chi(t)\bigl|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\bigr|_{2,|\tilde u_x|\omega}^2\,d\tau. \tag{3.12}
$$

Therefore, integrating (3.7) over $\mathbb{R}_+\times(0,t)$, substituting the above inequalities (3.8)-(3.12) into the resultant equality and then taking $N(T)+\delta$ suitably small, we obtain the desired estimate (3.3). □

Next, we obtain the estimate for the first-order derivatives of the solution for (2.7)-(2.9). As the existence of the higher-order derivatives of the solution is not supposed, we need to use the difference quotient for the rigorous derivation of the higher-order estimates. Since the argument using the difference quotient is similar to that in the paper [21, 22], we omit the details and proceed with the proof as if it verifies

$$\begin{aligned}
&(\varphi_1,\psi_1,\varphi_2,\psi_2)\in C\big([0,T];H^2(\mathbb{R}_+)\big), \qquad \varphi_{1x},\varphi_{2x}\in L^2\big([0,T];H^1(\mathbb{R}_+)\big),\\
&E_x\in L^2\big([0,T];L^2(\mathbb{R}_+)\big), \qquad \psi_{1x},\psi_{2x}\in L^2\big([0,T];H^2(\mathbb{R}_+)\big).
\end{aligned}$$

Lemma 3.3 *There exists a positive constant ε_0 such that if $N(T)+\delta<\varepsilon_0$, then*

$$\begin{aligned}
&\chi(t)\big\|(\varphi_{1x},\varphi_{2x})\big\|^2+\int_0^t\chi(\tau)\big(\big\|(\varphi_{1x},\varphi_{2x},E_x)\big\|^2+\varphi_{1x}(\tau,0)^2+\varphi_{2x}(\tau,0)^2\big)\,d\tau\\
&\quad\le C\big(\big\|(\varphi_{10},\varphi_{20})_x\big\|^2+\big|(\varphi_{10},\varphi_{20},\psi_{10},\psi_{20},E_0)\big|_{2,\omega}^2\big)+CL(t)^2+C\delta M(t)^2. \qquad (3.13)
\end{aligned}$$

Proof By differentiating the first and third equations of (2.7) in x, and then multiplying them by $\frac{\varphi_{1x}}{\rho_1^3}$ and $\frac{\varphi_{2x}}{\rho_2^3}$, respectively, one has for $i=1,2$,

$$\left(\frac{\varphi_{ix}^2}{2\rho_i^3}\right)_t+\left(\frac{u_i\varphi_{ix}^2}{2\rho_i^3}\right)_x-\tilde{u}_x\frac{\varphi_{ix}^2}{\rho_i^3}+\frac{\tilde{\rho}_x}{\rho_i^3}\varphi_{ix}\psi_{ix}+\frac{1}{\rho_i^2}\varphi_{ix}\psi_{ixx}=-f_{ix}\frac{\varphi_{ix}}{\rho_i^3},$$

which yields

$$\left(\frac{\varphi_{1x}^2}{2\rho_1^3}+\frac{\varphi_{2x}^2}{2\rho_2^3}\right)_t+\left(\frac{u_1\varphi_{1x}^2}{2\rho_1^3}+\frac{u_2\varphi_{2x}^2}{2\rho_2^3}\right)_x+\frac{1}{\rho_1^2}\varphi_{1x}\psi_{1xx}+\frac{1}{\rho_2^2}\varphi_{2x}\psi_{2xx}=R_{21} \qquad (3.14)$$

with $R_{21}=\tilde{u}_x\frac{\varphi_{1x}^2}{\rho_1^3}-\frac{\tilde{\rho}_x}{\rho_1^3}\varphi_{1x}\psi_{1x}+\tilde{u}_x\frac{\varphi_{2x}^2}{\rho_2^3}-\frac{\tilde{\rho}_x}{\rho_2^3}\varphi_{2x}\psi_{2x}-f_{1x}\frac{\varphi_{1x}}{\rho_1^3}-f_{2x}\frac{\varphi_{2x}}{\rho_2^3}$. On the other hand, multiplying the second and fourth equations of (2.7) by $\frac{\varphi_{1x}}{\rho_1^2}$ and $\frac{\varphi_{2x}}{\rho_2^2}$, respectively, gives

$$\begin{aligned}
&\left(\frac{\psi_i}{\rho_i}\varphi_{ix}\right)_t-\left(\frac{\psi_i}{\rho_i}\varphi_{it}+\frac{\tilde{\rho}_x\psi_i^2}{\rho_i}\right)_x-\psi_{ix}^2+\frac{P'(\rho_i)}{\rho_i^2}\varphi_{ix}^2-\frac{1}{\rho_i}\tilde{u}_x\varphi_i\psi_{ix}+\frac{\tilde{\rho}_{xx}}{\rho_i}\psi_i^2\\
&\qquad+\frac{2\tilde{\rho}_x}{\rho_i}\psi_i\psi_{ix}+\frac{\tilde{u}_x(\tilde{\rho}_x\varphi_i-\tilde{\rho}\varphi_{ix})\psi_i}{\rho_i^2}\\
&\quad=-g_i\frac{\varphi_{ix}}{\rho_i^2}+(-1)^{i-1}\frac{E}{\rho_i}\varphi_{ix}+\frac{\varphi_{ix}\psi_{ixx}}{\rho_i^2}, \quad i=1,2.
\end{aligned}$$

Further, we have

$$\begin{aligned}
&\left(\frac{\psi_1}{\rho_1}\varphi_{1x}+\frac{\psi_2}{\rho_2}\varphi_{2x}\right)_t-\left(\frac{\psi_1}{\rho_1}\varphi_{1t}+\frac{\tilde{\rho}_x\psi_1^2}{\rho_1}+\frac{\psi_2}{\rho_2}\varphi_{2t}+\frac{\tilde{\rho}_x\psi_2^2}{\rho_2}\right)_x-\psi_{1x}^2-\psi_{2x}^2\\
&\qquad+\frac{P'(\rho_1)}{\rho_1^2}\varphi_{1x}^2+\frac{P'(\rho_2)}{\rho_2^2}\varphi_{2x}^2\\
&\quad=\frac{E}{\rho_1}\varphi_{1x}-\frac{E}{\rho_2}\varphi_{2x}+R_{22}+\frac{\varphi_{1x}\psi_{1xx}}{\rho_1^2}+\frac{\varphi_{2x}\psi_{2xx}}{\rho_2^2}, \qquad (3.15)
\end{aligned}$$

here

$$
\begin{aligned}
R_{22} = {} & \frac{1}{\rho_1}\tilde{u}_x\varphi_1\psi_{1x} - \frac{\tilde{\rho}_{xx}}{\rho_1}\psi_1^2 - \frac{2\tilde{\rho}_x}{\rho_1}\psi_1\psi_{1x} - \frac{\tilde{u}_x(\tilde{\rho}_x\varphi_1 - \tilde{\rho}\varphi_{1x})\psi_1}{\rho_1^2} - g_1\frac{\varphi_{1x}}{\rho_1^2} \\
& + \frac{1}{\rho_2}\tilde{u}_x\varphi_2\psi_{2x} - \frac{\tilde{\rho}_{xx}}{\rho_2}\psi_2^2 - \frac{\tilde{u}_x(\tilde{\rho}_x\varphi_2 - \tilde{\rho}\varphi_{2x})\psi_2}{\rho_2^2} - g_2\frac{\varphi_{2x}}{\rho_2^2}.
\end{aligned}
$$

