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Thermal instability in a non-Darcy porous medium saturated with a nanofluid and with a convective boundary condition

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Abstract

In this paper, we investigate the effect of vertical throughflow on the onset of convection in a horizontal layer of a non-Darcy porous medium saturated with a nanofluid. A normal mode analysis is used to find solutions for the fluid layer confined between parallel plates with free-rigid boundaries. The criterion for the onset of stationary and oscillatory convection is derived. The analysis incorporates the effects of Brownian motion, thermophoresis and a convective boundary condition. The effects of the concentration Rayleigh number, Lewis number, Darcy number and modified diffusivity ratio on the stability of the system are investigated.

Keywords: throughflow; convective boundary condition; oscillatory convection; linear stability analysis

1 Introduction

In the last several years, an innovative technique for improving the heat transfer characteristics by adding ultra fine metallic particles in common fluids such as water and oil has been investigated. The term nanofluid refers to these kinds of fluids which have applications in automotive industries, energy saving devices, nuclear rectors, *etc.*, Choi [1]. Nanoparticles also have medical applications including cancer therapy and nano-drug delivery (Shaw and Murthy [2]). Buongiorno [3] noted that the absolute velocity of nanoparticles can be viewed as the sum of the base fluid velocity and a relative (slip) velocity. He concluded that in the absence of turbulence, Brownian diffusion and thermophoresis would be important. A lot of work has been done on nanofluids; see, for instance, Nadeem *et al.* [4], Malik *et al.* [5], Nadeem *et al.* [6] and the review by Das *et al.* [7].

The effect of a magnetic field on flow and heat transfer problems is important in industrial applications, such as in the buoyant upward gas-liquid flow in packed bed electrodes (Takahashi and Alkire [8]), sodium oxide-silicon dioxide glass melt flows (Guloyan [9]), reactive polymer flows in heterogeneous porous media [10], electrochemical generation of elemental bromine in porous electrode systems (Qi and Savinell [11]).

The convective boundary condition is more general and realistic in engineering and industrial processes such as transportation cooling processes, material drying, *etc.* It therefore seems appropriate to use the convective boundary condition to study other boundary layer flow situations. Aziz [12] studied the Blasius flow over a flat plate with a convection thermal boundary condition. Ishak [13] investigated the effects of suction and injection on a flat surface with convective boundary condition. The study of the boundary layer flows over flat surfaces under convective surface boundary condition has attracted the attention of many researchers, such as Makinde and Aziz [14], Yao *et al.* [15].

Nanofluids have great potential as coolants due to their enhanced thermal conductivities. The enhancement of effective thermal conductivity was confirmed by experiments conducted by many researchers (Masuda *et al.* [16]). Instability of nanofluids in natural convection was studied by Tzou [17]. Tzou [18] studied thermal instability of nanofluids in natural convection. Thermal instability in nanofluids in a porous medium has been a topic of interest due to potential applications of such flows in food and chemical processes, petroleum industry, bio-mechanics and geophysical problems. Buongiorno's model was applied to the problem of the onset of instability in a porous medium layer saturated with a nanofluid by Nield and Kuznetsov [19, 20]. Kuznetsov and Nield [21] used the Brinkman model to study thermal instability in a horizontal porous layer saturated with a nanofluid. Other related studies of thermal instability in a porous medium saturated with a nanofluid include those by Kuznetsov and Nield [22] and Nield and Kuznetsov [23–26].

In this study, we extend the work by Nield and Kuznetsov [26] to a non-Darcy porous medium saturated with a nanofluid. We have considered a convective boundary condition in place of an isothermal condition. The effect of the Biot number, magnetic parameter, Brinkman-Darcy parameter on thermal instability has been studied.

2 Mathematical formulation

We consider a horizontal layer of porous medium confined between the planes $z = 0$ and $z = H$ where the z-axis is vertically upwards. Each boundary wall is assumed to be permeable to the throughflow and perfectly thermally conducting. Radiation heat transfer between the sides of walls is negligible when compared with other modes of heat transfer. The size of nanoparticles is small as compared to the pore size of the matrix. The nanoparticles are spherical, and the nanofluid is incompressible and laminar. It is assumed that nanoparticles are suspended in the nanofluid using either a surfactant or surface charge technology, preventing the agglomeration and deposition of these on the porous medium. The porosity of the medium is denoted by ϵ and the permeability by K. The temperatures at the lower and upper wall are T_1 and T_0 with $T_1 > T_0$. The nanoparticle volume fractions are ϕ_0 at the lower wall and ϕ_1 at the upper wall, and it is assumed that the difference $\phi_1 - \phi_0$ is small in comparison with ϕ_0. A uniform magnetic field of strength B is imposed normal to the plate. Using the modified Brinkman model and the Oberbeck-Boussinesq approximation, the conservation equations for mass, momentum, energy and nanoparticles are as follows:

