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Asymptotic behavior of the time-dependent solution of an M/G/1 queueing model

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Abstract

We study the spectrum on the imaginary axis of the underlying operator which corresponds to the M/G/1 queueing model with exceptional service time for the first customer in each busy period that was described by infinitely many partial differential equations with integral boundary conditions and obtain that all points on the imaginary axis except 0 belong to the resolvent set of the operator and 0 is an eigenvalue of the operator and its adjoint operator. Thus, by combining these results with our previous results, we deduce that the time-dependent solution of the model converges strongly to its steady-state solution. Moreover, we show that our result on convergence is optimal.

MSC: 47A10; 47D99

Keywords: M/G/1 queueing model with exceptional service time for the first customer in each busy period; C_0-semigroup; eigenvalue; resolvent set

1 Introduction

According to Takagi [1], the M/G/1 queueing system with exceptional service time for the first customer in each busy period can be described by the following partial differential equations with integral boundary conditions:

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\lambda p_0(t) + \int_0^\infty Q_1(x,t)b_0(x)\,\mathrm{d}x + \int_0^\infty p_1(x,t)b(x)\,\mathrm{d}x, \tag{1.1}$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\bigl(\lambda + b(x)\bigr)p_1(x,t), \tag{1.2}$$

$$\frac{\partial p_k(x,t)}{\partial t} + \frac{\partial p_k(x,t)}{\partial x} = -\bigl(\lambda + b(x)\bigr)p_k(x,t) + \lambda p_{k-1}(x,t), \quad k \geq 2, \tag{1.3}$$

$$\frac{\partial Q_1(x,t)}{\partial t} + \frac{\partial Q_1(x,t)}{\partial x} = -\bigl(\lambda + b_0(x)\bigr)Q_1(x,t), \tag{1.4}$$

$$\frac{\partial Q_k(x,t)}{\partial t} + \frac{\partial Q_k(x,t)}{\partial x} = -\bigl(\lambda + b_0(x)\bigr)Q_k(x,t) + \lambda Q_{k-1}(x,t), \quad k \geq 2, \tag{1.5}$$

$$p_k(0,t) = \int_0^\infty Q_{k+1}(x,t)b_0(x)\,\mathrm{d}x + \int_0^\infty p_{k+1}(x,t)b(x)\,\mathrm{d}x, \quad k \geq 1, \tag{1.6}$$

$$Q_1(0,t) = \lambda p_0(t), \qquad Q_k(0,t) = 0, \quad k \geq 2, \tag{1.7}$$

$$p_0(0) = 1, \qquad p_k(x,0) = 0, \qquad Q_k(x,0) = 0, \quad k \geq 1, \tag{1.8}$$

where $(x,t) \in [0,\infty) \times [0,\infty)$; $p_0(t)$ represents the probability that there is no customer in the system and the server is idle at time t; $p_n(x,t)\,\mathrm{d}x$ $(n \geq 1)$ represents the probability that at time t there are n customers in the system and the server is busy with remaining service time lying between in $[x, x + \mathrm{d}x)$; $Q_n(x,t)\,\mathrm{d}x$ $(n \geq 1)$ represents the probability that at time t there are n customers in the system and the server is busy with the elapsed service time of the first service lying between x and $x + \mathrm{d}x$; λ represents the arrival rate of customers; $b(x)$ is the service rate at x; $b_0(x)$ is the exceptional service rate at x.

Many papers have been published about queueing systems with server vacations. But most works on vacation models have been limited to the analysis of steady-states. There are few treatments of transient behavior, see Welch [2], Minh [3], Takagi [1], Gupur [4, 5] for instance. In 1990, Takagi [1] first established the mathematical model of the M/G/1 queueing system with exceptional service time for the first customer in each busy period by using the supplementary variable technique, then studied the time-dependent solution of the model by using probability generating functions and got the Laplace transform of the probability generating function. Roughly speaking, he obtained the existence of a time-dependent solution of the model. In 2002, by using C_0-semigroup theory in functional analysis, Gupur [6] proved that the model has a unique positive time-dependent solution which satisfies the probability condition. In 2003, Gupur [4] considered the asymptotic behavior of the time-dependent solution of the model when $b(x)$ and $b_0(x)$ are constants. Firstly, he determined the resolvent set of the adjoint operator of the operator corresponding to the model; next he proved that 0 is an eigenvalue of the operator and its adjoint operator with geometric multiplicity one. Thus, by using Theorem 14 in Gupur, Li and Zhu [7] obtained that the time-dependent solution of the model converges strongly to its steady-state solution. In 2009, Zhang and Gupur [8] found that the operator has one eigenvalue on the left complex half-plane. In 2011, Lin and Gupur [9] proved that the operator has infinitely many eigenvalues on the left complex half-plane which converges to zero and therefore showed that the convergence of the time-dependent solution of the model obtained in Gupur [4] is the best result on the convergence, that is to say, it is impossible that the time-dependent solution exponentially (uniformly) converges to its steady-state solution. In the case that $b(x)$ and $b_0(x)$ are functions, any literature about asymptotic behavior of the above model has not been found. This paper is an effort on this subject.

According to Theorem 14 in Gupur, Li and Zhu [7], to obtain the asymptotic behavior of the time-dependent solution of the above model, we need to know the spectrum of the underlying operator on the imaginary axis. By investigating the above model and comparing with Gupur [10], one may find that the main difficult points of the above equations (1.1)-(1.8) are that there are infinitely many equations and boundary conditions. When studying the population equation, Greiner [11] put forward an idea to perturb the boundary condition which states 'one can introduce the maximal operator without the boundary condition and define a boundary operator, and by studying the spectrum of the boundary operator and the maximal operator can discuss the spectrum of the underlying operator which corresponds to the population equation.' In 2007, Haji and Radl [12] successfully applied Greiner's idea to the $M/M^B/1$ queueing model, in which both the service rate and arrival rate are constants, and studied the asymptotic behavior of its time-dependent solution. Gupur [5, 13] obtained the asymptotic behavior of the time-dependent solutions of two queueing models by using Greiner's idea. In this paper, firstly, by using probability generating functions, we prove that 0 is an eigenvalue of the underlying operator; next, by using the idea in Gupur [5, 13], the result in Haji and Radl [12] and Corollary 2.3 in Nagel [14], we deduce the resolvent set of the underlying operator; thirdly, we show that 0 is an eigenvalue of the adjoint operator of the underlying operator, and therefore, by using Theorem 14 in Gupur, Li and Zhu [7], we obtain that the time-dependent solution of the above model converges strongly to its steady-state solution. Finally, by Lin and Gupur [9] we show that our result on convergence is optimal, that is to say, it is impossible that the time-dependent solution of the model converges exponentially to its steady-state solution. Although the idea and method in Gupur [4] are quite different, the main result is a special case of our result.

In this paper, we use the notations in Gupur [5, 6, 13]. Take the state space as follows:

$$X = \left\{ (p,Q) \;\middle|\; \begin{array}{l} p \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ Q \in L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ \|(p,Q)\| = |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|Q_n\|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

It is obvious that X is a Banach space. In addition, X is also a Banach lattice under the following order relation:

$$(p,Q) \le (y,z) \quad \Longleftrightarrow \quad p_0 \le y_0, \qquad p_n(x) \le y_n(x), \qquad Q_n(x) \le z_n(x), \quad n \ge 1.$$

For convenience, we introduce

$$\begin{aligned}
B_1 g(x) &= -\frac{dg(x)}{dx} - \big(\lambda + b(x)\big)g(x), \quad g \in W^{1,1}[0,\infty), \\
B_2 g(x) &= -\frac{dg(x)}{dx} - \big(\lambda + b_0(x)\big)g(x), \quad g \in W^{1,1}[0,\infty), \\
\phi f(x) &= \int_0^\infty b_0(x) f(x)\,\mathrm{d}x, \quad f \in L^1[0,\infty), \\
\psi f(x) &= \int_0^\infty b(x) f(x)\,\mathrm{d}x, \quad f \in L^1[0,\infty).
\end{aligned}$$