Combining (3.14) and (3.15), we have

$$
\begin{aligned}
& \left(\frac{\varphi_{1x}^2}{2\rho_1^3} + \frac{\varphi_{1x}\psi_1}{\rho_1} + \frac{\varphi_{2x}^2}{2\rho_2^3} + \frac{\varphi_{2x}\psi_2}{\rho_2}\right)_t + \left(\frac{u_1\varphi_{1x}^2}{2\rho_1^3} - \frac{\varphi_{1t}\psi_1}{\rho_1} - \frac{\tilde{\rho}_x\psi_1^2}{\rho_1} + \frac{u_2\varphi_{2x}^2}{2\rho_2^3} \right. \\
& \qquad \left. - \frac{\varphi_{2t}\psi_2}{\rho_2} - \frac{\tilde{\rho}_x\psi_2^2}{\rho_2}\right)_x + \frac{P'(\rho_1)\varphi_{1x}^2}{\rho_1^2} + \frac{P'(\rho_2)\varphi_{2x}^2}{\rho_2^2} \\
& \quad = \psi_{1x}^2 + \psi_{2x}^2 + E\left(\frac{\varphi_{1x}}{\rho_1} - \frac{\varphi_{2x}}{\rho_2}\right) + R_{21} + R_{22}.
\end{aligned} \tag{3.16}
$$

The second term on the right-hand side of (3.16) can be rewritten as

$$
\begin{aligned}
E\left(\frac{\varphi_{1x}}{\rho_1} - \frac{\varphi_{2x}}{\rho_2}\right) &= \left[(\ln\rho_1 - \ln\rho_2)_x\right]E - \frac{(\rho_1-\rho_2)E\tilde{\rho}_x}{\rho_1\rho_2} \\
&= \left[(\ln\rho_1 - \ln\rho_2)E\right]_x - (\ln\rho_1 - \ln\rho_2)E_x + \frac{E_xE\tilde{\rho}_x}{\rho_1\rho_2}.
\end{aligned}
$$

Under the assumption (3.6) on the densities, it holds that

$$
(\ln\rho_i - \ln\rho_e)E_x \geq CE_x^2, \qquad \left|\frac{E_xE\tilde{\rho}_x}{\rho_1\rho_2}\right| \leq C|\tilde{\rho}_x|\left(E^2 + E_x^2\right). \tag{3.17}
$$

Moreover, owing to the Schwarz inequality with the aid of (2.5), R_{21} is estimated as

$$
\begin{aligned}
R_{21} & \leq |\tilde{u}_x|\left(\varphi_{1x}^2 + \psi_{1x}^2 + \varphi_{2x}^2 + \psi_{2x}^2\right) + |\tilde{\rho}_x|\left(\varphi_{1x}^2 + \psi_{1x}^2 + \varphi_{2x}^2 + \psi_{2x}^2\right) \\
& \quad + |\tilde{\rho}_{xx}|\left(\psi_1^2 + \varphi_{1x}^2 + \psi_2^2 + \varphi_{2x}^2\right) + |\tilde{u}_{xx}|\left(\varphi_1^2 + \varphi_{1x}^2 + \varphi_2^2 + \varphi_{2x}^2\right) \\
& \leq C\delta\left(\varphi_1^2 + \varphi_2^2 + \psi_1^2 + \psi_2^2 + \varphi_{1x}^2 + \varphi_{2x}^2 + \psi_{1x}^2 + \psi_{2x}^2\right).
\end{aligned} \tag{3.18}
$$

Similarly, we have

$$
\begin{aligned}
R_{22} & \leq \varepsilon\left(\varphi_{1x}^2 + \varphi_{2x}^2\right) + C_\varepsilon\left(\psi_{1x}^2\psi_1^2 + \psi_1^4 + \psi_{2x}^2\psi_2^2 + \psi_2^4\right) \\
& \quad + C\delta\left(\varphi_1^2 + \varphi_2^2 + \psi_1^2 + \psi_2^2 + \varphi_{1x}^2 + \varphi_{2x}^2 + \psi_{1x}^2 + \psi_{2x}^2\right).
\end{aligned} \tag{3.19}
$$

Multiplying (3.16) by a weight function $\chi(t)$, we get

$$
\begin{aligned}
& \left[\chi(t)\sum_{i=1}^{2}\left(\frac{\varphi_{ix}^2}{2\rho_i^3} + \frac{\varphi_{ix}\psi_i}{\rho_i}\right)\right]_t + \chi(t)\sum_{i=1}^{2}\left(\frac{u_i\varphi_{ix}^2}{2\rho_i^3} - \frac{\varphi_{it}\psi_i}{\rho_i} - \frac{\tilde{\rho}_x\psi_i^2}{\rho_i}\right)_x + \sum_{i=1}^{2}\frac{\chi(t)P'(\rho_i)\phi_{ix}^2}{\rho_i^2} \\
& \quad = \chi_t(t)\sum_{i=1}^{2}\left(\frac{\varphi_{ix}^2}{2\rho_i^3} + \frac{\varphi_{ix}\psi_i}{\rho_i}\right) + \chi(t)\left(\sum_{i=1}^{2}\psi_{ix}^2 + E\left(\frac{\varphi_{1x}}{\rho_1} - \frac{\varphi_{2x}}{\rho_2}\right)\right) \\
& \qquad + \chi(t)R_{21} + \chi(t)R_{22}.
\end{aligned} \tag{3.20}
$$

The boundary condition (2.9) gives

$$\int_0^\infty \left[\chi(t)\sum_{i=1}^2\left(\frac{u_i\varphi_{ix}^2}{2\rho_i^3}-\frac{\varphi_{it}\psi_i}{\rho_i}-\frac{\tilde{\rho}\psi_i^2}{\rho_i}\right)\right]_x dx \geq C\chi(t)\bigl(\varphi_{1x}(t,0)^2+\varphi_{2x}(t,0)^2\bigr) \tag{3.21}$$

and

$$\int_0^\infty \chi\bigl[(\ln\rho_1-\ln\rho_2)E\bigr]_x dx \leq C\chi\bigl(\varphi_1(t,0)^2+\varphi_2(t,0)^2+E(t,0)^2\bigr). \tag{3.22}$$

Integrating (3.20) over $[0,t]\times\mathbb{R}_+$, substituting (3.17), (3.18), (3.19), (3.21), (3.22) and the estimate

$$\begin{aligned}&\bigl|\varphi_1(t,x)\bigr|+\bigl|\varphi_2(t,x)\bigr|+\bigl|\psi_1(t,x)\bigr|+\bigl|\psi_2(t,x)\bigr|+\bigl|E(t,x)\bigr|\\&\quad\leq\bigl|\varphi_1(t,0)\bigr|+\bigl|\varphi_2(t,0)\bigr|+\bigl|E(t,0)\bigr|+\sqrt{x}\bigl\|(\varphi_{1x},\varphi_{2x},\psi_{1x},\psi_{2x},E_x)(t)\bigr\|,\end{aligned} \tag{3.23}$$

which is proved by the similar computation as in [21–24], in the resultant equality, and take ε and δ suitably small. These computations together with (3.3) give the desired estimate (3.13). □

Lemma 3.4 *There exists a positive constant ε_0 such that if $N(T)+\delta<\varepsilon_0$, then*

$$\begin{aligned}&\chi(t)\bigl\|(\psi_{1x},\psi_{2x})\bigr\|^2+\int_0^t\chi(\tau)\bigl\|(\psi_{1xx},\psi_{2xx})\bigr\|^2 d\tau\\&\quad\leq C\bigl(\bigl\|(\psi_{10},\psi_{20})_x\bigr\|^2+\bigl|(\varphi_{10},\varphi_{20},\psi_{10},\psi_{20},E_0)\bigr|_{2,\omega}^2\bigr)\\&\qquad+CL(t)^2+C\bigl(N(t)+\delta\bigr)M(t)^2.\end{aligned} \tag{3.24}$$