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\frac{\rho}{\epsilon}\frac{d\mathbf{v}}{dt} = -\nabla\mathbf{p} + \left[\phi\rho_p + (1-\phi)\left\{\rho\left(1-\beta(T-T_0)\right)\right\}\right]\mathbf{g} + \tilde{\mu}\nabla^2\mathbf{v} - \frac{\mu}{K}\mathbf{v} - \sigma_m B^2\mathbf{v}, \tag{2}$$

$$(\rho c)_m\frac{\partial T}{\partial t} + (\rho c)_f\mathbf{v}\cdot\nabla T = k_m\nabla^2 T + \epsilon(\rho c)_p\left(D_B\nabla\phi\cdot\nabla T + \frac{D_T}{T_1}\nabla T\cdot\nabla T\right), \tag{3}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{\epsilon}\mathbf{v}\cdot\nabla\phi = D_B\nabla^2\phi + \frac{D_T}{T_1}\nabla^2 T, \tag{4}$$

where $\mathbf{v} = (u, v, w)$ is the velocity vector, ρ is the density of the fluid, t is the time, $\mathbf{p}$ is the hydraulic pressure, ϕ is the volume fraction of nanoparticles, ρ_p is the density of nanoparticles, β is the coefficient of thermal expansion, $\tilde{\mu}$ is effective viscosity, μ is viscosity and σ_m is electric conductivity. In the energy equation, $(\rho c)_m$ is the heat capacity of the fluid in the porous medium, $(\rho c)_p$ is the heat capacity of nanoparticles and k_m is thermal conductivity. In the equation of continuity for nanoparticles, D_B is the Brownian diffusion coefficient, given by the Einstein-Stokes equation and D_T is the thermophoretic diffusion coefficient of nanoparticles.

We note that in equation (2),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{\epsilon}(\mathbf{v}\cdot\nabla), \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$$

is the convective derivative. We introduce non-dimensional variables as

$$\begin{aligned}
&(x', y', z') = \left(\frac{x, y, z}{H}\right), \qquad (u', v', w') = \left(\frac{u, v, w}{\kappa}\right)H, \qquad t' = \frac{t\kappa}{\sigma H^2},\\
&p' = \frac{pK}{\mu\kappa}, \qquad T' = \frac{T - T_1}{T_0 - T_1}, \qquad \phi' = \frac{\phi - \phi_0}{\phi_1 - \phi_0},
\end{aligned} \tag{5}$$

where $\kappa = k_m/(\rho c)_p$ is thermal diffusivity of the fluid. It is assumed that the wall is heated by convection from a hot fluid with temperature T_w and heat transfer coefficient h_f. Under these conditions, the thermal field is written as

$$-k_m \frac{\partial T}{\partial z}\bigg|_{z=0} = h_f(T_0 - T_w), \qquad T|_{z=H} = T_1, \tag{6}$$

while the boundary conditions for velocity and nanoparticle concentration are

$$w(0) = V, \qquad \phi(0) = \phi_0, \qquad w(H) = V \quad \text{and} \quad \phi(H) = \phi_1. \tag{7}$$

After substituting equations (5) into (1)-(4), the resulting nondimensional equations are written as follows (ignoring primes):

$$\nabla\cdot\mathbf{v} = 0, \tag{8}$$

$$\frac{1}{Va}\frac{d\mathbf{v}}{dt} = -\nabla\mathbf{p} - Rm\hat{e}_z + RaT\hat{e}_z - Rn\phi\hat{e}_z + \widetilde{Da}\nabla^2\mathbf{v} - \mathbf{v} - M\mathbf{v}, \tag{9}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \nabla^2 T + \frac{N_B}{Le}\nabla\phi\cdot\nabla T + \frac{N_A N_B}{Le}\nabla T\cdot\nabla T, \tag{10}$$

$$\frac{1}{\sigma}\frac{\partial\phi}{\partial t} + \frac{1}{\epsilon}\mathbf{v}\cdot\nabla\phi = \frac{1}{Le}\nabla^2\phi + \frac{N_A}{Le}\nabla^2 T. \tag{11}$$

The boundary conditions are written as

$$\begin{aligned}
&w = Q, \qquad \frac{\partial T}{\partial z} = -Bi(1 - T), \qquad \phi = 0 \quad \text{at } z = 0,\\
&w = Q, \qquad T = 0, \qquad \phi = 1 \quad \text{at } z = 0.
\end{aligned} \tag{12}$$

The dimensionless parameters in equations (8)-(12) are the Prandtl number Pr, the Darcy number Da, the Vadasz number Va, the density Rayleigh number Rm, the Rayleigh-Darcy number Ra, the concentration Rayleigh number Rn, the Brinkman-Darcy number

$\widetilde{Da}$, the magnetic parameter M, the Lewis number Le, the modified diffusivity ratio N_A, the modified particle-density increment N_B, the Peclet number Q and the Biot number Bi. These parameters are defined, respectively, by

$$Pr = \mu/\rho\kappa, \qquad Da = K/H^2, \qquad Va = \frac{\epsilon Pr}{Da},$$
$$Rm = \frac{\rho_p\phi_0 + \rho(1-\phi_0)gH}{\mu\kappa}, \qquad Ra = \frac{\rho g\alpha H(T_0 - T_1)}{\mu\kappa}, \qquad Rn = \frac{(\rho_p - \rho)(\phi_1 - \phi_0)gH}{\mu\kappa},$$
$$\widetilde{Da} = \frac{\tilde{\mu}K}{\mu H^2}, \qquad M = \frac{\sigma_m B_0^2\kappa}{\mu}, \qquad Le = \frac{\kappa}{D_B}, \qquad N_A = \frac{D_T(T_0 - T_1)}{D_B T_1(\phi_1 - \phi_0)},$$
$$N_B = \frac{(\rho c)_p(\phi_1 - \phi_0)}{(\rho c)_f}, \qquad Q = \frac{HV}{\alpha_m}, \qquad Bi = \frac{h_f H}{k_m}.$$

We note here that the parameter Rm is a measure of the basic static pressure gradient.