We define

$$\begin{aligned}
A_m(p,Q) = {} & \left(\begin{pmatrix} -\lambda & \psi & 0 & 0 & \cdots \\ 0 & B_1 & 0 & 0 & \cdots \\ 0 & \lambda & B_1 & 0 & \cdots \\ 0 & 0 & \lambda & B_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} \right. \\
& \left. + \begin{pmatrix} \phi & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix} \right.,
\end{aligned}$$

$$\left.\left.\begin{pmatrix} B_2 & 0 & 0 & \cdots \\ \lambda & B_2 & 0 & \cdots \\ 0 & \lambda & B_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix}\right)\right),$$

$$D(A_m) = \left\{ (p,Q) \in X \;\middle|\; \begin{array}{l} \frac{\mathrm{d}p_n(x)}{\mathrm{d}x} \in L^1[0,\infty), \frac{\mathrm{d}Q_n(x)}{\mathrm{d}x} \in L^1[0,\infty), n \geq 1, \\ p_n(x) \text{ and } Q_n(x) \text{ are absolutely continuous functions} \\ \text{and } \sum_{n=1}^{\infty} \| \frac{\mathrm{d}p_n}{\mathrm{d}x} \|_{L^1[0,\infty)} < \infty, \sum_{n=1}^{\infty} \| \frac{\mathrm{d}Q_n}{\mathrm{d}x} \|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

We choose a boundary space as

$$\partial X = l^1 \times l^1$$

and define the boundary operators

$$L : D(A_m) \to \partial X, \qquad \Phi : D(A_m) \to \partial X,$$

$$L(p,Q) = L\left(\begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix}\right) = \left(\begin{pmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} Q_1(0) \\ Q_2(0) \\ Q_3(0) \\ \vdots \end{pmatrix}\right),$$

$$\Phi(p,Q) = \left(\begin{pmatrix} 0 & 0 & \psi & 0 & 0 & \cdots \\ 0 & 0 & 0 & \psi & 0 & \cdots \\ 0 & 0 & 0 & 0 & \psi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & \phi & 0 & 0 & \cdots \\ 0 & 0 & \phi & 0 & \cdots \\ 0 & 0 & 0 & \phi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}\right).$$

Now we introduce the underlying operator $(A, D(A))$ by

$$Ap = A_m p, \qquad D(A) = \{p \in D(A_m) \mid Lp = \Phi p\}.$$

Then the system of the above equations (1.1)-(1.8) can be written as an abstract Cauchy problem in the Banach space X, which is just the form given in Gupur [6]

$$\begin{cases} \frac{\mathrm{d}(p,Q)(t)}{\mathrm{d}t} = A(p,Q)(t), \quad t \in (0,\infty), \\ (p,Q)(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}\right). \end{cases} \tag{1.9}$$

Gupur [6] has proved the following result for the system (1.9).

Theorem 1.1 *The operator* $(A, D(A))$ *generates a positive contraction* C_0*-semigroup* $T(t)$ *and the system* (1.9) *has a unique positive time-dependent solution* $(p,Q)(x,t) =$

$T(t)(p,Q)(0)$ *which satisfies*

$$\begin{aligned}\big\|(p,Q)(\cdot,t)\big\| &= p_0(t)+\sum_{n=1}^{\infty}\int_0^{\infty}p_n(x,t)\,\mathrm{d}x\\&\quad+\sum_{n=1}^{\infty}\int_0^{\infty}Q_n(x,t)\,\mathrm{d}x=1,\quad \forall t\in[0,\infty).\end{aligned}$$

2 Main results

Lemma 2.1 *If* $\int_0^\infty \lambda x b(x)e^{-\int_0^x b(\xi)\,d\xi}\,dx<1$, *then* 0 *is an eigenvalue of* A *with geometric multiplicity one*.

Proof We consider the equation $A(p,Q)=0$, *i.e.*,

$$\lambda p_0=\int_0^\infty Q_1(x)b_0(x)\,\mathrm{d}x+\int_0^\infty p_1(x)b(x)\,\mathrm{d}x, \tag{2.1}$$

$$\frac{\mathrm{d}p_1(x)}{\mathrm{d}x}=-\big(\lambda+b(x)\big)p_1(x), \tag{2.2}$$

$$\frac{\mathrm{d}p_n(x)}{\mathrm{d}x}=-\big(\lambda+b(x)\big)p_n(x)+\lambda p_{n-1}(x),\quad \forall n\geq 2, \tag{2.3}$$

$$\frac{\mathrm{d}Q_1(x)}{\mathrm{d}x}=-\big(\lambda+b_0(x)\big)Q_1(x), \tag{2.4}$$

$$\frac{\mathrm{d}Q_n(x)}{\mathrm{d}x}=-\big(\lambda+b_0(x)\big)Q_n(x)+\lambda Q_{n-1}(x),\quad \forall n\geq 2, \tag{2.5}$$

$$p_n(0)=\int_0^\infty Q_{n+1}(x)b_0(x)\,\mathrm{d}x+\int_0^\infty p_{n+1}(x)b(x)\,\mathrm{d}x,\quad n\geq 1, \tag{2.6}$$

$$Q_1(0)=\lambda p_0,\qquad Q_k(0)=0,\quad k\geq 2. \tag{2.7}$$

By solving (2.2)-(2.5), we have

$$p_1(x)=a_1e^{-\lambda x-\int_0^x b(\xi)\,\mathrm{d}\xi}, \tag{2.8}$$

$$\begin{aligned}p_n(x)&=a_ne^{-\lambda x-\int_0^x b(\xi)\,\mathrm{d}\xi}\\&\quad+\lambda e^{-\lambda x-\int_0^x b(\xi)\,\mathrm{d}\xi}\int_0^x p_{n-1}(s)e^{\lambda s+\int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s,\quad n\geq 2,\end{aligned} \tag{2.9}$$

$$Q_1(x)=b_1e^{-\lambda x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}, \tag{2.10}$$

$$\begin{aligned}Q_n(x)&=b_ne^{-\lambda x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}\\&\quad+\lambda e^{-\lambda x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}\int_0^x Q_{n-1}(s)e^{\lambda s+\int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s,\quad n\geq 2.\end{aligned} \tag{2.11}$$

Through using (2.8)-(2.11) repeatedly, we deduce

$$p_n(x)=e^{-\lambda x-\int_0^x b(\xi)\,\mathrm{d}\xi}\sum_{k=1}^{n}\frac{(\lambda x)^{k-1}}{(k-1)!}a_{n+1-k},\quad n\geq 1, \tag{2.12}$$

$$Q_n(x)=e^{-\lambda x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}\sum_{k=1}^{n}\frac{(\lambda x)^{k-1}}{(k-1)!}b_{n+1-k},\quad n\geq 1. \tag{2.13}$$

By combining (2.10) and (2.11) with (2.7) and using (2.13), we deduce

$$\begin{aligned}
&Q_n(x) = \lambda p_0 \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x - \int_0^x b_0(\xi)\,d\xi}, \quad n \geq 1 &(2.14)\\
&\quad\Longrightarrow\\
&\sum_{n=1}^{\infty} \|Q_n\|_{L^1[0,\infty)} = \sum_{n=1}^{\infty} \int_0^{\infty} \left| \lambda p_0 \frac{(\lambda x)^{n-1}}{(n-1)!} \right| e^{-\lambda x - \int_0^x b_0(\xi)\,d\xi}\,dx\\
&\qquad = \lambda |p_0| \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x - \int_0^x b_0(\xi)\,d\xi}\,dx\\
&\qquad = \lambda |p_0| \int_0^{\infty} e^{\lambda x} e^{-\lambda x - \int_0^x b_0(\xi)\,d\xi}\,dx\\
&\qquad = \lambda |p_0| \int_0^{\infty} e^{-\int_0^x b_0(\xi)\,d\xi}\,d\xi < \infty. &(2.15)
\end{aligned}$$

It is difficult to determine directly all a_k and to verify $\sum_{k=1}^{\infty} \|p_k\|_{L^1[0,\infty)} < \infty$. In the following, we use another method. We introduce the probability generating function $P(x,z) = \sum_{n=1}^{\infty} p_n(x) z^n$ for all complex variables $|z| < 1$. Theorem 1.1 ensures that $P(x,z)$ is well defined. (2.2) and (2.3) give

$$\begin{aligned}
&\frac{\partial \sum_{n=1}^{\infty} p_n(x) z^n}{\partial x} = -\sum_{n=1}^{\infty} \big(\lambda + b(x)\big) p_n(x) z^n + \lambda \sum_{n=2}^{\infty} p_{n-1}(x) z^n,\\
&\frac{\partial P(x,z)}{\partial x} = -\big(\lambda + b(x)\big) \sum_{n=1}^{\infty} p_n(x) z^n + \lambda z \sum_{n=1}^{\infty} p_n(x) z^n,\\
&\frac{\partial P(x,z)}{\partial x} = -\big(\lambda + b(x)\big) P(x,z) + \lambda z P(x,z) = \big[\lambda(z-1) - b(x)\big] P(x,z)\\
&\quad\Longrightarrow\\
&P(x,z) = P(0,z) e^{\lambda x(z-1) - \int_0^x b(\xi)\,\mathrm{d}\xi}. &(2.16)
\end{aligned}$$