Proof Multiplying $(2.7)_2$ by $-\frac{\psi_{1xx}}{\rho_1}$, and $(2.7)_4$ by $-\frac{\psi_{2xx}}{\rho_2}$, respectively, we have for $i=1,2$,

$$\begin{aligned}&\left(\frac{\psi_{ix}^2}{2}\right)_t-\left(\psi_{it}\psi_{ix}+\frac{u_i\psi_{ix}^2}{2}\right)_x+\frac{\psi_{ixx}^2}{\rho_i}\\&\quad=-\frac{\psi_{ix}^3}{2}-\frac{\tilde{u}_x\psi_{ix}^2}{2}+\frac{P'(\rho_i)}{\rho_i}\varphi_{ix}\psi_{ixx}+\frac{g_i\psi_{ixx}}{\rho_i}+(-1)^iE\psi_{ixx},\end{aligned}$$

which yields

$$\left(\frac{\psi_{1x}^2}{2}+\frac{\psi_{2x}^2}{2}\right)_t-\sum_{i=1}^2\left(\psi_{it}\psi_{ix}+\frac{u_i\psi_{ix}^2}{2}\right)_x+\frac{\psi_{1xx}^2}{\rho_1}+\frac{\psi_{2xx}^2}{\rho_2}=-E(\psi_{1xx}-\psi_{2xx})+R_3 \tag{3.25}$$

with

$$R_3=-\frac{\psi_{1x}^3}{2}-\frac{\tilde{u}_x\psi_{1x}^2}{2}+\frac{P'(\rho_1)}{\rho_1}\varphi_{1x}\psi_{1xx}-\frac{\psi_{2x}^3}{2}-\frac{\tilde{u}_x\psi_{2x}^2}{2}+\frac{P'(\rho_2)}{\rho_2}\varphi_{2x}\psi_{2xx}+\frac{g_1\psi_{1xx}}{\rho_1}+\frac{g_2\psi_{2xx}}{\rho_2}.$$

Note that $-E\psi_{1xx}+E\psi_{2xx}=-(E(\psi_1-\psi_2)_x)_x+E_x(\psi_1-\psi_2)_x$, and the function R_3 is estimated by using (2.5) and Schwarz inequality as

$$\begin{aligned}R_3&\leq\varepsilon\bigl(\psi_{1xx}^2+\psi_{2xx}^2\bigr)+C_\varepsilon\bigl(\psi_{1x}^2+\psi_{2x}^2+\varphi_{1x}^2+\varphi_{2x}^2+\psi_{1x}^4+\psi_{2x}^4\bigr)\\&\quad+C_\varepsilon|\tilde{u}_x|\bigl(\varphi_1^2+\varphi_2^2+\psi_1^2+\psi_2^2+\psi_{1x}^2+\psi_{2x}^2\bigr)+C_\varepsilon|\tilde{\rho}_x|\bigl(\varphi_1^2+\varphi_2^2\bigr),\end{aligned} \tag{3.26}$$

where ε is an arbitrary positive constant and C_ε is a positive constant depending on ε. Then, multiplying (3.25) by a weight function $\chi(t)$, we get

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\left(\chi\frac{\psi_{ix}^2}{2}\right)_t-\left(\chi\left(\psi_{it}\psi_{ix}+\frac{u_i\psi_{ix}^2}{2}\right)\right)_x+\chi\frac{\psi_{ixx}^2}{\rho_i}\right]\\
&\quad=\chi_t\sum_{i=1}^{2}\frac{\psi_{ix}^2}{4}-\chi\left[E(\psi_{1xx}-\psi_{2xx})-R_3\right].
\end{aligned}\tag{3.27}$$

Integrate (3.27) over $[0,t]\times\mathbb{R}_+$, substitute (3.26) as well as the estimate

$$\begin{aligned}
\int_0^\infty\left(\psi_{1x}^4+\psi_{2x}^4\right)dx&\le C\left(\|\psi_{1x}\|_1^2\|\psi_{1x}\|^2+\|\psi_{2x}\|_1^2\|\psi_{2x}\|^2\right)\\
&\le CN(t)\left\|(\psi_{1x},\psi_{1xx},\psi_{2x},\psi_{2xx})\right\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t\chi\left|E(\psi_1-\psi_2)_x(0,\tau)\right|d\tau\\
&\quad\le\int_0^t\chi\left|E(0,\tau)\right|^2d\tau+\int_0^t\chi\left|(\psi_1-\psi_2)_x(0,\tau)\right|^2d\tau\\
&\quad\le\int_0^t\chi\left|E(0,\tau)\right|^2d\tau+C_\varepsilon\int_0^t\chi\left\|(\psi_1,\psi_2)_x\right\|^2d\tau+\varepsilon\int_0^t\chi\left\|(\psi_1,\psi_2)_{xx}\right\|^2d\tau
\end{aligned}$$

in the resultant equality, and take ε suitably small. These computations together with (3.3), (3.13) and (3.23) give the desired estimate (3.24). □

Proof of Proposition 3.1 Summing up the estimates (3.3), (3.13) and (3.24) and taking $N(T)+\delta$ suitably small, we have

$$\begin{aligned}
&\chi(t)\left[\sum_{i=1}^{2}\left(\left|(\varphi_i,\psi_i)\right|_{2,w}^2+\sum_{i=1}^{2}\left\|(\varphi_{ix},\psi_{ix})\right\|^2\right)+|E|_{2,w}^2\right]\\
&\qquad+\int_0^t\chi(\tau)\left[\sum_{i=1}^{2}\left(\left|(\varphi_i,\psi_i)\right|_{2,w_x}^2+\left|\psi_{ix}(\tau)\right|_{2,w}^2+\varphi_i(\tau,0)^2+\varphi_{ix}(\tau,0)^2+\left\|(\varphi_{ix},\psi_{ixx})\right\|\right)\right.\\
&\qquad\left.+\left|E(\tau)\right|_{2,w_x}^2+\|E_x\|^2+E(\tau,0)^2\right]d\tau\\
&\quad\le C\left(\left|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\right|_{2,w}^2+\left\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20})_x\right\|^2\right)+CL(t)^2.
\end{aligned}\tag{3.28}$$