2.1 Basic solution

A time-independent quiescent solution is obtained in the z direction only and has the form

$$\mathbf{v} = v_b \equiv (0, 0, Q), \qquad T = T_b(z), \qquad \phi = \phi_b(z). \tag{13}$$

Assumptions that Le is very large (of order 10^2 to 10^3, see Buongiorno [3]), equations (9)-(11) now reduce to

$$\widetilde{Da}\frac{d^2 v_b}{dz^2} - \frac{dP_b}{dz} - Rm + RaT_b - Rn\phi_b - v_b - Mv_b = 0, \tag{14}$$
$$\frac{d^2 T_b}{dz^2} + \frac{N_B}{Le}\frac{d\phi_b}{dz}\frac{dT_b}{dz} + \frac{N_A N_B}{Le}\left(\frac{dT_b}{dz}\right)^2 = 0, \tag{15}$$
$$\frac{d^2\phi_b}{dz^2} + N_A\frac{d^2 T_b}{dz^2} = 0. \tag{16}$$

Solving equations (15) and (16) with boundary condition (12) gives

$$T_b = -\frac{Bi(e^{Qz} - e^Q)}{Q - Bi(1 - e^Q)}, \tag{17}$$
$$\phi_b = \frac{e^{\lambda z} - 1}{e^{\lambda} - 1}\left(1 + \frac{\delta(e^Q - 1)}{Q(Q - \lambda)}\right) + \frac{\delta}{Q(Q - \lambda)(e^{\lambda} - 1)}\left(1 - e^{Qz}\left(e^{\lambda} - 1\right)\right), \tag{18}$$

where $\lambda = \frac{QLe}{\varepsilon}$ and $\delta = \frac{N_A BiQ^2}{Q - Bi(1 - e^Q)}$. In the limit $Q \to 0$, we obtain

$$T_b = \frac{Bi}{1 + Bi}(1 - z), \qquad \phi_b = z. \tag{19}$$

As $Bi \to \infty$, the thermal boundary condition at the lower plate reduces to $T(0) = 1$ (isothermal condition), and the base solution becomes

$$T_b = 1 - z \quad \text{and} \quad \phi_b = z; \tag{20}$$

the same results were obtained by Nield and Kuznetsov [26].

2.2 Perturbation solution

To study the stability of the system, we superimpose infinitesimal perturbations on the basic state solution,

$$\mathbf{v} = \mathbf{v}_b + \mathbf{v}', \qquad p = p_b + p', \qquad T = T_b + T', \qquad \phi = \phi_b + \phi'. \tag{21}$$

Substituting (21) in equations (8)-(11), and linearizing by neglecting products of primed quantities, we get the following equations:

$$\left(\frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial y} + \frac{\partial w'}{\partial z}\right) = 0, \tag{22}$$

$$\frac{1}{Va}\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} + RaT'\hat{e}_z - Rn\phi'\hat{e}_z + \widetilde{Da}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) - w' - Mw', \tag{23}$$

$$\begin{aligned}\frac{\partial T}{\partial t} + Q\frac{\partial T'}{\partial z} + \frac{dT_b}{dz}w' &= \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} + \frac{\partial^2 T'}{\partial z^2}\right) \\ &\quad + \frac{N_B}{Le}\left(\frac{d\phi_b}{dz}\frac{\partial T'}{\partial z} + \frac{dT_b}{dz}\frac{\partial \phi'}{\partial z}\right) + \frac{2N_A N_B}{Le}\frac{dT_b}{dz}\frac{\partial T'}{\partial z},\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{1}{\sigma}\frac{\partial \phi'}{\partial t} + \frac{Q}{\epsilon}\frac{\partial \phi'}{\partial z} + \frac{1}{\epsilon}\frac{d\phi_b}{dz}w' &= \frac{1}{Le}\left(\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{\partial^2 \phi'}{\partial z^2}\right) \\ &\quad + \frac{N_A}{Le}\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} + \frac{\partial^2 T'}{\partial z^2}\right),\end{aligned} \tag{25}$$

with the boundary conditions

$$\begin{aligned}&w' = 0, \qquad \frac{\partial T'}{\partial z} = BiT', \qquad \phi' = 0 \quad \text{at } z = 0, \\ &w' = 0, \qquad T' = 0, \qquad \phi' = 0 \quad \text{at } z = 1.\end{aligned} \tag{26}$$

The derivatives of T_b and ϕ_b are

$$\frac{dT_b}{dz} = -\frac{BiQe^{Qz}}{Q - Bi(1 - e^Q)}, \tag{27}$$

$$\frac{d\phi_b}{dz} = \frac{\lambda e^{\lambda z}}{e^\lambda - 1}\left(1 - \frac{\delta(e^Q - 1)}{Q(Q - \lambda)}\right) - \frac{\delta Q}{Q - \lambda}e^{Qz}. \tag{28}$$