By applying (2.6), (2.16), (2.14), (2.1), $\int_0^{\infty} b(x) e^{-\int_0^x b(\xi)\,d\xi}\,dx = 1$, $\int_0^{\infty} b_0(x) e^{-\int_0^x b_0(\xi)\,d\xi}\,dx = 1$ and the L'Hospital rule it follows that

$$\begin{aligned}
P(0,z) &= \sum_{n=1}^{\infty} p_n(0) z^n = \sum_{n=1}^{\infty} \left(\int_0^{\infty} Q_{n+1}(x) b_0(x)\,\mathrm{d}x + \int_0^{\infty} p_{n+1}(x) b(x)\,\mathrm{d}x \right) z^n\\
&= \int_0^{\infty} b_0(x) \sum_{n=1}^{\infty} Q_{n+1}(x) z^n\,dx + \int_0^{\infty} b(x) \sum_{n=1}^{\infty} p_{n+1}(x) z^n\,dx\\
&= \int_0^{\infty} b_0(x) \sum_{n=1}^{\infty} \lambda p_0 \frac{(\lambda x)^n}{n!} z^n e^{-\lambda x - \int_0^x b_0(\xi)\,d\xi}\,dx\\
&\quad + \int_0^{\infty} b(x) \frac{1}{z} \sum_{n=1}^{\infty} p_{n+1}(x) z^{n+1}\,dx\\
&= \lambda p_0 \int_0^{\infty} b_0(x) e^{\lambda x z} e^{-\lambda x - \int_0^x b_0(\xi)\,d\xi}\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty b(x)\frac{1}{z}\left\{\sum_{n=1}^\infty p_n(x)z^n - p_1(x)z\right\}dx \\
&= \lambda p_0\int_0^\infty b_0(x)e^{\lambda x(z-1)-\int_0^x b_0(\xi)\,d\xi}\,dx \\
&\quad + \frac{1}{z}\int_0^\infty b(x)\sum_{n=1}^\infty p_n(x)z^n\,dz - \int_0^\infty b(x)p_1(x)\,dx \\
&= \lambda p_0\int_0^\infty b_0(x)e^{\lambda x(z-1)-\int_0^x b_0(\xi)\,d\xi}\,dx \\
&\quad + \frac{1}{z}\int_0^\infty b(x)\sum_{n=1}^\infty p_n(x)z^n\,dz - \left[\lambda p_0 - \int_0^\infty \lambda p_0 b_0(x)e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx\right] \\
&= \lambda p_0\int_0^\infty b_0(x)e^{\lambda x(z-1)-\int_0^x b_0(\xi)\,d\xi}\,dx - \lambda p_0 \\
&\quad + \lambda p_0\int_0^\infty b_0(x)e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx + \frac{1}{z}\int_0^\infty b(x)P(0,z)e^{\lambda x(z-1)-\int_0^x b(\xi)\,d\xi}\,dx \\
&\Longrightarrow \\
P(0,z) &= \frac{\int_0^\infty b_0(x)e^{\lambda x(z-1)-\int_0^x b_0(\xi)\,d\xi}\,dx - 1 + \int_0^\infty b_0(x)e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx}{z - \int_0^\infty b(x)e^{\lambda x(z-1)-\int_0^x b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x}\lambda p_0 z \\
&\Longrightarrow \\
\lim_{z\to 1}P(0,z) &= \frac{\int_0^\infty \lambda x b_0(x)e^{-\int_0^x b_0(\xi)\,d\xi}\,dx + \int_0^\infty b_0(x)e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx}{1 - \int_0^\infty \lambda x b(x)e^{-\int_0^x b(\xi)\,d\xi}\,dx}\lambda p_0.
\end{aligned}
\tag{2.17}
$$

(2.16) and (2.17) give

$$
\begin{aligned}
\sum_{n=1}^\infty p_n(x) &= \lim_{z\to 1}P(x,z) \\
&= \frac{\int_0^\infty \lambda x b_0(x)e^{-\int_0^x b_0(\xi)\,d\xi}\,dx + \int_0^\infty b_0(x)e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx}{1 - \int_0^\infty \lambda x b(x)e^{-\int_0^x b(\xi)\,d\xi}\,dx}\lambda|p_0| \\
&\quad \times \int_0^\infty e^{-\int_0^x b(\xi)\,d\xi}\,dx \\
&< \infty.
\end{aligned}
\tag{2.18}
$$

(2.18) and (2.15) show that 0 is an eigenvalue of A. Moreover, from (2.12), (2.14), (2.1) and (2.6), it is easy to see that the eigenvectors corresponding to zero span one dimensional linear space, that is, the geometric multiplicity of 0 is one. □

According to Theorem 14 in Gupur, Li and Zhu [7], we know that in order to obtain the asymptotic behavior of the time-dependent solution of the system (1.9), we need the spectrum of A on the imaginary axis. Through investigating the system (1.9), we find that the infinite number of equations and the boundary conditions are the difficult points. Greiner [11] put forward an idea to study the spectrum of A by perturbing boundary conditions. And by using the Greiner idea, Haji and Radl [12] gave a result which was described by the Dirichlet operator. In the following, by applying the result, we deduce the resolvent set of

A on the imaginary axis. To do this, define $(A_0, D(A_0))$ as

$$A_0 p = A_m p, \qquad D(A_0) = \{p \in D(A_m) \mid Lp = 0\}$$

and discuss the inverse of A_0. For any given $(y,z) \in X$, consider the equation $(\gamma I - A_0)(p,Q) = (y,z)$, that is,

$$(\gamma + \lambda)p_0 = y_0 + \int_0^\infty Q_1(x)b_0(x)\,\mathrm{d}x + \int_0^\infty p_1(x)b(x)\,\mathrm{d}x, \tag{2.19}$$

$$\frac{\mathrm{d}p_1(x)}{\mathrm{d}x} = -(\gamma + \lambda + b(x))p_1(x) + y_1(x), \tag{2.20}$$

$$\frac{\mathrm{d}p_n(x)}{\mathrm{d}x} = -(\gamma + \lambda + b(x))p_n(x) + \lambda p_{n-1}(x) + y_n(x), \quad \forall n \geq 2, \tag{2.21}$$

$$\frac{\mathrm{d}Q_1(x)}{\mathrm{d}x} = -(\gamma + \lambda + b_0(x))Q_1(x) + z_1(x), \tag{2.22}$$

$$\frac{\mathrm{d}Q_n(x)}{\mathrm{d}x} = -(\gamma + \lambda + b_0(x))Q_n(x) + \lambda Q_{n-1}(x) + z_n(x), \quad \forall n \geq 2, \tag{2.23}$$

$$p_n(0) = 0, \qquad Q_n(0) = 0, \quad n \geq 1. \tag{2.24}$$

By (2.19)-(2.24) it is easy to calculate

$$\begin{aligned} p_0 = {} & \frac{y_0}{\gamma + \lambda} + \frac{1}{\gamma + \lambda}\int_0^\infty b(x)e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi}\int_0^x y_1(s)e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s\,\mathrm{d}x \\ & + \frac{1}{\gamma + \lambda}\int_0^\infty b_0(x)e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}\int_0^x z_1(s)e^{(\gamma+\lambda)s + \int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s\,\mathrm{d}x, \end{aligned} \tag{2.25}$$

$$p_1(x) = e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi}\int_0^x y_1(s)e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \tag{2.26}$$

$$\begin{aligned} p_n(x) = {} & e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi}\int_0^x y_n(s)e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s \\ & + \lambda e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\mathrm{d}\xi}\int_0^x p_{n-1}(s)e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \quad n \geq 2, \end{aligned} \tag{2.27}$$

$$Q_1(x) = e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}\int_0^x z_1(s)e^{(\gamma+\lambda)s + \int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \tag{2.28}$$

$$\begin{aligned} Q_n(x) = {} & e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}\int_0^x z_n(s)e^{(\gamma+\lambda)s + \int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s \\ & + \lambda e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}\int_0^x Q_{n-1}(s)e^{(\gamma+\lambda)s + \int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \quad n \geq 2. \end{aligned} \tag{2.29}$$

