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Existence and nonexistence of entire positive solutions for (p,q)-Laplacian elliptic system with a gradient term

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Abstract

This work is concerned with the entire positive solutions for a (p,q)-Laplacian elliptic system of equations with a gradient term. We find the sufficient condition for nonexistence of entire large positive solutions and existence of infinitely many entire solutions, which are large or bounded.

1 Introduction

In this paper, we consider a class of (p,q)-Laplacian elliptic system of equations with a gradient term

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + m_1(|x|)|\nabla u|^{p-1} = a(|x|)f(u,v), & \text{in } R^N, \\ \operatorname{div}(|\nabla v|^{p-2}\nabla v) + m_2(|x|)|\nabla v|^{q-1} = b(|x|)g(u,v), & \text{in } R^N, \end{cases} \tag{1.1}$$

where $N > 2$, $p \geq 2$, $q \geq 2$, the nonlinearities $f, g : [0,\infty) \times [0,\infty) \to (0,\infty)$ are positive, continuous and nondecreasing functions for each variable, $m_1(|x|)$ and $m_2(|x|)$ are continuous functions, and the potentials $a, b \in C(R^N)$ are c-positive functions (or circumferentially positive) in a domain $\Omega \subset R^N$ which are nonnegative in Ω and satisfy the following:

- If $x_0 \in \Omega$ and $a(x_0) = 0$, then there exists a domain Ω_0 such that $x_0 \in \Omega_0 \subset \Omega$ and $a(x) > 0$ for all $x \in \partial\Omega_0$.

Problem (1.1) arises in the theory of quasiregular and quasiconformal mappings, stochastic control and non-Newtonian fluids, *etc.* In the non-Newtonian theory, the quantity (p,q) is a characteristic of the medium. Media with $(p,q) > (2,2)$ are called dilatant fluids, while $(p,q) < (2,2)$ are called pseudoplastics. If $(p,q) = (2,2)$, they are Newtonian fluids.

We are concerned only with the entire positive solutions of problem (1.1). An entire large (or explosive) solution of problem (1.1) means a pair of functions $(u,v) \in C^{1+\theta}(R^N) \times C^{1+\theta}(R^N)$ for $\theta \in (0,1)$ solving problem (1.1) in the weak sense and $u(x) \to \infty$, $v(x) \to \infty$ as $|x| \to \infty$.

In recent years, existence and nonexistence of entire solutions for the semilinear elliptic system

$$\begin{cases} \triangle u + f(x,u,v) = 0, & \text{in } R^N, \\ \triangle v + f(x,u,v) = 0, & \text{in } R^N, \end{cases}$$

have been studied by many authors; see [1–3] and the references therein. For example, Ghergu and Radulescu [1], Lair and Wood [2], Kawano and Kusano [3] discussed the entire solutions under proper conditions. For other works for a single equation, we refer to [4, 5] and the references therein. Moreover, a comprehensive discussion on entire solutions for a large class of semilinear systems

$$\begin{cases} \triangle u = a(|x|)f(v), & \text{in } R^N, \\ \triangle v = b(|x|)g(u), & \text{in } R^N \end{cases}$$

can be found in Ghergu and Radulescu [1]. Later, Yang [6] extended their results to a class of quasilinear elliptic systems. To our best knowledge, problem (1.1) of equations with a gradient term has not been sufficiently investigated. Only a few papers have dealt with this problem (1.1). In [7], Ghergu and Radulescu studied the existence of blow-up solutions for the system

$$\begin{cases} \triangle u + |\nabla u| = a(|x|)f(v), & \text{in } \Omega, \\ \triangle v + |\nabla v| = b(|x|)g(u), & \text{in } \Omega. \end{cases}$$

They proved that boundary blow-up solutions fail to exist if f and g are sublinear, whereas this result holds if $\Omega = R^N$ is bounded and a, b are slow decay at infinity. They also showed the existence of infinitely blow-up solutions in $\Omega = R^N$ if a, b are of fast decay and f, g satisfy a sublinear-type growth condition at infinity. In [8], Cirstea and Radulescu studied a related problem. Recently, Zhang and Liu [9] studied the semilinear elliptic systems with a gradient term

$$\begin{cases} \triangle u + |\nabla u| = a(|x|)f(u,v), & \text{in } R^N, \\ \triangle v + |\nabla v| = b(|x|)g(u,v), & \text{in } R^N \end{cases}$$

and obtained the sufficient condition of nonexistence and existence of positive entire solutions. Furthermore, for the single equation with a gradient term, we read [10–12] and the references therein.