First, we prove the estimate (3.1). Noting the Poincaré-type inequality (3.23), and substituting $\omega(x)=(1+x)^\beta$ and $\chi(t)=(1+t)^\xi$ in (3.28) for $\beta\in[0,\alpha]$ and $\xi\ge 0$ gives

$$\begin{aligned}
&(1+t)^\xi\left(\left\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)(t)\right\|_{a,\beta}^2+\left\|(\varphi_1,\varphi_2,\psi_1,\psi_2)(t)\right\|_1^2\right)\\
&\qquad+\int_0^t(1+\tau)^\xi\left(\varphi_1(\tau,0)^2+\varphi_2(\tau,0)^2+\varphi_{1x}(\tau,0)^2+\varphi_{2x}(\tau,0)^2+E(\tau,0)^2\right)d\tau
\end{aligned}$$

$$\begin{aligned}
&+ \beta \int_0^t (1+\tau)^{\xi} \left\|(\varphi_1, \psi_1, \varphi_2, \psi_2, E)(\tau)\right\|_{\mathrm{a},\beta-1}^2 \, d\tau \\
&+ \int_0^t (1+\tau)^{\xi} \left(\left\|(\psi_{1x}, \psi_{2x})(\tau)\right\|_{\mathrm{a},\beta}^2 + \left\|(\varphi_{1x}, \varphi_{2x}, \psi_{1xx}, \psi_{2xx})(\tau)\right\|^2 \right) d\tau \\
\le{} & C\left(\left\|(\varphi_{10}, \psi_{10}, \varphi_{20}, E_x \psi_{20})\right\|_1^2 + \left\|(\varphi_{10}, \psi_{10}, \varphi_{20}, \psi_{20}, E_0)\right\|_{\mathrm{a},\beta}^2 \right) \\
&+ C\beta(\beta-1) \int_0^t (1+\tau)^{\xi} \left\|(\psi_1, \psi_2)\right\|_{\mathrm{a},\beta-2}^2 \, d\tau \\
&+ C\xi \int_0^t (1+\tau)^{\xi-1} \left(\left\|(\varphi_1, \psi_1, \varphi_2, \psi_2, E)\right\|_{\mathrm{a},\beta}^2 + \left\|(\varphi_{1x}, \psi_{1x}, \varphi_{2x}, \psi_{2x})\right\|^2 \right) d\tau. \qquad (3.29)
\end{aligned}$$

Therefore, applying an induction to (3.29) gives the desired estimate (3.1). Since this computation is similar those in [23, 25], we omit the details.

Next, we prove the estimate (3.2). Substitute $\omega(x) = e^{\beta x}$ and $\chi(t) = e^{\alpha t}$ in (3.28) for $\beta < \lambda$ to obtain

$$\begin{aligned}
& e^{\alpha t} \left(\left\|(\varphi_1, \psi_1, \varphi_2, \psi_2, E)(t)\right\|_{\mathrm{e},\beta}^2 + \left\|(\varphi_{1x}, \varphi_{2x}, \psi_{1x}, \psi_{2x})(t)\right\|^2 \right) \\
&+ \int_0^t e^{\alpha\tau} \left(\varphi_1(\tau,0)^2 + \varphi_2(\tau,0)^2 + \varphi_{1x}(\tau,0)^2 + \varphi_{2x}(\tau,0)^2 + E(\tau,0)^2 \right) d\tau \\
&+ \int_0^t e^{\alpha\tau} \left(\beta \left\|(\varphi_1, \psi_1, \varphi_2, \psi_2, E)(\tau)\right\|_{\mathrm{e},\beta}^2 \right. \\
&+ \left. \left\|(\varphi_{1x}, \varphi_{2x}, E_x, \psi_{1xx}, \psi_{2xx})(\tau)\right\|^2 + \left\|(\psi_{1x}, \psi_{2x})(\tau)\right\|_{\mathrm{e},\beta}^2 \right) d\tau \\
\le{} & C\left(\left\|(\varphi_{10}, \psi_{10}, \varphi_{20}, \psi_{20})_x\right\|^2 + \left\|(\varphi_{10}, \psi_{10}, \varphi_{20}, \psi_{20}, E_0)\right\|_{\mathrm{e},\beta}^2 \right) \\
&+ C\left(\alpha + \beta^2\right) \int_0^t e^{\alpha\tau} \left\|(\varphi_1, \psi_1, \varphi_2, \psi_2, E)\right\|_{\mathrm{e},\beta}^2 \, d\tau \\
&+ C\alpha \int_0^t e^{\alpha\tau} \left\|(\varphi_{1x}, \psi_{1x}, \varphi_{2x}, \psi_{2x})\right\|^2 \, d\tau \\
&+ C\delta \int_0^t e^{\alpha\tau} \left(\varphi_1(\tau,0)^2 + \varphi_2(\tau,0)^2 + E(\tau,0)^2 \right. \\
&+ \left. \left\|(\varphi_{1x}, \varphi_{2x}, E_x, \psi_{1x}, \psi_{2x})(\tau)\right\|^2 \right) d\tau. \qquad (3.30)
\end{aligned}$$

Here, we have used the Poincaré-type inequality (3.23) again. Thus, taking δ, β and α suitably small, we obtain the desired *a priori* estimate (3.2). □

4 *A priori* estimate for $M_+ = 1$

In the section we proceed to consider the transonic case $M_+ = 1$. To state the *a priori* estimate of the solution precisely, here we use the notations:

$$\begin{aligned}
N_1(t) &= \sup_{0 \le \tau \le t} \left\|(1+x)^{\alpha/2}(\varphi_1, \psi_1, \varphi_2, \psi_2)(\tau)\right\|_1, \\
M_1(t)^2 &= \int_0^t (1+t)^{\xi} \left\|(\varphi_{1x}, \varphi_{2x}, E_x, \psi_{1x}, \psi_{2x}, \psi_{1xx}, \psi_{2xx})(\tau)\right\|_{\mathrm{a},\beta}^2 \, d\tau.
\end{aligned}$$

Proposition 4.1 *Suppose that the same assumption as in Theorem* 1.1 *holds. Suppose that* $(\varphi_1, \psi_1, \varphi_2, \psi_2, E) \in X^1_{(1+x)^\alpha}(0,T)$ *is a solution to* (2.7)-(2.9) *for certain positive constants* α

and T. Then there exist positive constants ε_0 and C such that if $N_1(T)+\delta\le\varepsilon_0$, then the solution $(\varphi_1,\psi_1,\varphi_2,\psi_2,E)$ satisfies the estimate

$$\begin{aligned}
&(1+t)^{\alpha/2+\varepsilon}\big(\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2)(t)\big\|_1^2+\big\|E(t)\big\|^2\big)\\
&\qquad+\int_0^t(1+\tau)^{\alpha/2+\varepsilon}\big|(\varphi_1,\varphi_2,\varphi_{1x},\varphi_{2x},E)(\tau,0)\big|^2\,d\tau\\
&\qquad+\int_0^t(1+\tau)^{\alpha/2+\varepsilon}\big(\big\|(\varphi_{1x},\varphi_{2x},E_x)(\tau)\big\|^2+\big\|(\psi_{1x},\psi_{2x})(\tau)\big\|_1^2\big)\,d\tau\\
&\quad\le C\big(\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20})_x\big\|^2+\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\big\|_{\mathrm{a},\alpha}^2\big)(1+t)^{\varepsilon}.
\end{aligned}\tag{4.1}$$

In order to prove Proposition 4.1, we need to get a lower estimate for $\tilde{u}_x$ and the Mach number $\tilde{M}$ on the stationary solution $(\tilde{\rho},\tilde{u})$ defined by $\tilde{M}(x):=\frac{|\tilde{u}(x)|}{\sqrt{P'(\tilde{\rho}(x))}}$.

Lemma 4.2 (see [23]) *The stationary solution $\tilde{u}(x)$ satisfies*

$$\tilde{u}_x(x)\ge A\left(\frac{u_+}{u_b}\right)^{\gamma+2}\frac{\delta^2}{(1+B_x)^2},\qquad A:=\frac{(\gamma+1)\rho_+}{2},\qquad B:=\delta A$$

for $x\in(0,\infty)$. Moreover, there exists a positive constant C such that

$$\frac{\gamma+1}{2|u_+|}\frac{\delta}{1+Bx}-C\frac{\delta^2}{(1+Bx)^2}\le\tilde{M}(x)-1\le C\frac{\delta}{1+Bx}.$$

Based on Lemma 4.2, we obtain the weighted L^2 estimate of $(\varphi_1,\psi_1,\varphi_2,\psi_2,E)$.