If we set

$$Ef(x) = e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi}\int_0^x f(s)e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \quad \forall f \in L^1[0,\infty),$$

$$E_0 f(x) = e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}\int_0^x f(s)e^{(\gamma+\lambda)s + \int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \quad \forall f \in L^1[0,\infty),$$

then the above equations (2.25)-(2.29) give, if the resolvent of A_0 exists,

$$(\gamma I - A_0)^{-1}(y,z) = \left(\begin{pmatrix} \frac{1}{\gamma+\lambda} & \frac{1}{\gamma+\lambda}\psi E & 0 & 0 & \cdots \\ 0 & E & 0 & 0 & \cdots \\ 0 & \lambda E^2 & E & 0 & \cdots \\ 0 & \lambda^2 E^3 & \lambda E^2 & E & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} y_0 \\ y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \end{pmatrix} \right.$$

$$+ \begin{pmatrix} \frac{1}{\gamma+\lambda}\phi E_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3(x) \\ \vdots \end{pmatrix},$$

$$\left. \begin{pmatrix} E_0 & 0 & 0 & \cdots \\ \lambda E_0^2 & E_0 & 0 & \cdots \\ \lambda^2 E_0^3 & \lambda E_0^2 & E_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3(x) \\ \vdots \end{pmatrix} \right).$$

From which together with the definition of the resolvent set we have the following result.

Lemma 2.2 *Let* $b(x), b_0(x) : [0,\infty) \to [0,\infty)$ *be measurable,* $0 < \inf_{x\in[0,\infty)} b(x) \le \sup_{x\in[0,\infty)} b(x) < \infty$ *and* $0 < \inf_{x\in[0,\infty)} b_0(x) \le \sup_{x\in[0,\infty)} b_0(x) < \infty$. *Then*

$$\left\{ \gamma \in \mathbb{C} \;\middle|\; \begin{array}{l} \operatorname{Re}\gamma + \lambda > 0, \\ \operatorname{Re}\gamma + \inf_{x\in[0,\infty)} b(x) > 0, \\ \operatorname{Re}\gamma + \inf_{x\in[0,\infty)} b_0(x) > 0 \end{array} \right\} \subset \rho(A_0).$$

Proof For any $f \in L^1[0,\infty)$, by using integration by parts, we estimate

$$\begin{aligned}
\int_0^\infty \left|Ef(x)\right| \mathrm{d}x &= \int_0^\infty \left| e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi} \int_0^x e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi} f(s)\,\mathrm{d}s \right| \mathrm{d}x \\
&\le \int_0^\infty e^{-(\operatorname{Re}\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi} \int_0^x e^{(\operatorname{Re}\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi} \left|f(s)\right| \mathrm{d}s\,\mathrm{d}x \\
&\le \frac{-1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)} \\
&\quad \times \int_0^\infty \int_0^x e^{(\operatorname{Re}\gamma+\lambda)s + \int_0^s b(\xi)\,d\xi} \left|f(s)\right| ds\, de^{-(\operatorname{Re}\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \\
&= \frac{-1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)} \\
&\quad \times \left[e^{-(\operatorname{Re}\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi} \int_0^x e^{(\operatorname{Re}\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi} \left|f(s)\right| \mathrm{d}s \right|_{x=0}^{x=\infty} \\
&\quad - \left. \int_0^\infty e^{-(\operatorname{Re}\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi} e^{(\operatorname{Re}\gamma+\lambda)x + \int_0^x b(\xi)\,\mathrm{d}\xi} \left|f(x)\right| \mathrm{d}x \right] \\
&= \frac{1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)} \|f\|_{L^1[0,\infty)}
\end{aligned}$$

$$\Longrightarrow$$

$$\|E\| \leq \frac{1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)}. \tag{2.30}$$

Similarly,

$$\|E_0\| \leq \frac{1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b_0(x)}. \tag{2.31}$$

From (2.30), (2.31), $\|\phi\| \leq \sup_{x\in[0,\infty)} b_0(x)$ and $\|\psi\| \leq \sup_{x\in[0,\infty)} b(x)$ we deduce, for $(y,z) \in X$,

$$\begin{aligned}
&\left\|(\gamma I - A_0)^{-1}(y,z)\right\| \\
&\quad= \left|\frac{y_0}{\gamma+\lambda} + \frac{1}{\gamma+\lambda}\psi E y_1 + \frac{1}{\gamma+\lambda}\phi E_0 z_1\right| \\
&\qquad+ \|Ey_1\|_{L^1[0,\infty)} + \left\|\lambda E^2 y_1 + E y_2\right\|_{L^1[0,\infty)} + \left\|\lambda^2 E^3 y_1 + \lambda E^2 y_2 + E y_3\right\|_{L^1[0,\infty)} \\
&\qquad+ \left\|\lambda^3 E^4 y_1 + \lambda^2 E^3 y_2 + \lambda E^2 y_3 + E y_4\right\|_{L^1[0,\infty)} \\
&\qquad+ \cdots \\
&\qquad+ \|E_0 z_1\|_{L^1[0,\infty)} + \left\|\lambda E_0^2 z_1 + E_0 z_2\right\|_{L^1[0,\infty)} + \left\|\lambda^2 E_0^3 z_1 + \lambda E_0^2 z_2 + E_0 z_3\right\|_{L^1[0,\infty)} \\
&\qquad+ \cdots \\
&\quad\leq \frac{1}{|\gamma+\lambda|}|y_0| + \frac{1}{|\gamma+\lambda|}|\psi E y_1| + \frac{1}{|\gamma+\lambda|}|\phi E_0 z_1| \\
&\qquad+ \|Ey_1\|_{L^1[0,\infty)} + \left\|\lambda E^2 y_1\right\|_{L^1[0,\infty)} + \|Ey_2\|_{L^1[0,\infty)} + \left\|\lambda^2 E^3 y_1\right\|_{L^1[0,\infty)} \\
&\qquad+ \left\|\lambda E^2 y_2\right\|_{L^1[0,\infty)} + \|Ey_3\|_{L^1[0,\infty)} + \left\|\lambda^3 E^4 y_1\right\|_{L^1[0,\infty)} \\
&\qquad+ \left\|\lambda^2 E^3 y_2\right\|_{L^1[0,\infty)} + \left\|\lambda E^2 y_3\right\|_{L^1[0,\infty)} + \|Ey_4\|_{L^1[0,\infty)} \\
&\qquad+ \cdots \\
&\qquad+ \|E_0 z_1\|_{L^1[0,\infty)} + \left\|\lambda E_0^2 z_1\right\|_{L^1[0,\infty)} + \|E_0 z_2\|_{L^1[0,\infty)} + \left\|\lambda^2 E_0^3 z_1\right\|_{L^1[0,\infty)} \\
&\qquad+ \left\|\lambda E_0^2 z_2\right\|_{L^1[0,\infty)} + \|E_0 z_3\|_{L^1[0,\infty)} \\
&\qquad+ \cdots \\
&\quad\leq \frac{1}{|\gamma+\lambda|}|y_0| + \frac{1}{|\gamma+\lambda|}\|\psi\|\|E\|\|y_1\|_{L^1[0,\infty)} + \frac{1}{|\gamma+\lambda|}\|\phi\|\|E_0\|\|z_1\|_{L^1[0,\infty)} \\
&\qquad+ \|E\|\|y_1\|_{L^1[0,\infty)} + \lambda\|E\|^2\|y_1\|_{L^1[0,\infty)} + \|E\|\|y_2\|_{L^1[0,\infty)} \\
&\qquad+ \lambda^2\|E\|^3\|y_1\|_{L^1[0,\infty)} + \lambda\|E\|^2\|y_2\|_{L^1[0,\infty)} + \|Ey_3\|_{L^1[0,\infty)} \\
&\qquad+ \lambda^3\|E\|^4\|y_1\|_{L^1[0,\infty)} + \lambda^2\|E\|^3\|y_2\|_{L^1[0,\infty)} + \lambda\|E\|^2\|y_3\|_{L^1[0,\infty)} + \|E\|\|y_4\|_{L^1[0,\infty)} \\
&\qquad+ \cdots \\
&\qquad+ \|E_0\|\|z_1\|_{L^1[0,\infty)} + \lambda\|E_0\|^2\|z_1\|_{L^1[0,\infty)} + \|E_0\|\|z_2\|_{L^1[0,\infty)} \\
&\qquad+ \lambda^2\|E_0\|^3\|z_1\|_{L^1[0,\infty)} + \lambda\|E_0\|^2\|z_2\|_{L^1[0,\infty)} + \|E_0\|\|z_3\|_{L^1[0,\infty)} \\
&\qquad+ \cdots \\
&\quad= \frac{1}{|\gamma+\lambda|}|y_0| + \frac{1}{|\gamma+\lambda|}\|\psi\|\|E\|\|y_1\|_{L^1[0,\infty)} + \frac{1}{|\gamma+\lambda|}\|\phi\|\|E_0\|\|z_1\|_{L^1[0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\lambda^{n-1}\|E\|^{n}\|y_1\|_{L^1[0,\infty)}+\sum_{n=1}^{\infty}\lambda^{n-1}\|E\|^{n}\|y_2\|_{L^1[0,\infty)}\\
&+\sum_{n=1}^{\infty}\lambda^{n-1}\|E\|^{n}\|y_3\|_{L^1[0,\infty)}+\sum_{n=1}^{\infty}\lambda^{n-1}\|E\|^{n}\|y_4\|_{L^1[0,\infty)}\\
&+\cdots\\
&+\sum_{n=1}^{\infty}\lambda^{n-1}\|E_0\|^{n}\|z_1\|_{L^1[0,\infty)}+\sum_{n=1}^{\infty}\lambda^{n-1}\|E_0\|^{n}\|z_2\|_{L^1[0,\infty)}+\sum_{n=1}^{\infty}\lambda^{n-1}\|E_0\|^{n}\|z_3\|_{L^1[0,\infty)}\\
&+\cdots\\
={}&\frac{1}{|\gamma+\lambda|}|y_0|+\frac{1}{|\gamma+\lambda|}\|\psi\|\|E\|\|y_1\|_{L^1[0,\infty)}+\frac{1}{|\gamma+\lambda|}\|\phi\|\|\|E_0\|\|z_1\|_{L^1[0,\infty)}\\
&+\sum_{n=1}^{\infty}\lambda^{n-1}\|E\|^{n}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\sum_{n=1}^{\infty}\lambda^{n-1}\|E_0\|^{n}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)}\\
\le{}&\frac{1}{|\gamma+\lambda|}|y_0|+\frac{1}{|\gamma+\lambda|}\frac{\sup_{x\in[0,\infty)}b(x)}{\operatorname{Re}\gamma+\lambda+\inf_{x\in[0,\infty)}b(x)}\|y_1\|_{L^1[0,\infty)}\\
&+\frac{1}{|\gamma+\lambda|}\frac{\sup_{x\in[0,\infty)}b_0(x)}{\operatorname{Re}\gamma+\lambda+\inf_{x\in[0,\infty)}b_0(x)}\|z_1\|_{L^1[0,\infty)}\\
&+\frac{1}{\operatorname{Re}\gamma+\inf_{x\in[0,\infty)}b(x)}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{1}{\operatorname{Re}\gamma+\inf_{x\in[0,\infty)}b_0(x)}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)}\\
={}&\sup\biggl\{\frac{1}{|\gamma+\lambda|},\frac{1}{|\gamma+\lambda|}\frac{\sup_{x\in[0,\infty)}b(x)}{\operatorname{Re}\gamma+\lambda+\inf_{x\in[0,\infty)}b(x)}+\frac{1}{\operatorname{Re}\gamma+\inf_{x\in[0,\infty)}b(x)},\\
&\frac{1}{|\gamma+\lambda|}\frac{\sup_{x\in[0,\infty)}b_0(x)}{\operatorname{Re}\gamma+\lambda+\inf_{x\in[0,\infty)}b_0(x)}+\frac{1}{\operatorname{Re}\gamma+\inf_{x\in[0,\infty)}b_0(x)}\biggr\}\|(y,z)\|,
\end{aligned}
$$