For regular fluids, the parameters Rn, N_A and N_B are zero and the third term in equation (25) is absent since $d\phi_b/dz = 0$. For $\widetilde{Da} = 0$, and in the absence of a magnetic field, the equations reduce to the familiar Horton-Roger-Lapwood problem with throughflow. Taking the curl of equation (23) and simplifying, we obtain

$$\frac{1}{Va}\frac{\partial}{\partial t}\nabla^2 w' - \widetilde{Da}\nabla^4 w' + (1 + M)\nabla^2 w' = Ra\nabla_H^2 T' + Rn\nabla_H^2 \phi'. \tag{29}$$

3 Normal modes and stability analysis

Differential equations (24), (25), (29) and boundary conditions (26) constitute a linear boundary value problem that can be solved using the method of normal modes. The perturbation quantities are of the form

$$\left[w', T', \phi'\right] = \left[W(z), \Theta(z), \Phi(z)\right] e^{ilx+imy+nt}, \tag{30}$$

where l and m are wave numbers in the x and y directions and n is the growth rate of the disturbances. Substituting into the differential equations, we obtain

$$\left(D^2 - \alpha^2\right)\left(\frac{s}{Va} + 1 + M - \widetilde{Da}\left(D^2 - \alpha^2\right)\right) W + Ra\alpha^2 \Theta - Rn\alpha^2 \Phi = 0, \tag{31}$$

$$-\frac{dT_b}{dz} W + \left(D^2 + \frac{N_b}{Le}\frac{d\phi_b}{dz} D + \frac{2N_A N_B}{Le}\frac{dT_b}{dz} D - QD - \alpha^2 - s\right)\Theta$$
$$+ \frac{N_B}{Le}\frac{dT_b}{dz} D\Phi = 0, \tag{32}$$

$$\left(QD + \frac{1}{\epsilon}\frac{d\phi_b}{dz}\right) W - \frac{N_A}{Le}\left(D^2 - \alpha^2\right)\Theta - \left(\frac{1}{Le}\left(D^2 - \alpha^2\right) - \frac{s}{\sigma}\right)\Phi = 0, \tag{33}$$

subject to the boundary conditions

$$\begin{aligned} &W = 0, \qquad \frac{\partial \Theta}{\partial z} = Bi\Theta, \qquad \Phi = 0 \quad \text{at } z = 0, \\ &W = 0, \qquad \Theta = 0, \qquad \Phi = 0 \quad \text{at } z = 1. \end{aligned} \tag{34}$$

Here $D = d/dz$ and $\alpha = (l^2 + m^2)^{1/2}$ is a dimensionless resultant wavenumber. We use a Galerkin-type weighted residuals method to obtain approximate solutions. We choose trial functions $W_p, \Theta_p, \Phi_p; p = 1, 2, 3, \ldots$, that satisfy the boundary conditions and consider

$$W = \sum_{p=1}^{N} A_p W_p, \qquad \Theta = \sum_{p=1}^{N} B_p \Theta_p, \qquad \Phi = \sum_{p=1}^{N} C_p \Phi_p. \tag{35}$$

Substituting equation (35) into equations (31)-(33) gives a system of $3N$ algebraic equations in $3N$ unknowns. The vanishing of the determinant of coefficients produces an eigenvalue equation for the system. Regarding Ra as the eigenvalue, we find Ra in terms of the other parameters. It is interesting to note that the convective boundary condition is applicable for rigid-free and rigid-rigid boundary conditions. In this study we mainly focus on rigid-free boundary and discuss the case of stationary and oscillatory convection. The vanishing of the shear-stresses tangent to the surface and continuity equation gives the boundary conditions

$$\begin{aligned} &W = 0, \qquad D^2 W = 0, \qquad \Theta = 0, \qquad \Phi = 0 \quad \text{at } z = 0, \\ &W = 0, \qquad D^2 W = 0, \qquad \Theta = 0, \qquad \Phi = 0 \quad \text{at } z = 1. \end{aligned} \tag{36}$$

We assume that the solutions to W, Θ and Φ are of the form

$$\begin{aligned} &W = z^{1+p}(1-z)(3-2z), \qquad \Theta = z^{1+p}(1-z), \\ &\Phi = z^p(1-z), \quad p = 1, 2, 3, \ldots, \end{aligned} \tag{37}$$

which satisfies the boundary conditions. Using the above transformation in equations (31)-(33) and integrating with respect to z from $z = 0$ to $z = 1$, the system of equations may be written as

$$\begin{bmatrix} -(\frac{s}{Va}+1+M)-\frac{36}{5}\widetilde{Da} & \frac{1}{12}Ra\alpha^2 & -\frac{1}{6}Rn\alpha^2 \\ J_{TW} & -\frac{1}{2}+\frac{N_B}{Le}J_{\Phi\Theta}+\frac{2N_AN_B}{Le}J_{T\Theta}-\frac{J}{2} & \frac{N_b}{Le}J_{T\Phi} \\ \frac{1}{\varepsilon}J_{\Phi W} & \frac{N_A}{Le}\frac{J}{2} & \frac{1}{Le}\frac{J}{2}+\frac{s}{\sigma} \end{bmatrix}\begin{bmatrix} W_0 \\ \Theta_0 \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (38)$$

where $J = 2 + \alpha^2/6$ and $J_{TW}, J_{\Phi\Theta}, J_{T\Theta}, J_{T\Phi}$ and $J_{\Phi W}$ are