Lemma 4.3 *There exists a positive constant ε_0 such that if $N_1(T)+\delta<\varepsilon_0$, then*

$$\begin{aligned}
&(1+t)^{\xi}\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\big\|_{\mathrm{a},\beta}^2+\int_0^t(1+\tau)^{\xi}\big(\varphi_1(\tau,0)^2+\varphi_2(\tau,0)^2+E(\tau,0)^2\big)\,d\tau\\
&\qquad+\int_0^t(1+\tau)^{\xi}\big(\beta\delta^2\big\|(\varphi_1,\varphi_2,\psi_1,\psi_2)(\tau)\big\|_{\mathrm{a},\beta-2}^2\\
&\qquad+\beta\big\|E(\tau)\big\|_{\mathrm{a},\beta-1}^2+\big\|(\psi_{1x},\psi_{2x})(\tau)\big\|_{\mathrm{a},\beta}^2\big)\,d\tau\\
&\quad\le C\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\big\|_{\mathrm{a},\beta}^2+C\xi\int_0^t(1+\tau)^{\xi-1}\big\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\big\|_{\mathrm{a},\beta}^2\,d\tau\\
&\qquad+C\delta\int_0^t(1+\tau)^{\xi}\big\|(\varphi_{1x},\varphi_{2x},E_x,\psi_{1xx},\psi_{2xx})\big\|_{\mathrm{a},\beta}^2\,d\tau
\end{aligned}\tag{4.2}$$

for $\beta\in[0,\alpha]$ and $\xi\ge0$.

Proof First, from (2.7), similar as (3.4), we also have

$$\begin{aligned}
&\left(\rho_1\mathcal{E}_1+\rho_2\mathcal{E}_2+\frac{1}{2}\mathcal{E}^2\right)_t+\Big[\rho_1u_1\mathcal{E}_1+\big(P(\rho_1)-P(\tilde{\rho})\big)\psi_1-\psi_1\psi_{1x}+\rho_2u_2\mathcal{E}_2\\
&\qquad+\big(P(\rho_2)-P(\tilde{\rho})\big)\psi_2-\psi_2\psi_{2x}+\frac{\tilde{u}}{2}E^2\Big]_x+\tilde{u}_x\Big[P(\rho_1)-P(\tilde{\rho})-P'(\tilde{\rho})\varphi_1+\rho_1\psi_1^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + P(\rho_2) - P(\tilde\rho) - P'(\tilde\rho)\varphi_2 + \rho_2\psi_2^2 - \frac{1}{2}E^2\bigg] + \left(\psi_{1x}^2 + \psi_{2x}^2\right) \\
&= -\frac{\tilde u_{xx}}{\tilde\rho}[\varphi_1\psi_1 + \varphi_2\psi_2].
\end{aligned} \tag{4.3}
$$

Notice that from the second and fourth equations of (1.1), one has

$$
2E = \partial_t(u_1 - u_2) + \frac{1}{2}\partial_x\left(u_1^2 - u_2^2\right) + \left[\frac{P'(\rho_1)\partial_x\rho_1}{\rho_1} - \frac{P'(\rho_2)\partial_x\rho_2}{\rho_2}\right] - \left[\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right],
$$

which implies

$$
\begin{aligned}
\frac{\tilde u_x}{2}E^2 &= \left(\frac{\tilde u_x}{4}(u_1 - u_2)E\right)_t - \frac{\tilde u_x}{4}(u_1 - u_2)E_t + \frac{\tilde u_x}{8}E\partial_x\left(u_1^2 - u_2^2\right) \\
&\quad + \frac{A\gamma\bar u_x}{4(\gamma-1)}\left(\rho_1^{\gamma-1} - \rho_2^{\gamma-1}\right)_x E + \frac{\tilde u_x}{4}E\left[\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right] \\
&= \left(\frac{\tilde u_x}{4}(u_1 - u_2)E\right)_t + \left(\frac{\tilde u\tilde u_x}{4}E(\psi_1 - \psi_2)\right)_x + \frac{\tilde u}{8}(\rho_1 + \rho_2)(\psi_1 - \psi_2)^2 \\
&\quad - \frac{\tilde u\tilde u_{xx}}{4}E(\psi_1 - \psi_2) + \frac{\tilde u_x}{8}E_x\left(\psi_1^2 - \psi_2^2\right) + \frac{A\gamma\bar u_x}{4(\gamma-1)}\left(\rho_1^{\gamma-1} - \rho_2^{\gamma-1}\right)_x E \\
&\quad + \frac{\tilde u_x}{4}E\left[\left(\frac{\psi_1^2}{2} - \frac{\psi_2^2}{2}\right)_x + \left(\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right)\right].
\end{aligned}
$$

Plugging the above equality into (4.3), we arrive at

$$
\begin{aligned}
&\left(\rho_1\mathcal{E}_1 + \rho_2\mathcal{E}_2 - \frac{\tilde u_x}{4}(u_1 - u_2)E + \frac{1}{2}E^2\right)_t + \bigg(\rho_1 u_1\mathcal{E}_1 + \rho_2 u_2\mathcal{E}_2 \\
&\quad + \left(P(\rho_1) - P(\tilde\rho)\right)\psi_1 - \psi_1\psi_{1x} \\
&\quad + \left(P(\rho_2) - P(\tilde\rho)\right)\psi_2 - \psi_2\psi_{2x} + \frac{\tilde u\tilde u_x}{4}(\psi_1 - \psi_2)E + \frac{\tilde u}{2}E^2\bigg)_x + \psi_{1x}^2 + \psi_{2x}^2 \\
&\quad + \tilde u_x\bigg[P(\rho_1) - P(\tilde\rho) - P'(\tilde\rho)\varphi_1 + \rho_1\psi_1^2 - \frac{\rho_1}{8}(\psi_1 - \psi_2)^2 + P(\rho_2) \\
&\quad - P(\tilde\rho) - P'(\tilde\rho)\varphi_2 + \rho_2\psi_2^2 - \frac{\rho_2}{8}(\psi_1 - \psi_2)^2\bigg] \\
&= -\frac{\tilde u_{xx}}{\tilde\rho}[\varphi_1\psi_1 + \varphi_2\psi_2] + R_4,
\end{aligned} \tag{4.4}
$$

where

$$
\begin{aligned}
R_4 &= -\frac{\tilde u\tilde u_{xx}}{4}E(\psi_1 - \psi_2) + \frac{\tilde u_x}{8}E_x\left(\psi_1^2 - \psi_2^2\right) + \frac{\tilde u_x}{4}E(\psi_1\psi_{1x} - \psi_2\psi_{2x}) \\
&\quad + \frac{A\gamma\tilde u_x}{4(\gamma-1)}\left(\rho_1^{\gamma-1} - \rho_2^{\gamma-1}\right)_x E + \frac{\tilde u_x}{4}\left[\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right]E \\
&=: R_{41} + R_{42} + R_{43} + R_{44} + R_{45}.
\end{aligned}
$$

Further, multiplying (4.4) by a weight function $W(t,x) := (1+Bx)^{\beta}(1+t)^{\xi}$, we have

$$\begin{aligned}
&\left(W\rho_1\mathcal{E}_1 + W\rho_2\mathcal{E}_2 - \frac{\tilde{u}_x}{4}W(u_1-u_2)E + \frac{1}{2}WE^2\right)_t \\
&\quad + \left(-WG_2 - W\psi_1\psi_{1x} - W\psi_2\psi_{2x} + \frac{1}{2}W_x\psi_1^2 + \frac{1}{2}W_x\psi_2^2\right)_x \\
&\quad + W_xG_2 + W\psi_{1x}^2 + W\psi_{2x}^2 + G_3 \\
&= W_t\left(\rho_1\mathcal{E}_1 + \rho_2\mathcal{E}_2 - \frac{\tilde{u}_x}{4}(u_1-u_2)E + \frac{1}{2}E^2\right) - W\frac{\tilde{u}_{xx}}{\tilde{\rho}}[\varphi_1\psi_1 + \varphi_2\psi_2] + WR_4. \qquad (4.5)
\end{aligned}$$