which means that the result of this lemma is right. □

Lemma 2.3 *For $\gamma\in\rho(A_0)$ we have*

$$
\begin{aligned}
&(p,Q)\in\ker(\gamma I-A_m)\quad\Longleftrightarrow\\
&p_0=\frac{1}{\gamma+\lambda}\biggl[b_1\int_0^{\infty}b_0(x)e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x+a_1\int_0^{\infty}b(x)e^{-(\gamma+\lambda)x-\int_0^x b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x\biggr], &&(2.32)\\
&p_n(x)=e^{-(\gamma+\lambda)x-\int_0^x b(\xi)\,\mathrm{d}\xi}\sum_{k=0}^{n-1}\frac{(\lambda x)^k}{k!}a_{n-k},\quad n\ge 1, &&(2.33)\\
&Q_n(x)=e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}\sum_{k=0}^{n-1}\frac{(\lambda x)^k}{k!}b_{n-k},\quad n\ge 1, &&(2.34)\\
&(a_1,a_2,\ldots)\in l^1,\qquad (b_1,b_2,\ldots)\in l^1. &&(2.35)
\end{aligned}
$$

Proof If $(p,Q)\in\ker(\gamma I-A_m)$, then $(\gamma I-A_m)(p,Q)=0$, which is equivalent to

$$(\gamma+\lambda)p_0=\int_0^{\infty}Q_1(x)b_0(x)\,\mathrm{d}x+\int_0^{\infty}p_1(x)b(x)\,\mathrm{d}x, \tag{2.36}$$

$$\frac{\mathrm{d}p_1(x)}{\mathrm{d}x}=-\bigl(\gamma+\lambda+b(x)\bigr)p_1(x), \tag{2.37}$$

$$\frac{\mathrm{d}p_n(x)}{\mathrm{d}x} = -\big(\gamma + \lambda + b(x)\big)p_n(x) + \lambda p_{n-1}(x), \quad \forall n \geq 2, \tag{2.38}$$

$$\frac{\mathrm{d}Q_1(x)}{\mathrm{d}x} = -\big(\gamma + \lambda + b_0(x)\big)Q_1(x), \tag{2.39}$$

$$\frac{\mathrm{d}Q_n(x)}{\mathrm{d}x} = -\big(\gamma + \lambda + b_0(x)\big)Q_n(x) + \lambda Q_{n-1}(x), \quad \forall n \geq 2. \tag{2.40}$$

By solving (2.37)-(2.40) we have

$$p_1(x) = a_1 e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi}, \tag{2.41}$$

$$\begin{aligned} p_n(x) &= a_n e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi} \\ &\quad + \lambda e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi} \int_0^x p_{n-1}(s) e^{(\gamma+\lambda)s + \int_0^s b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \quad n \geq 2, \end{aligned} \tag{2.42}$$

$$Q_1(x) = b_1 e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}, \tag{2.43}$$

$$\begin{aligned} Q_n(x) &= b_n e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi} \\ &\quad + \lambda e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi} \int_0^x Q_{n-1}(s) e^{(\gamma+\lambda)s + \int_0^s b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s, \quad n \geq 2. \end{aligned} \tag{2.44}$$

Through inserting (2.41) and (2.43) into (2.36), it follows that

$$\begin{aligned} p_0 &= \frac{a_1}{\gamma+\lambda} \int_0^\infty b(x) e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x \\ &\quad + \frac{b_1}{\gamma+\lambda} \int_0^\infty b_0(x) e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x. \end{aligned} \tag{2.45}$$

By using (2.41), (2.42), (2.43) and (2.44) repeatedly, we deduce

$$\begin{aligned} p_2(x) &= a_2 e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} + \lambda e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \int_0^x a_1\,ds \\ &= e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} [a_2 + \lambda x a_1], \end{aligned} \tag{2.46}$$

$$\begin{aligned} p_3(x) &= a_3 e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} + \lambda e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \int_0^x [a_2 + \lambda s a_1]\,ds \\ &= e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \left[a_3 + \lambda x a_2 + \frac{(\lambda x)^2}{2} a_1\right], \end{aligned} \tag{2.47}$$

$$\begin{aligned} p_4(x) &= a_4 e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} + \lambda e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \\ &\quad \times \int_0^x \left[a_3 + \lambda s a_2 + \frac{(\lambda s)^2}{2} a_1\right] ds \\ &= e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \left[a_4 + \lambda x a_3 + \frac{(\lambda x)^2}{2} a_2 + \frac{(\lambda x)^3}{3!} a_1\right], \end{aligned} \tag{2.48}$$

$$\ldots$$

$$p_n(x) = e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k}, \quad n \geq 1, \tag{2.49}$$

$$\begin{aligned} Q_2(x) &= b_2 e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,d\xi} + \lambda e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,d\xi} \int_0^x b_1\,ds \\ &= e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,d\xi} [b_2 + \lambda x b_1], \end{aligned} \tag{2.50}$$

$$Q_3(x) = b_3 e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,d\xi} + \lambda e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,d\xi} \int_0^x [b_2 + \lambda s b_1]\,ds$$
$$= e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,d\xi}\left[b_3 + \lambda x b_2 + \frac{(\lambda x)^2}{2} b_1\right], \tag{2.51}$$
$$\cdots$$
$$Q_n(x) = e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} b_{n-k}, \quad n \geq 1. \tag{2.52}$$