$$J_{TW} = \int_0^1 \frac{dT_b}{dz} W\,dz = \frac{Bi}{Q^4(Q - Bi(1-e^Q))}\bigl[6Q^2 + 30Q + 48 - e^Q\bigl(Q^3 - 18Q + 48\bigr)\bigr],$$

$$J_{\Phi\Theta} = \int_0^1 \frac{d\phi_b}{dz} D\Theta\,dz = -\frac{\delta}{Q^2(Q-\lambda)}\bigl[2Q + 6 - e^Q\bigl(Q^2 - 4Q + 6\bigr)\bigr] + \frac{1}{\lambda^2(e^\lambda - 1)}\left(1 + \frac{\delta(e^Q-1)}{Q-\lambda}\right)\bigl[2\lambda + 6 - e^\lambda\bigl(\lambda^2 - 4\lambda + 6\bigr)\bigr],$$

$$J_{T\Theta} = \int_0^1 \frac{dT_b}{dz}\Theta\,dz = -\frac{Bi}{Q^2(Q - Bi(1-e^Q))}\bigl[2Q + 6 - e^Q\bigl(Q^2 - 4Q + 6\bigr)\bigr],$$

$$J_{T\Phi} = \int_0^1 \frac{dT_b}{dz}\Phi\,dz = \frac{Bi}{Q^2(Q - Bi(1-e^Q))}\bigl[Q + 2 + e^Q(Q-2)\bigr],$$

$$J_{\Phi W} = \int_0^1 \frac{d\phi_b}{dz} DW\,dz = \frac{\delta}{Q^4(Q-\lambda)}\bigl[6Q^2 + 30Q + 48 - e^Q\bigl(Q^3 - 16Q + 48\bigr)\bigr] - \frac{1}{\lambda^4(e^\lambda - 1)}\left(1 + \frac{\delta(e^Q-1)}{Q-\lambda}\right)\bigl[6\lambda^2 + 30\lambda + 48 - e^\lambda\bigl(\lambda^3 - 16\lambda + 48\bigr)\bigr].$$

For the non-trivial solution, the determinant of the augmented matrix is equal to zero, s is a dimensionless growth factor. We put $s = i\omega$, where ω is real and is the dimensionless frequency.

3.1 Stationary convection

In the case of non-oscillatory convection, $\omega = 0$. We considered the first approximation, *i.e.*, $N = 1$, which gives the non-oscillatory stability boundary as

$$Ra_s - Rn\left[Rn\frac{\frac{N_A}{Le}\frac{J}{2}J_{TW} + \frac{J_{\phi W}}{\varepsilon}A_1}{A_2}\right] = \frac{(1 + M + \frac{36}{5}\widetilde{Da})[\frac{J}{Le}A_1 - \frac{N_A}{Le}\frac{J}{2}\frac{N_B}{Le}J_{T\phi}]}{\frac{\alpha^2}{12}A_2}. \quad (39)$$

From equation (39), we get the stationary Rayleigh number as

$$Ra_s = \frac{(1 + M + \frac{36}{5}\widetilde{Da})[\frac{J}{Le}A_1 - \frac{N_A}{Le}\frac{J}{2}\frac{N_B}{Le}J_{T\phi}]}{\frac{\alpha^2}{12}A_2} - 2Rn\frac{\frac{N_A}{Le}\frac{J}{2}J_{TW} + \frac{J_{\phi W}}{\varepsilon}A_1}{A_2}, \quad (40)$$

where A_1 and A_2 are given by

$$A_1 = \frac{N_B}{Le}J_{\phi\theta} + \frac{2N_AN_B}{Le}J_{T\theta} - \frac{J}{2},$$

$$A_2 = \frac{J}{Le}J_{TW} + \frac{1}{\varepsilon}\frac{N_B}{Le}J_{T\phi}J_{\phi W}.$$

The critical cell size at the onset of instability is obtained when

$$\left(\frac{\partial Ra}{\partial\alpha}\right)_{\alpha=\alpha_c} = 0. \tag{41}$$

Solving the above equation, we get a polynomial in α^2 of order 2 and the critical α_c^2 is written as

$$\alpha_c^2\Big|_s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_3}}{2a_2}, \tag{42}$$

where the values of a_1, a_2 and a_3 are given by

$$\begin{aligned}
a_1 &= -\frac{1}{18}\frac{J_{TW}}{Le}\left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left[\frac{2}{Le}A_1 - \frac{N_A}{Le}\frac{J}{2}\frac{N_B}{Le}J_{T\phi}\right], \\
a_2 &= -\frac{A_2}{6}\left(\frac{2J_{TW}}{Le} + \frac{J_{\phi W}}{\varepsilon}\frac{N_B}{Le}J_{T\phi}\right)\left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left[\frac{2}{Le}A_1 - \frac{N_A}{Le}\frac{J}{2}\frac{N_B}{Le}J_{T\phi}\right], \\
a_3 &= -\frac{A_2}{432}\left(\frac{2J_{TW}}{Le} + \frac{J_{\phi W}}{\varepsilon}\frac{N_B}{Le}J_{T\phi}\right)\left[\frac{1}{Le}\left(1 + M + \frac{36}{5}\widetilde{Da}\right) + Rn\left(\frac{N_A}{Le}J_{TW} - \frac{J_{\phi W}}{\varepsilon}\right)\right] \\
&\quad - \frac{1}{432}\frac{J_{TW}}{Le}\left[\frac{2}{Le}A_1 - \frac{N_A}{Le}\frac{J}{2}\frac{N_B}{Le}J_{T\phi}\right].
\end{aligned}$$

We calculate the corresponding critical Rayleigh number $(Ra_s)_{\text{crit}}$ using the above critical value of α for stationary convection.