Where G_2 and G_3 are defined

$$\begin{aligned}
G_2 &= -\rho_1u_1\mathcal{E}_1 - \rho_2u_2\mathcal{E}_2 - \big(P(\rho_1) - P(\tilde{\rho})\big)\psi_1 - \big(P(\rho_2) - P(\tilde{\rho})\big)\psi_2 \\
&\quad - \frac{\tilde{u}\tilde{u}_x}{4}(\psi_1-\psi_2)E - \frac{\tilde{u}}{2}E^2
\end{aligned}$$

and

$$\begin{aligned}
G_3 &:= W\tilde{u}_x\bigg[P(\rho_1) - P(\tilde{\rho}) - P'(\tilde{\rho})\varphi_1 + \rho_1\psi_1^2 - \frac{\rho_1}{8}(\psi_1-\psi_2)^2 + P(\rho_2) - P(\tilde{\rho}) \\
&\quad - P'(\tilde{\rho})\varphi_2 + \rho_2\psi_2^2 - \frac{\rho_2}{8}(\psi_1-\psi_2)^2\bigg] - \frac{1}{2}W_{xx}\psi_1^2 - \frac{1}{2}W_{xx}\psi_2^2.
\end{aligned}$$

By the same computation as in deriving (3.9), we rewrite the terms G_2 and G_3 to $G_2 = G_{21} + G_{22}$, $G_3 = G_{31} + G_{32}$ with

$$\begin{aligned}
G_{21} &= \sum_{i=1}^{2}\left[\left(\frac{P'(\rho_+)^{3/2}}{2\rho_+}\varphi_i^2 + \frac{\rho_+\sqrt{P'(\rho_+)}}{2}\psi_i^2\right)(\tilde{M}-1) + \frac{P'(\tilde{\rho})}{2\tilde{\rho}}\left(\sqrt{P'(\tilde{\rho})}\varphi_i - \tilde{\rho}\psi_i\right)^2\right] \\
&\quad - \frac{\tilde{u}\tilde{u}_x}{4}(\psi_1-\psi_2)E - \frac{1}{2}u_+E^2, \\
G_{22} &= -\sum_{i=1}^{2}\Bigg[\frac{\tilde{\rho}P'(\tilde{\rho})\tilde{u}}{2}\left(\frac{1}{\rho_i^2} - \frac{1}{\tilde{\rho}^2}\right)\varphi_i^2 + P(\tilde{\rho})\tilde{u}\left[\Phi\left(\frac{\tilde{\rho}}{\rho_i}\right) - \frac{\gamma}{2}\left(\frac{\tilde{\rho}}{\rho_i}-1\right)^2\right] \\
&\quad + (\rho_iu_i - \tilde{\rho}\tilde{u})\mathcal{E}_i + \big(P(\rho_i) - P(\tilde{\rho}) - P'(\tilde{\rho})\varphi_i\big)\psi_i - \Bigg[\left(\frac{P'(\tilde{\rho})^{3/2}}{2\tilde{\rho}} - \frac{P'(\rho_+)^{3/2}}{2}\rho_+\right)\varphi_i^2 \\
&\quad + \left(\frac{\tilde{\rho}\sqrt{P'(\tilde{\rho})}}{2} - \frac{\rho_+\sqrt{P'(\rho_+)}}{2}\right)\psi_i^2\Bigg](\tilde{M}-1)\Bigg] - \frac{1}{2}(\tilde{u}-u_+)E^2, \\
G_{31} &= W\tilde{u}_x\left(\rho_+\psi_1^2 + \frac{1}{2}P''(\rho_+)\varphi_1^2\right) - \frac{1}{2}W_{xx}\psi_1^2 \\
&\quad + W\tilde{u}_x\left(\rho_+\psi_2^2 + \frac{1}{2}P''(\rho_+)\varphi_2^2\right) - \frac{1}{2}W_{xx}\psi_2^2
\end{aligned}$$

and

$$G_{32} = W\tilde{u}_x\sum_{i=1}^{2}\left[(\rho_i-\rho_+)\psi_i^2 + \frac{1}{2}\big(P''(\tilde{\rho}) - P''(\rho_+)\big)\varphi_i^2 + P(\rho_i) - P(\tilde{\rho}) - P'(\tilde{\rho})\varphi_i - \frac{1}{2}P''(\tilde{\rho})\varphi_i^2\right].$$

By utilizing Lemma 4.2 with the aid of the fact that $\beta < \alpha^*$ and $u_+ < 0$, we obtain the lower estimate of $W_x G_{21} + G_{31}$ as

$$\begin{aligned} W_x G_{21} + G_{31} \geq{}& \sum_{i=1}^{2} \biggl[\frac{K\gamma\rho_+^{\gamma-2}A}{4} \biggl[(\gamma+1)\beta + 2\biggl(\frac{u_+}{u_b}\biggr)^{\gamma+2}(\gamma-1) \biggr] \delta^2(1+t)^{\xi}(1+Bx)^{\beta-2}\varphi_i^2 \\ &+ \frac{\rho_+ A}{4} \biggl[4\biggl(\frac{u_+}{u_b}\biggr)^{\gamma+2} - (\gamma+1)\beta(\beta-2) \biggr] \delta^2(1+t)^{\xi}(1+Bx)^{\beta-2}\psi_i^2 \biggr] \\ &- C\beta\delta^3(1+t)^{\xi}(1+Bx)^{\beta-3}\bigl(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2\bigr) \\ \geq{}& C\delta^2(1-C\delta)(1+t)^{\xi}(1+Bx)^{\beta-2}\bigl(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2\bigr) \\ &+ C\beta(1+t)^{\xi}(1+Bx)^{\beta-1}E^2 \end{aligned} \tag{4.6}$$

for $\beta \in (0,\alpha]$. On the other hand, the estimates (2.6), (3.5) and (3.6) yield

$$\begin{aligned} |W_x G_{22} + G_{32}| \leq{}& C\bigl(N_1(t)+\delta^2\bigr)\delta(1+t)^{\xi}(1+Bx)^{\beta-2}\bigl(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2\bigr) \\ &+ C\delta\beta(1+t)^{\xi}(1+Bx)^{\beta-1}E^2. \end{aligned} \tag{4.7}$$

For the first term on the right-hand side of (4.5), we estimate it as

$$\begin{aligned} &\biggl| \int_0^t \int_{\mathbb{R}_+} W\frac{\tilde{u}_{xx}}{\tilde{\rho}}[\varphi_1\psi_1+\varphi_2\psi_2]\,dx\,d\tau \biggr| \\ &\quad\leq \biggl| \int_0^t \int_{\mathbb{R}_+} W\frac{\tilde{u}_{xx}}{\tilde{\rho}}\bigl[\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2\bigr]\,dx\,d\tau \biggr| \\ &\quad\leq C\delta \int_0^t (1+\tau)^{\xi}\bigl(\varphi_1(0,\tau)^2+\varphi_2(0,\tau)^2\bigr)\,d\tau \\ &\qquad+ C\delta \int_0^t (1+\tau)^{\xi} \int_{\mathbb{R}_+} (1+Bx)^{\beta}\bigl(\varphi_{1x}^2+\psi_{1x}^2+\varphi_{2x}^2+\psi_{2x}^2\bigr)\,dx\,d\tau. \end{aligned} \tag{4.8}$$