Since $(p, Q) \in \ker(\gamma I - A_m)$, by the imbedding theorem in Adams [15],

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} \left|p_n(0)\right| \leq \sum_{n=1}^{\infty} \|p_n\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=1}^{\infty} \left\{ \|p_n\|_{L^1[0,\infty)} + \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} \right\} \\ &< \infty,\end{aligned} \tag{2.53}$$

$$\begin{aligned}\sum_{n=1}^{\infty} |b_n| &= \sum_{n=1}^{\infty} \left|Q_n(0)\right| \leq \sum_{n=1}^{\infty} \|Q_n\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=1}^{\infty} \left\{ \|Q_n\|_{L^1[0,\infty)} + \left\| \frac{dQ_n}{dx} \right\|_{L^1[0,\infty)} \right\} \\ &< \infty.\end{aligned} \tag{2.54}$$

(2.45)-(2.54) show that (2.32)-(2.35) are true.

Conversely, if (2.32)-(2.35) hold, then by using the formulas

$$\int_0^\infty e^{-Cx} x^k \,\mathrm{d}x = \frac{k!}{C^{k+1}}, \quad C > 0, k \in \mathbb{N},$$
$$\int_0^\infty b(x) e^{-\int_0^x b(\xi)\,d\xi}\,dx = 1, \qquad \int_0^\infty b_0(x) e^{-\int_0^x b_0(\xi)\,d\xi}\,dx = 1,$$

integration by parts and the Fubini theorem, we estimate

$$\begin{aligned}\|p_n\|_{L^1[0,\infty)} &= \int_0^\infty \left| e^{-(\gamma+\lambda)x-\int_0^x b(\xi)\,\mathrm{d}\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k} \right| \mathrm{d}x \\ &\leq \int_0^\infty e^{-(\operatorname{Re}\gamma+\lambda)x-\int_0^x b(\xi)\,\mathrm{d}\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} |a_{n-k}|\,\mathrm{d}x \\ &\leq \sum_{k=0}^{n-1} \frac{\lambda^k}{k!} |a_{n-k}| \int_0^\infty e^{-(\operatorname{Re}\gamma+\lambda+\inf_{x\in[0,\infty)} b(x))x} x^k\,\mathrm{d}x \\ &= \sum_{k=0}^{n-1} \frac{\lambda^k}{k!} |a_{n-k}| \frac{k!}{(\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x))^{k+1}} \\ &= \sum_{k=0}^{n-1} \frac{\lambda^k}{(\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x))^{k+1}} |a_{n-k}|, \quad n \geq 1 \\ &\Longrightarrow\end{aligned}$$

$$\begin{aligned}
&|p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} \\
&\quad \le \frac{|a_1|}{|\gamma + \lambda|} \int_0^\infty b(x) e^{-\int_0^x b(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x \\
&\qquad + \frac{|b_1|}{|\gamma + \lambda|} \int_0^\infty b_0(x) e^{-\int_0^x b_0(\xi)\,\mathrm{d}\xi}\,\mathrm{d}x \\
&\qquad + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)} \left(\frac{\lambda}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)}\right)^k |a_{n-k}| \\
&\quad = \frac{|a_1| + |b_1|}{|\gamma + \lambda|} \\
&\qquad + \frac{1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b(x)}\right)^k \sum_{n=k+1}^{\infty} |a_{n-k}| \\
&\quad = \frac{|a_1| + |b_1|}{|\gamma + \lambda|} + \frac{1}{\operatorname{Re}\gamma + \inf_{x\in[0,\infty)} b(x)} \sum_{n=1}^{\infty} |a_n| \\
&\quad < \infty, \qquad (2.55)
\end{aligned}$$

$$\begin{aligned}
&\|Q_n\|_{L^1[0,\infty)} \le \sum_{k=0}^{n-1} \frac{\lambda^k}{(\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b_0(x))^{k+1}} |b_{n-k}|, \quad n \ge 1 \\
&\qquad \Longrightarrow \\
&\sum_{n=1}^{\infty} \|Q_n\|_{L^1[0,\infty)} \le \frac{1}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b_0(x)} \\
&\qquad \times \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \left(\frac{\lambda}{\operatorname{Re}\gamma + \lambda + \inf_{x\in[0,\infty)} b_0(x)}\right)^k |b_{n-k}| \\
&\qquad = \frac{1}{\operatorname{Re}\gamma + \inf_{x\in[0,\infty)} b_0(x)} \sum_{n=1}^{\infty} |b_n| \\
&\qquad < \infty. \qquad (2.56)
\end{aligned}$$

(2.33) and (2.34) give

$$\begin{aligned}
\frac{dp_n(x)}{dx} &= -\big(\gamma + \lambda + b(x)\big) e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k} \\
&\quad + e^{-(\gamma+\lambda)x - \int_0^x b(\xi)\,d\xi} \sum_{k=1}^{n-1} \frac{\lambda^k x^{k-1}}{(k-1)!} a_{n-k} \\
&= -\big(\gamma + \lambda + b(x)\big) p_n(x) + \lambda p_{n-1}(x), \quad n \ge 2, \qquad (2.57)
\end{aligned}$$

$$\frac{dp_1(x)}{dx} = -\big(\gamma + \lambda + b(x)\big) p_1(x), \qquad (2.58)$$

$$\begin{aligned}
\frac{dQ_n(x)}{dx} &= -\big(\gamma + \lambda + b_0(x)\big) e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} b_{n-k} \\
&\quad + e^{-(\gamma+\lambda)x - \int_0^x b_0(\xi)\,d\xi} \sum_{k=1}^{n-1} \frac{\lambda^k x^{k-1}}{(k-1)!} b_{n-k}
\end{aligned}$$

$$= -\big(\gamma + \lambda + b_0(x)\big)Q_n(x) + \lambda Q_{n-1}(x), \quad n \geq 2, \tag{2.59}$$

$$\frac{dQ_1(x)}{dx} = -\big(\gamma + \lambda + b_0(x)\big)Q_1(x). \tag{2.60}$$

By combining (2.57), (2.58), (2.59) and (2.60) with (2.55) and (2.56), we derive

$$\sum_{n=1}^{\infty}\left\|\frac{\mathrm{d}p_n(x)}{\mathrm{d}x}\right\|_{L^1[0,\infty)} \leq \Big[\operatorname{Re}\gamma + 2\lambda + \sup_{x\in[0,\infty)} b(x) + |\operatorname{Im}\gamma|\Big]\sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)} < \infty, \tag{2.61}$$

$$\sum_{n=1}^{\infty}\left\|\frac{dQ_n}{dx}\right\|_{L^1[0,\infty)} \leq \Big[\operatorname{Re}\gamma + 2\lambda + \sup_{x\in[0,\infty)} b_0(x) + |\operatorname{Im}\gamma|\Big]\sum_{n=1}^{\infty}\|Q_n\|_{L^1[0,\infty)} < \infty. \tag{2.62}$$

(2.55)-(2.62) mean that $(p, Q) \in D(A_m)$ and $(\gamma I - A_m)(p, Q) = 0$. □

It is not difficult to see that L is surjective. Moreover,

$$L\big|_{\ker(\gamma I - A_m)} : \ker(\gamma I - A_m) \to \partial X$$

is invertible for $\gamma \in \rho(A_0)$. For $\forall\gamma \in \rho(A_0)$ we define the Dirichlet operator as

$$D_\gamma := (L|_{\ker(\gamma I - A_m)})^{-1} : \partial X \to \ker(\gamma I - A_m).$$