3.2 Oscillatory convection

In case of the oscillatory convection, $\omega \neq 0$. The stability boundaries are

$$\begin{aligned}
0 &= \omega^2\left[\frac{1}{Va}\left(-\frac{J}{12Le} + \frac{1}{\sigma}A_1\right) - \frac{1}{12\sigma}\left(\frac{n}{Va} + 1 + M\right)\right] \\
&\quad + \left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left[\frac{J}{Le}A_1 - \frac{N_AN_B}{Le}\frac{J}{2}J_{T\phi}J\right] - Ra\left[\frac{\alpha^2}{12}\left(-\frac{J}{Le}J_{TW} - \frac{N_B}{\varepsilon Le}J_{\phi W}J_{T\phi}\right)\right] \\
&\quad - Rn\left[\frac{\alpha^2}{6}Rn\left(-\frac{JN_A}{2Le}J_{TW} - \frac{J_{\phi W}}{\varepsilon}A_1\right)\right],
\end{aligned} \tag{43}$$

$$\begin{aligned}
0 &= Ra - Rn\left[\frac{6\varepsilon J_{TW}}{\sigma J_{\phi W}}\right] - \frac{1}{Va\alpha^2 J_{TW}}\omega^2 - \frac{12\sigma}{Va\alpha^2 J_{Tw}}\left[\frac{J}{Le}A_1 - \frac{N_AN_B}{Le}\frac{J}{2}J_{T\phi}\right] \\
&\quad + \frac{12\sigma}{\alpha^2 J_{Tw}}\left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left[-\frac{J}{12Le} + \frac{1}{\sigma}A_1\right].
\end{aligned} \tag{44}$$

In order for ω to be real, it is necessary that

$$\begin{aligned}
Ra - Rn\left[\frac{\sigma J_{\phi W}}{6\varepsilon J_{TW}}\right] &\geq \frac{12\sigma}{Va\alpha^2 J_{Tw}}\left[\frac{J}{Le}A_1 - \frac{N_AN_B}{Le}\frac{J}{2}J_{T\phi}\right] \\
&\quad + \frac{12\sigma}{\alpha^2 J_{Tw}}\left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left[-\frac{J}{12Le} + \frac{1}{\sigma}A_1\right].
\end{aligned} \tag{45}$$

From equations (43) and (44), Ra (independent of ω) and ω (independent of Ra) are written as

$$Ra_{\text{osc}} = \frac{12\sigma}{Va\alpha^2 J_{Tw}}\left[\frac{J}{Le}A_1 - \frac{N_A N_B}{Le}\frac{J}{2}J_{T\phi}\right] + \frac{12\sigma}{\alpha^2 J_{Tw}}\left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left[-\frac{J}{12Le} + \frac{1}{\sigma}A_1\right] + \frac{\omega^2}{Va\alpha^2 J_{Tw}} + Rn\frac{6\varepsilon J_{TW}}{\sigma J_{\phi W}}, \tag{46}$$

$$\omega^2 = \frac{\frac{\sigma}{J_{TW}A_2}\left[\frac{\alpha^2 J_{\phi W}}{72\varepsilon}Rn + \frac{J}{VaLe}A_1 - \frac{N_a N_V J}{2VaLe^2}J_{T\phi} + (1 + M + \frac{36}{5}\widetilde{Da})(-\frac{J}{12Le} + \frac{1}{\sigma}A_1)\right]}{\frac{1}{Va}(-\frac{J}{12Le} + \frac{1}{\sigma}A_1) - \frac{1}{12\sigma}(1 + M + \frac{36}{5}\widetilde{Da}) + \frac{1}{12VaJ_{TW}}A_2}. \tag{47}$$

The critical cell size at the onset of instability is obtained by solving $\frac{\partial Ra}{\partial \alpha} = 0$. The critical value of α for oscillatory convection is calculated from

$$\alpha_c^4\big|_{\text{osc}} = -72Le \times \left[\left(A_1 - \frac{N_A}{Le}\frac{N_B}{Le}J_{T\phi}\right) + Va\left(1 + M + \frac{36}{5}\widetilde{Da}\right)\left(-\frac{1}{6Le} + \frac{1}{\sigma}A_1\right) + \frac{\omega^2}{6\sigma}\right]. \tag{48}$$

The critical Rayleigh number $(Ra_{\text{osc}})_{\text{crit}}$ is obtained using the above critical value of α for oscillatory convection.

4 Results and discussion

The critical Rayleigh numbers for stationary and oscillatory convection are calculated from equations (40) and (46), respectively. The influence of the Biot number, magnetic parameter, Darcy number, porosity of the medium on the stationary Rayleigh number and oscillatory Rayleigh number are shown in Figures 1-6, where we have also shown the critical stationary and oscillatory Rayleigh numbers for different Darcy and Biot numbers. In the present problem, we mainly focused on the influence of Biot number on stationary and oscillatory critical Rayleigh numbers.