Similarly, we get

$$\begin{aligned} &\biggl| \int_0^t \int_{\mathbb{R}_+} WR_{41}\,dx\,d\tau \biggr| \\ &\quad\leq \biggl| \int_0^t \int_{\mathbb{R}_+} W\tilde{u}_{xx}\bigl[E^2+\psi_1^2+\psi_2^2\bigr]\,dx\,d\tau \biggr| \\ &\quad\leq C\delta \int_0^t (1+\tau)^{\xi}E(0,\tau)^2\,d\tau \\ &\qquad+ C\delta \int_0^t (1+\tau)^{\xi} \int_{\mathbb{R}_+} (1+Bx)^{\beta}\bigl(\psi_{1x}^2+\psi_{2x}^2+E_x^2\bigr)\,dx\,d\tau. \end{aligned} \tag{4.9}$$

In the same way, we estimate R_{42} and R_{43} as follows:

$$\biggl| \int_0^t \int_{\mathbb{R}_+} WR_{42}\,dx\,d\tau \biggr| \leq C\delta \int_0^t (1+\tau)^{\xi} \int_{\mathbb{R}_+} (1+Bx)^{\beta}\bigl(\psi_{1x}^2+\psi_{2x}^2+E_x^2\bigr)\,dx\,d\tau \tag{4.10}$$

and

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}_+} WR_{43}\,dx\,d\tau\right| \\
&\quad\le C\delta\int_0^t (1+\tau)^\xi E(0,\tau)^2\,d\tau + C\delta\int_0^t (1+\tau)^\xi\int_{\mathbb{R}_+}(1+Bx)^\beta E_x^2\,dx\,d\tau \\
&\qquad + CN_1(t)\int_0^t (1+\tau)^\xi\int_{\mathbb{R}_+}(1+Bx)^\beta\left(\psi_{1x}^2+\psi_{2x}^2\right)dx\,d\tau. \qquad (4.11)
\end{aligned}$$

Since $(\rho_1^{\gamma-1}-\rho_2^{\gamma-1})_x = (\gamma-1)[(\rho_1^{\gamma-2}-\rho_2^{\gamma-2})\tilde{\rho}_x + \rho_1^{\gamma-2}\varphi_{1x} - \rho_2^{\gamma-2}\varphi_{2x}]$, we have

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}_+} WR_{44}\,dx\,d\tau\right| \\
&\quad\le C\delta\int_0^t (1+\tau)^\xi E(0,\tau)^2\,d\tau \\
&\qquad + C\delta\int_0^t (1+\tau)^\xi\int_{\mathbb{R}_+}(1+Bx)^\beta\left(\varphi_{1x}^2+\varphi_{2x}^2+E_x^2\right)dx\,d\tau. \qquad (4.12)
\end{aligned}$$

Moreover, it is easy to compute $R_{45} = \tilde{u}_x(\frac{E\psi_{1xx}}{\rho_1} - \frac{E\psi_{2xx}}{\rho_2}) + \frac{\tilde{u}_{xx}\tilde{u}_x}{\rho_1\rho_2}E(\varphi_1-\varphi_2)$, which implies

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}_+} WR_{45}\,dx\,d\tau\right| \\
&\quad\le C\delta\int_0^t (1+\tau)^\xi\left(E(0,\tau)^2+\varphi_1(0,\tau)^2+\varphi_2(0,\tau)^2\right)d\tau \\
&\qquad + C\delta\int_0^t (1+\tau)^\xi\int_{\mathbb{R}_+}(1+Bx)^\beta\left(\varphi_{1x}^2+\varphi_{2x}^2+E_x^2+\psi_{1xx}^2+\psi_{2xx}^2\right)dx\,d\tau. \qquad (4.13)
\end{aligned}$$

Finally, integrate (4.5) over $\mathbb{R}_+\times(0,t)$, substitute (4.6)-(4.13) in the resultant equality, and take $N_1(t)$ and δ suitably small. This procedure yields the desired estimate (4.2) for $\beta\in(0,\alpha]$.

Next, we prove (4.2) for $\beta=0$. Substituting $W=(1+t)^\xi$ in (4.5) and integrating the resultant equality over $\mathbb{R}_+\times(0,t)$, we get

$$\begin{aligned}
&(1+t)^\xi\left\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)(t)\right\|^2 \\
&\qquad + \int_0^t (1+\tau)^\xi\left(\sum_{i=1}^2\varphi_i(\tau,0)^2 + E(\tau,0)^2 + \left\|(\psi_{1x},\psi_{2x})\right\|^2\right)d\tau \\
&\quad\le C\left\|(\varphi_{10},\psi_{10},\varphi_{10},\psi_{10},E_0)\right\|^2 + C\xi\int_0^t (1+\tau)^{\xi-1}\left\|(\varphi_1,\psi_1,\varphi_2,\psi_2,E)\right\|^2d\tau \\
&\qquad + C\delta\int_0^t (1+\tau)^\xi\left\|(\varphi_{1x},\varphi_{2x},E_x,\psi_{1xx},\psi_{2xx})\right\|^2dx\,d\tau.
\end{aligned}$$

Here, we have used the fact that $G_2\ge 0$ holds. Therefore, we obtain the estimate (4.2) for the case of $\beta=0$. □

In order to complete the proof of Proposition 4.1, we need to obtain the weighted estimate of $(\varphi_{1x},\psi_{1x},\varphi_{2x},\psi_{2x})$.

Lemma 4.4 *There exists a positive constant ε_0 such that if $N_1(T)+\delta<\varepsilon_0$, then for $\beta\in[0,\alpha]$ and $\xi\geq 0$,*

$$\begin{aligned}&(1+t)^{\xi}\big\|(\varphi_{1x},\psi_{1x},\varphi_{2x},\psi_{2x})\big\|_{a,\beta}^{2}+\int_0^t(1+\tau)^{\xi}\big(\varphi_{1x}(\tau,0)^2+\varphi_{2x}(\tau,0)^2\big)\,d\tau\\&\quad+\int_0^t(1+\tau)^{\xi}\big\|(\varphi_{1x},\psi_{1xx},\varphi_{2x},\psi_{2xx},E_x)(\tau)\big\|_{a,\beta}^{2}\,d\tau\\&\quad\leq C\big(\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0)\big\|_{a,\beta}^{2}+\big\|(\varphi_{10x},\psi_{10x},\varphi_{20x},\psi_{20x})\big\|_{a,\beta}^{2}\big)\\&\qquad+C\xi\int_0^t(1+\tau)^{\xi-1}\big\|(\varphi_1,\psi_1,\varphi_{1x},\psi_{1x},\varphi_2,\psi_2,\varphi_{2x},\psi_{2x})(\tau)\big\|_{a,\beta}^{2}\,d\tau.\end{aligned}\tag{4.14}$$