Lemma 2.3 gives the explicit form of D_γ for $\gamma \in \rho(A_0)$

$$D_\gamma(\vec{a}, \vec{b}) = \left(\begin{pmatrix} \frac{1}{\gamma+\lambda}\psi\epsilon_0 & 0 & 0 & \cdots \\ \epsilon_0 & 0 & 0 & \cdots \\ \epsilon_1 & \epsilon_0 & 0 & \cdots \\ \epsilon_2 & \epsilon_1 & \epsilon_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{1}{\gamma+\lambda}\phi\delta_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix}, \begin{pmatrix} \delta_0 & 0 & 0 & 0 & \cdots \\ \delta_1 & \delta_0 & 0 & 0 & \cdots \\ \delta_2 & \delta_1 & \delta_0 & 0 & \cdots \\ \delta_3 & \delta_2 & \delta_1 & \delta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix}\right), \tag{2.63}$$

where

$$\epsilon_k = \frac{(\lambda x)^k}{k!}e^{-(\gamma+\lambda)x-\int_0^x b(\xi)\,\mathrm{d}\xi}, \quad k \geq 0,$$

$$\delta_k = \frac{(\lambda x)^k}{k!}e^{-(\gamma+\lambda)x-\int_0^x b_0(\xi)\,\mathrm{d}\xi}, \quad k \geq 0.$$

From (2.63) and the definition of Φ, it is easy to determine the expression of ΦD_γ for $\gamma \in \rho(A_0)$.

$$\Phi D_\gamma(\vec{a},\vec{b}) = \left(\begin{pmatrix} \psi\epsilon_1 & \psi\epsilon_0 & 0 & 0 & \cdots \\ \psi\epsilon_2 & \psi\epsilon_1 & \psi\epsilon_0 & 0 & \cdots \\ \psi\epsilon_3 & \psi\epsilon_2 & \psi\epsilon_1 & \psi\epsilon_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} \right.$$
$$+ \begin{pmatrix} \phi\delta_1 & \phi\delta_0 & 0 & 0 & \cdots \\ \phi\delta_2 & \phi\delta_1 & \phi\delta_0 & 0 & \cdots \\ \phi\delta_3 & \phi\delta_2 & \phi\delta_1 & \phi\delta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix},$$
$$\left. \begin{pmatrix} \frac{\lambda}{\gamma+\lambda}\psi\epsilon_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{\gamma+\lambda}\phi\delta_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \right).$$

Haji and Radl [12] gave the following result through which we deduce the resolvent set of A on the imaginary axis.

Lemma 2.4 *If* $\gamma \in \rho(A_0)$ *and* $1 \notin \sigma(\Phi D_\gamma)$, *then*

$$\gamma \in \sigma(A) \quad \Longleftrightarrow \quad 1 \in \sigma(\Phi D_\gamma).$$

By using Lemma 2.4 and Nagel [14], page 297, we derive the following result.

Lemma 2.5 *Let* $b(x), b_0(x) : [0,\infty) \to [0,\infty)$ *be measurable,* $0 < \inf_{x\in[0,\infty)} b(x) \leq \sup_{x\in[0,\infty)} b(x) < \infty$ *and* $0 < \inf_{x\in[0,\infty)} b_0(x) \leq \sup_{x\in[0,\infty)} b_0(x) < \infty$. *Then all points on the imaginary axis except zero belong to the resolvent set of* A.

Proof Take $\gamma = im$, $m \in \mathbb{R} \setminus \{0\}$, $\vec{a} = (a_1, a_2, \ldots) \in l^1$ and $\vec{b} = (b_1, b_2, \ldots) \in l^1$. Then by the Riemann-Lebesgue lemma,

$$\lim_{m\to\infty} \int_0^\infty f(x)\cos(mx)\,dx = 0,$$
$$\lim_{m\to\infty} \int_0^\infty f(x)\sin(mx)\,dx = 0, \quad f \in L^1[0,\infty), f(x) \geq 0,$$

we know there exists $\mathcal{M} > 0$ such that $|m| > \mathcal{M}$

$$\begin{aligned} \left| \int_0^\infty f(x)e^{-imx}\,dx \right|^2 &= \left(\int_0^\infty f(x)\cos(mx)\,dx \right)^2 + \left(\int_0^\infty f(x)\sin(mx)\,dx \right)^2 \\ &< \left(\int_0^\infty f(x)\,dx \right)^2 \\ &\Longrightarrow \\ \left| \int_0^\infty f(x)e^{-imx}\,dx \right| &< \int_0^\infty f(x)\,dx. \end{aligned} \tag{2.64}$$

By replacing $f(x)$ in (2.64) with $f(x) = e^{-\lambda x-\int_0^x b(\xi)\,d\xi}$, $f(x) = e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}$ and using the fact

$$\left|\int_0^\infty \frac{(\lambda x)^n}{n!} e^{-(im+\lambda)x-\int_0^x b(\xi)\,d\xi}\,dx\right| \le \int_0^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x-\int_0^x b(\xi)\,d\xi}\,dx, \quad n \ge 1,$$
$$\left|\int_0^\infty \frac{(\lambda x)^n}{n!} e^{-(im+\lambda)x-\int_0^x b_0(\xi)\,d\xi}\,dx\right| \le \int_0^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx, \quad n \ge 1$$

we derive, for $|m| > \mathcal{M}$,

$$\begin{aligned}
\|\Phi D_\gamma(\vec{a},\vec{b})\| &= |\psi\epsilon_1 a_1 + \psi\epsilon_0 a_2 + \phi\delta_1 b_1 + \phi\delta_0 b_2| \\
&\quad + |\psi\epsilon_2 a_1 + \psi\epsilon_1 a_2 + \psi\epsilon_0 a_3 + \phi\delta_2 b_1 + \phi\delta_1 b_2 + \phi\delta_0 b_3| \\
&\quad + |\psi\epsilon_3 a_1 + \psi\epsilon_2 a_2 + \psi\epsilon_1 a_3 + \psi\epsilon_0 a_4 \\
&\quad + \phi\delta_3 b_1 + \phi\delta_2 b_2 + \phi\delta_1 b_3 + \phi\delta_0 b_4| \\
&\quad + \cdots \\
&\quad + \left|\sum_{j=0}^n \psi\epsilon_j a_{n+1-j} + \sum_{j=0}^n \phi\delta_j b_{n+1-j}\right| \\
&\quad + \cdots \\
&\quad + \left|\frac{\lambda}{\gamma+\lambda}\psi\epsilon_0 a_1 + \frac{\lambda}{\gamma+\lambda}\phi\delta_0 b_1\right| \\
&\le \sum_{n=1}^\infty |\psi\epsilon_n||a_1| + \sum_{n=0}^\infty |\psi\epsilon_n||a_2| + \sum_{n=0}^\infty |\psi\epsilon_n||a_3| + \cdots \\
&\quad + \sum_{n=1}^\infty |\phi\delta_n||b_1| + \sum_{n=0}^\infty |\phi\delta_n||b_2| + \sum_{n=0}^\infty |\phi\delta_n||b_3| + \cdots \\
&\quad + \frac{\lambda}{\sqrt{m^2+\lambda^2}}|\psi\epsilon_0||a_1| + \frac{\lambda}{\sqrt{m^2+\lambda^2}}|\phi\delta_0||b_1| \\
&< \sum_{n=1}^\infty |\psi\epsilon_n||a_1| + \sum_{n=0}^\infty |\psi\epsilon_n| \sum_{k=2}^\infty |a_k| \\
&\quad + \sum_{n=1}^\infty |\phi\delta_n||b_1| + \sum_{n=0}^\infty |\phi\delta_n| \sum_{k=2}^\infty |b_k| \\
&\quad + |\psi\epsilon_0||a_1| + |\phi\delta_0||b_1| \\
&= \sum_{n=0}^\infty |\psi\epsilon_n| \sum_{k=1}^\infty |a_k| + \sum_{n=0}^\infty |\phi\delta_n| \sum_{k=1}^\infty |b_k| \\
&= \sum_{n=0}^\infty \left|\int_0^\infty b(x)\frac{(\lambda x)^n}{n!} e^{-(im+\lambda)x-\int_0^x b(\xi)\,d\xi}\,dx\right| \sum_{k=1}^\infty |a_k| \\
&\quad + \sum_{n=0}^\infty \left|\int_0^\infty b_0(x)\frac{(\lambda x)^n}{n!} e^{-(im+\lambda)x-\int_0^x b_0(\xi)\,d\xi}\,dx\right| \sum_{k=1}^\infty |b_k| \\
&< \sum_{n=0}^\infty \int_0^\infty b(x)\frac{(\lambda x)^n}{n!} e^{-\lambda x-\int_0^x b(\xi)\,d\xi}\,dx \sum_{k=1}^\infty |a_k|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\int_0^{\infty} b_0(x)\frac{(\lambda x)^n}{n!}e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|b_k| \\
&=\int_0^{\infty} b(x)\sum_{n=0}^{\infty}\frac{(\lambda x)^n}{n!}e^{-\lambda x-\int_0^x b(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|a_k| \\
&\quad+\int_0^{\infty} b_0(x)\sum_{n=0}^{\infty}\frac{(\lambda x)^n}{n!}e^{-\lambda x-\int_0^x b_0(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|b_k| \\
&=\int_0^{\infty} b(x)e^{\lambda x}e^{-\lambda x-\int_0^x b(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|a_k| \\
&\quad+\int_0^{\infty} b_0(x)e^{\lambda x}e^{-\lambda x-\int_0^x b(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|b_k| \\
&=\int_0^{\infty} b(x)e^{-\int_0^x b(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|a_k| \\
&\quad+\int_0^{\infty} b_0(x)e^{-\int_0^x b_0(\xi)\,d\xi}\,dx\sum_{k=1}^{\infty}|b_k| \\
&=-e^{-\int_0^x b(\xi)\,d\xi}\Big|_0^{\infty}\sum_{k=1}^{\infty}|a_k|-e^{-\int_0^x b_0(\xi)\,d\xi}\Big|_0^{\infty}\sum_{k=1}^{\infty}|b_k| \\
&=\sum_{k=1}^{\infty}|a_1|+\sum_{k=1}^{\infty}|b_k| \\
&=\big\|(\vec{a},\vec{b})\big\| \\
&\Longrightarrow
\end{aligned}
$$