The effects of the Biot number, magnetic parameter, Darcy number and porosity on the stationary Rayleigh number are shown in Figure 1, where it is evident that Ra_s increases with the magnetic parameter but decreases with the Biot number. Hence the magnetic parameter exerts a stabilizing influence on the stationary convection regime, but Biot number destabilizes stationary convection. The Darcy number has a stabilizing effect on

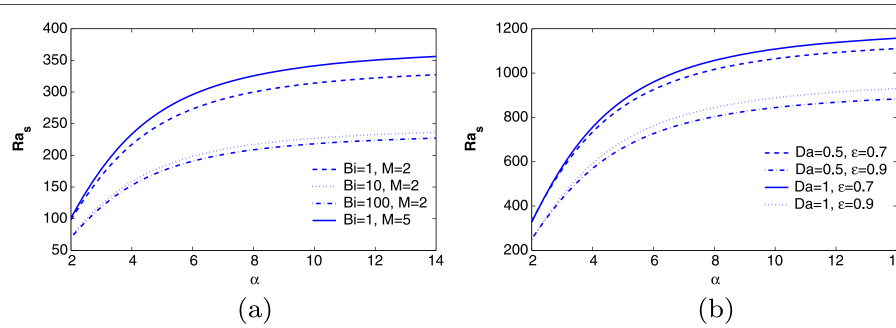


Figure 1 Variation of the stationary Rayleigh number with wave number for different value of (a) *Bi*, *M* (*Da* = 1, *ε* = 0.7) and (b) *Da*, *ε* (*Bi* = 1, *M* = 2) with *Va* = 0.3, *Rn* = –3, N_B = 10, *Le* = 500, N_A = 5, *σ* = 3.

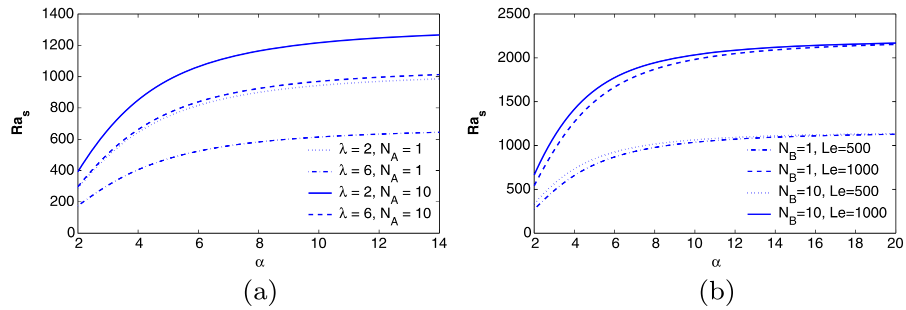


Figure 2 Variation of the stationary Rayleigh number with wave number for different value of (a) λ, N_A (N_B = 10, *Le* = 500) and (b) N_B, *Le* (λ = 2, N_A = 5) with *Va* = 0.3, *Rn* = –3, *Da* = 1, *M* = 2, *ε* = 0.7, N_B = 10, *Le* = 500, N_A = 5, *σ* = 3, *Bi* = 1.

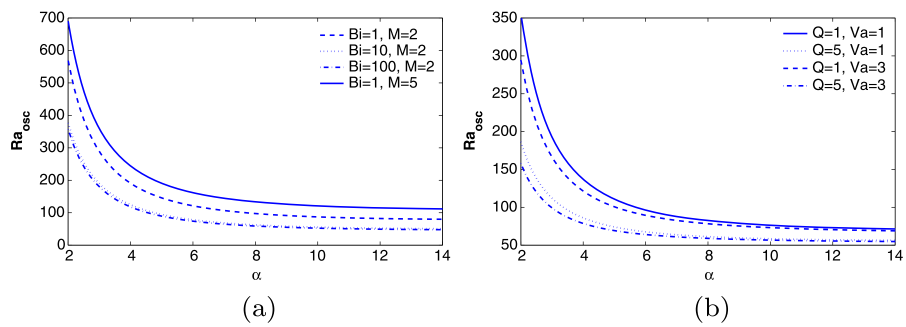


Figure 3 Variation of the oscillatory Rayleigh number with wave number for different value of (a) *Bi*, *M* (*Q* = 1, *Va* = 0.3) and (b) *Q*, *Va* (*Bi* = 1, *M* = 2) with *Rn* = –3, *Da* = 1, N_B = 10, *Le* = 500, N_A = 5, *σ* = 3.