Proof Since the derivation of the estimate (4.14) is similar to that of (3.13) and (3.24), we only give the outline of the proof. Multiplying (3.16) by $W=(1+Bx)^{\beta}(1+t)^{\xi}$, we have

$$\begin{aligned}&\sum_{i=1}^{2}\bigg[\bigg[W\bigg(\frac{\varphi_{ix}^2}{2\rho_i^3}+\frac{\varphi_{ix}\psi_i}{\rho_i}\bigg)\bigg]_t+\bigg[W\bigg(\frac{u_i\varphi_{ix}^2}{2\rho_i^3}-\frac{\varphi_{it}\psi_i}{\rho_i}-\frac{\tilde{\rho}_x\psi_i^2}{\rho_i}\bigg)\bigg]_x+W\frac{P'(\rho_i)\varphi_{ix}^2}{\rho_i^2}\bigg]\\&\quad=\sum_{i=1}^{2}\bigg[W_t\bigg(\frac{\varphi_{ix}^2}{2\rho_i^3}+\frac{\varphi_{ix}\psi_i}{\rho_i}\bigg)+W\bigg(\psi_{ix}^2+(-1)^{i+1}E\frac{\varphi_{ix}}{\rho_i}+R_{21}+R_{22}\bigg)\\&\qquad-W_x\bigg(\frac{u_i\varphi_{ix}^2}{2\rho_i^3}-\frac{\varphi_{it}\psi_i}{\rho_i}-\frac{\tilde{\rho}_x\psi_i^2}{\rho_i}\bigg)\bigg].\end{aligned}\tag{4.15}$$

Integrating (4.15) over $\mathbb{R}_+\times(0,t)$ and substituting (4.2) gives the estimate for φ_{1x} and φ_{2x} as

$$\begin{aligned}&(1+t)^{\xi}\big\|(\varphi_{1x},\varphi_{2x})\big\|_{a,\beta}^{2}+\int_0^t(1+\tau)^{\xi}\big(\varphi_{1x}(\tau,0)^2+\varphi_{2x}(\tau,0)^2+\big\|(\varphi_{1x},\varphi_{2x},E_x)\big\|_{a,\beta}^{2}\big)\,d\tau\\&\quad\leq C\big\|(\varphi_{10},\psi_{10},\varphi_{20},\psi_{20},E_0,\varphi_{10x},\varphi_{20x})\big\|_{a,\beta}^{2}\\&\qquad+C\xi\int_0^t(1+\tau)^{\xi-1}\big\|(\varphi_1,\psi_1,\varphi_{1x},\varphi_2,\psi_2,\varphi_{2x})\big\|_{a,\beta}^{2}\,d\tau+C\big(N_1(t)+\delta\big)M_1(t)^2.\end{aligned}\tag{4.16}$$

Here, we have used the inequalities

$$\begin{aligned}&\big((1+Bx)^{\beta}\big)_x\bigg|\frac{u_1\varphi_{1x}^2}{2\rho_1^3}-\frac{\varphi_{1t}\psi_1}{\rho_1}-\frac{\tilde{\rho}_x\psi_1^2}{\rho_1}+\frac{u_2\varphi_{2x}^2}{2\rho_2^3}-\frac{\varphi_{2t}\psi_2}{\rho_2}-\frac{\tilde{\rho}_x\psi_2^2}{\rho_2}\bigg|\\&\quad\leq C\delta(1+Bx)^{\beta}\big(\varphi_{1x}^2+\varphi_{2x}^2\big)+C(1+Bx)^{\beta}\big(\psi_{1x}^2+\psi_{2x}^2\big)\\&\qquad+C\beta(1+Bx)^{\beta-2}\big(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2\big)\end{aligned}$$

and

$$\begin{aligned}(1+Bx)^{\beta}|R_{21}+R_{22}|&\leq(\varepsilon+C\delta)(1+Bx)^{\beta}\big(\varphi_{1x}^2+\varphi_{2x}^2\big)+C_{\varepsilon}(1+Bx)^{\beta}\big(\psi_{1x}^2+\psi_{2x}^2\big)\\&\quad+C\delta(1+Bx)^{\beta}\big(\varphi_1^2+\psi_1^2+\varphi_2^2+\psi_2^2\big),\end{aligned}$$

where ε is an arbitrary positive constant. We note that the third term on the right-hand side of the above inequality is estimated by applying the Poincaré-type inequality (3.23) for the case of $\beta=0$.

Next, we prove the estimate for (ψ_{1x}, ψ_{2x}). Multiply (3.25) by $W = (1+t)^{\xi}(1+Bx)^{\beta}$ to get

$$\begin{aligned}
&\left(W\frac{\psi_{1x}^2}{2} + W\frac{\psi_{2x}^2}{2}\right)_t - \sum_{i=1}^{2}\left(W\psi_{it}\psi_{ix} + W\frac{u_i\psi_{ix}^2}{2}\right)_x + W\sum_{i=1}^{2}\frac{\psi_{ixx}^2}{\rho_i} \\
&\quad = \frac{1}{2}W_t\sum_{i=1}^{2}\frac{\psi_{ix}^2}{2} - W_x\sum_{i=1}^{2}\left(\psi_{it}\psi_{ix} + \frac{u_i\psi_{ix}^2}{2}\right) - EW\psi_{1xx} + EW\psi_{2xx} + WR_3. \qquad (4.17)
\end{aligned}$$

Integrate (4.17) in $\mathbb{R}_+ \times (0,t)$ and substitute (4.2) and (4.16) in the resultant equality with the inequalities

$$\begin{aligned}
&\left((1+Bx)^{\beta}\right)_x\left|\psi_{1t}\psi_{1x} + \frac{u_1\psi_{1x}^2}{2} + \psi_{2t}\psi_{2x} + \frac{u_2\psi_{2x}^2}{2}\right| + (1+Bx)^{\beta}|R_3| \\
&\quad \le \varepsilon(1+Bx)^{\beta}\sum_{i=1}^{2}\psi_{ixx}^2 + C_{\varepsilon}(1+Bx)^{\beta}\sum_{i=1}^{2}\left(\varphi_{ix}^2 + \psi_{ix}^2 + \psi_{ix}^4\right) \\
&\qquad + C_{\varepsilon}\delta(1+Bx)^{\beta-4}\sum_{i=1}^{2}\left(\varphi_i^2 + \psi_i^2\right)
\end{aligned}$$

and

$$\int_0^{\infty}(1+Bx)^{\beta}\left(\psi_{1x}^4 + \psi_{2x}^4\right)dx \le CN_1(t)\left\|(\psi_{1x}, \psi_{1xx}, \psi_{2x}, \psi_{2xx})\right\|_{\mathrm{a},\beta}^2.$$

This procedure yields

$$\begin{aligned}
&(1+t)^{\xi}\left\|(\psi_{1x}, \psi_{2x})\right\|_{\mathrm{a},\beta}^2 + \int_0^t (1+\tau)^{\xi}\left\|(\psi_{1xx}, \psi_{2xx})\right\|_{\mathrm{a},\beta}^2 d\tau \\
&\quad \le C\left(\left\|(\varphi_{10}, \psi_{10}, \varphi_{20}, \psi_{20}, E_0)\right\|_{\mathrm{a},\beta}^2 + \left\|(\varphi_{10x}, \psi_{10x}, \varphi_{20x}, \psi_{20x})\right\|_{\mathrm{a},\beta}^2\right) \\
&\qquad + C\xi\int_0^t (1+\tau)^{\xi-1}\sum_{i=1}^{2}\left\|(\varphi_i, \psi_i, \varphi_{ix}, \psi_{ix})\right\|_{\mathrm{a},\beta}^2 d\tau + C\left(N_1(t)+\delta\right)M_1(t)^2. \qquad (4.18)
\end{aligned}$$

Finally, adding (4.16) to (4.18) and taking $N_1(t) + \delta$ suitably small give the desired estimate (4.14). □

By the same inductive argument as in deriving (4.1), we can prove Proposition 4.1, which immediately yields the decay estimate (1.12).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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