$$\|\Phi D_\gamma\|<1. \tag{2.65}$$

(2.65) means that when $|m|>\mathcal{M}$, the spectral radius $r(\Phi D_\gamma)\leq\|\Phi D_\gamma\|<1$, which implies $1\notin\sigma(\Phi D_\gamma)$ for $|m|>\mathcal{M}$, and therefore by Lemma 2.4, we know $\gamma=im\notin\sigma(A)$ for $|m|>\mathcal{M}$, that is,

$$\big\{im \mid |m|>\mathcal{M}\big\}\subset\rho(A),\qquad \big\{im \mid |m|\leq\mathcal{M}\big\}\subset\sigma(A)\cap i\mathbb{R}. \tag{2.66}$$

On the other hand, since $T(t)$ is positive uniformly bounded by Theorem 1.1, by Corollary 2.3 in Nagel [14], page 297, we know that $\sigma(A)\cap i\mathbb{R}$ is imaginary additively cyclic, which states that $im\in\sigma(A)\cap i\mathbb{R}\Rightarrow imk\in\sigma(A)\cap i\mathbb{R}$ for all integer k, from which together with (2.66) and Lemma 2.1 we conclude $\sigma(A)\cap i\mathbb{R}=\{0\}$. □

It is not difficult to prove X^*, dual space of X, is as follows:

$$X^*=\left\{(p^*,Q^*)\;\middle|\;\begin{array}{l} p^*\in\mathbb{R}\times L^{\infty}[0,\infty)\times L^{\infty}[0,\infty)\times\cdots, \\ Q^*\in L^{\infty}[0,\infty)\times L^{\infty}[0,\infty)\times L^{\infty}[0,\infty)\times\cdots, \\ |||(p^*,Q^*)|||=\max\left\{\begin{array}{l}\sup\{|p_0^*|,\sup_{n\geq1}\|p_n^*\|_{L^{\infty}[0,\infty)}\}, \\ \sup_{n\geq1}\|Q_n^*\|_{L^{\infty}[0,\infty)}\end{array}\right\}<\infty\end{array}\right\}.$$

It is obvious that X^* is a Banach space. Gupur [4] gave the expression of A^*, the adjoint operator of A as follows:

$$A^*(p^*,Q^*) = (G+F+\Re)(p^*,Q^*), \quad (p^*,Q^*) \in D(G),$$

where

$$G(p^*,Q^*) = \left(\begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots \\ 0 & \frac{d}{dx}-(\lambda+b(x)) & 0 & 0 & \cdots \\ 0 & 0 & \frac{d}{dx}-(\lambda+b(x)) & 0 & \cdots \\ 0 & 0 & 0 & \frac{d}{dx}-(\lambda+b(x)) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^*(x) \\ p_2^*(x) \\ p_3^*(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \frac{d}{dx}-(\lambda+b_0(x)) & 0 & 0 & \cdots \\ 0 & \frac{d}{dx}-(\lambda+b_0(x)) & 0 & \cdots \\ 0 & 0 & \frac{d}{dx}-(\lambda+b_0(x)) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_1^*(x) \\ Q_2^*(x) \\ Q_3^*(x) \\ \vdots \end{pmatrix} \right),$$

$$D(G) = \left\{ (p^*,Q^*) \in X^* \,\middle|\, \begin{array}{l} \frac{dp_k^*(x)}{dx} \text{ and } \frac{dQ_k^*(x)}{dx} \text{ exist and} \\ p_k^*(\infty) = Q_k^*(\infty) = \alpha, k \geq 1 \end{array} \right\},$$

$$F(p^*,Q^*) = \left(\begin{pmatrix} 0 & 0 & 0 & \cdots \\ b(x) & 0 & 0 & \cdots \\ 0 & b(x) & 0 & \cdots \\ 0 & 0 & b(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^*(0) \\ p_2^*(0) \\ p_3^*(0) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} b_0(x) & 0 & 0 & 0 & \cdots \\ 0 & b_0(x) & 0 & 0 & \cdots \\ 0 & 0 & b_0(x) & 0 & \cdots \\ 0 & 0 & 0 & b_0(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^*(0) \\ p_2^*(0) \\ p_3^*(0) \\ \vdots \end{pmatrix} \right),$$

$$\Re(p^*,Q^*) = \left(\begin{pmatrix} \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_1^*(0) \\ Q_2^*(0) \\ Q_3^*(0) \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^*(x) \\ p_2^*(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_1^*(x) \\ Q_2^*(x) \\ Q_3^*(x) \\ \vdots \end{pmatrix} \right).$$

Since $T(t)$ is uniformly bounded, by Arendt and Batty [16] and Lemma 2.1, we know that 0 is an eigenvalue of A^*. Furthermore, by replacing μ and η in Lemma 3 in Gupur [4] with $b(x)$ and $b_0(x)$, respectively, we deduce the following result.

Lemma 2.6 *If* $\int_0^\infty \lambda x b(x) e^{-\int_0^x b(\xi)\,d\xi}\,dx < 1$, *then* 0 *is an eigenvalue of* A^* *with geometric multiplicity one*.

Since Theorem 1.1, Lemma 2.1, Lemma 2.5 and Lemma 2.6 satisfy the conditions of Theorem 14 in Gupur, Li and Zhu [7], the following conclusion is the direct result of Theorem 14 in Gupur, Li and Zhu [7].

Theorem 2.7 *Let* $b(x), b_0(x) : [0,\infty) \to [0,\infty)$ *be measurable*, $0 < \inf_{x\in[0,\infty)} b(x) \le \sup_{x\in[0,\infty)} b(x) < \infty$ *and* $0 < \inf_{x\in[0,\infty)} b_0(x) \le \sup_{x\in[0,\infty)} b_0(x) < \infty$. *If* $\int_0^\infty \lambda x b(x) \times e^{-\int_0^x b(\xi)\,d\xi}\,dx < 1$, *then the time-dependent solution of the system* (1.9) *converges strongly to its steady-state solution*, *that is*,

$$\lim_{t\to\infty} \bigl\| (p,Q)(\cdot,t) - \bigl\langle (p^*,Q^*), \bigl(p(0),Q(0)\bigr)\bigr\rangle (p,Q)(\cdot) \bigr\| = 0,$$

where (p^*,Q^*) *and* (p,Q) *are the eigenvectors in Lemma* 2.6 *and Lemma* 2.1, *respectively*.

When $b(x) = \mu$ and $b_0(x) = \eta$, Lin and Gupur [9] proved that if $\sqrt{\lambda} < \sqrt{\mu} < \sqrt{\lambda} + \sqrt{\eta}$, then $(2\sqrt{\lambda\mu} - \lambda - \mu)\theta$ are eigenvalues of A with geometric multiplicity one for all $\theta \in (0,1)$. Which means that the result in Theorem 2.7 is optimal, that is to say, it is impossible that the time-dependent solution of the system (1.9) exponentially converges to its steady-state solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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