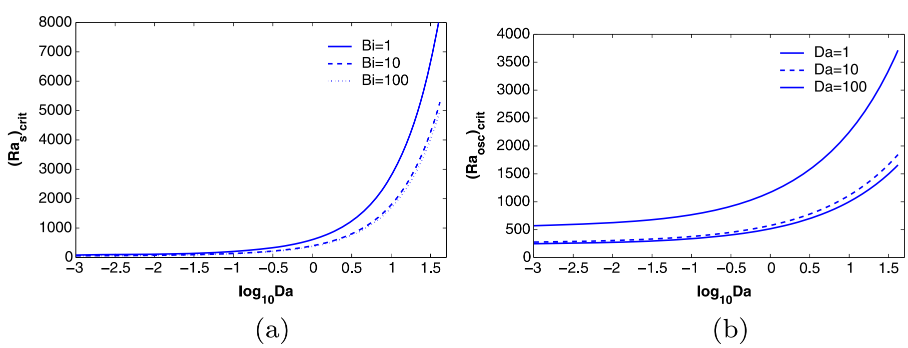


Figure 4 Influence of the Biot number on (a) $(Ra_s)_{\mathrm{crit}}$ and (b) $(Ra_{\mathrm{osc}})_{\mathrm{crit}}$ as a function of *Da* with *Va* = 0.3, *Rn* = –3, *M* = 2, N_B = 10, *Le* = 500, N_A = 5, *σ* = 3, *ε* = 0.7.

stationary convection, while the porosity parameter has a destabilizing influence on the stationary convection for fixed Biot numbers. This finding is in line with the results reported by Chand and Rana [27].

Figure 2 shows the influence of the modified diffusivity ratio λ, the modified particle-density increment and the Lewis number on the stationary Rayleigh number. The stationary Rayleigh number increases with N_A, N_B and the Lewis number, and this helps to stabilize the stationary convection regime. On the other hand, λ destabilizes stationary

Figure 5 Effect of the Biot number on critical α_s as a function of Da with $Va = 0.3$, $Rn = -3$, $M = 2$, $N_B = 10$, $Le = 500$, $N_A = 5$, $\sigma = 3$, $\varepsilon = 0.7$.

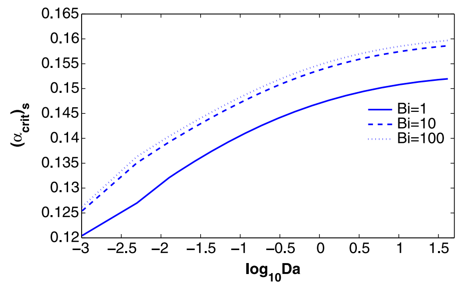
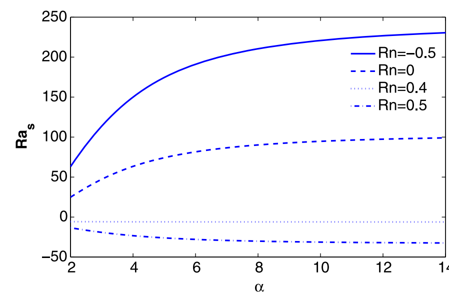


Figure 6 Effect of the Biot number on critical α_s as a function of Da with $Va = 0.3$, $Rn = -3$, $M = 2$, $N_B = 10$, $Le = 500$, $N_A = 5$, $\sigma = 3$, $\varepsilon = 0.7$.



convection. Thus, in the absence of the Darcy-Brinkman and magnetic parameters, the stability boundary depends on N_B and N_A as earlier suggested by Nield and Kuznetsov [26]. The throughflow (due to Peclet number Q) also assists in increasing the critical stationary Rayleigh number.

The influence of the Biot number, magnetic parameter, Peclet number and Vadasz number on the oscillatory Rayleigh number is shown in Figure 3. The critical oscillatory Rayleigh number increases with the magnetic parameter while decreasing with parameters Bi, Va and Q. Hence the magnetic parameter stabilizes the oscillatory convection, while the other parameters are destabilizing to the oscillatory regime.

The critical stationary and oscillatory Rayleigh number as a function of the Darcy number is shown in Figure 4. The critical stationary Rayleigh number decreases with the Biot number. For $Da = 0$, the critical Rayleigh number has a minimum value that depends on the Biot number as shown in Figure 5. For larger Biot numbers (isothermal condition), the critical Rayleigh number is small. The critical value of Ra_s is 64.9325, 44.4513 and 42.3685 for $Bi = 1, 10$ and 100, respectively. The critical Ra_{osc} values, on the other hand, are 562.3439, 286.1277 and 258.3124 for $Bi = 1, 10$ and 100, respectively.

We have defined Rn so that it is positive when the particle density increases upwards (the destabilizing situation). From Figure 6, we note that Ra takes a negative value when Rn is sufficiently large. In this case, the destabilizing effect of nanoparticle concentration is so large that the bottom of the fluid layer must be cooled relative to the top to produce a state of neutral stability as earlier found by Kuznetsov and Nield [21] in the absence of a magnetic field and higher Biot numbers. In the present problem, a state of neutral stability appeared when $Rn = 0.4$.

5 Conclusion

In this study we used linear stability to investigate the onset of thermal instability in a non-Darcy porous medium saturated with a nanofluid, and with a convective boundary condition. We have determined the effects of various embedded parameters such as the Biot number, the magnetic parameter and the Darcy number on the critical Rayleigh number for the onset of both oscillatory and stationary thermal instabilities. We have shown that increasing the Darcy number and the magnetic parameter has the effect of increasing the critical Rayleigh number for the onset of thermal instabilities in the case of stationary convection, while increasing the Biot number and the porosity is destabilizing to the stationary regime. The modified diffusivity ratio, particle density increment and the Lewis number help to stabilize stationary convection. Oscillatory convection was found not to be as sensitive to the fluid and physical parameters as stationary convection.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors were involved in carrying out this study.

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