Motivated by the results of the above cited papers, we study the nonexistence and existence of positive entire solutions for system (1.1) deeply, and the results of the semilinear systems are extended to the quasilinear ones. In [13], the authors studied the existence and nonexistence of entire large positive solutions of (p,q)-Laplacian system (1.1) with $f(u,v) = \varphi(v)$, $g(u,v) = \psi(u)$. However, they obtained different results under the suitable conditions. In this paper, our main purpose is to establish new results under new conditions for system (1.1). Roughly speaking, we find that the entire large positive solutions fail to exist if f, g are sublinear and a, b have fast decay at infinity, while f, g satisfy some growth conditions at infinity, and a, b are of slow decay or fast decay at infinity, then the system has many infinitely entire solutions, which are large or bounded. Unfortunately, it remains unknown whether an analogous result holds for system (1.1) with different gradient power $m_1(|x|)|\nabla u|^{\alpha}$, $m_2(|x|)|\nabla u|^{\alpha}$ for $0 < \alpha < p-1$.

2 Main results and proof

We now state and prove the main results of this paper.

In order to describe our results conveniently, let us define $\Phi_p(x) = |x|^{p-2}x$, $\Phi_q(x) = |x|^{q-2}x$ and let $\Phi_{p'}(x)$, $\Phi_{q'}(x)$ denote inverse functions of Φ_p and Φ_q, where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$. Moreover, we define

$$
\begin{aligned}
&M_1(t) = e^{\int_0^t m_1(z)\,dz} t^{N-1}, \qquad M_2(t) = e^{\int_0^t m_2(z)\,dz} t^{N-1}, \\
&F(\infty) = \lim_{r\to\infty} F(r), \qquad F(r) = \int_0^r \frac{ds}{\Phi_{p'}(f(s,s)) + \Phi_{q'}(f(s,s))}, \\
&H_p a(\infty) = \lim_{r\to\infty} H_p a(r), \qquad H_p a(r) = \int_0^r \Phi_{p'}\left(M_1^{-1}(t)\int_0^t M_1(s)a(s)\right)ds, \\
&H_q b(\infty) = \lim_{r\to\infty} H_q b(r), \qquad H_q b(r) = \int_0^r \Phi_{q'}\left(M_2^{-1}(t)\int_0^t M_2(s)b(s)\right)ds,
\end{aligned}
$$

where $r > 0$.

Firstly, we give a nonexistence result of a positive entire radial large solution of system (1.1).

Theorem 1 *Suppose that f and g satisfy*

$$
\max\left\{\sup_{s+t\ge 1}\frac{f(s,t)}{(s+t)^{m-1}}, \sup_{s+t\ge 1}\frac{g(s,t)}{(s+t)^{m-1}}\right\} < +\infty, \tag{2.1}
$$

and a, b satisfy the decay conditions

$$
\int_0^\infty \Phi_{p'}\bigl(a(t)\bigr)\,dt < +\infty, \qquad \int_0^\infty \Phi_{q'}\bigl(b(t)\bigr)\,dt < +\infty, \tag{2.2}
$$

where $m = \min\{p,q\}$, then problem (1.1) *has no positive entire radial large solution.*

Proof Our proof is by the method of contradiction. That is, we assume that system (1.1) has the positive entire radial large solution (u,v). From (1.1), we know that

$$
\begin{aligned}
&\bigl(M_1(t)\Phi_p\bigl(u'\bigr)\bigr)' = M_1(t)a(t)f\bigl(u(t),v(t)\bigr), \quad t \ge 0, \\
&\bigl(M_2(t)\Phi_p\bigl(v'\bigr)\bigr)' = M_2(t)b(t)g\bigl(u(t),v(t)\bigr), \quad t \ge 0.
\end{aligned}
$$

Now, we set

$$
U(r) = \max_{0\le t\le r} u(r), \qquad V(r) = \max_{0\le t\le r} v(r).
$$

It is easy to see that (U,V) are positive and nondecreasing functions. Moreover, we have $U \ge u$, $V \ge v$ and $U(r), V(r) \to +\infty$ as $r \to +\infty$. It follows from (2.1) that there exists C_0

such that

$$\max\{f(s,t),g(s,t)\} < C_0(s+t)^{m-1}, \quad s+t \geq 1 \tag{2.3}$$

and

$$\max\{f(s,t),g(s,t)\} < C_0, \quad s+t \leq 1. \tag{2.4}$$

Combining (2.3) and (2.4), we can get

$$\max\{f(s,t),g(s,t)\} < C_0(1+s+t)^{m-1}, \quad s+t \geq 0. \tag{2.5}$$

Then we have

$$\begin{aligned} f\big(u(r),v(r)\big) &\leq C_0\big(1+u(r)+u(r)\big)^{m-1} \\ &\leq C_0\big(1+U(r)+V(r)\big)^{m-1}, \quad r \geq 0. \end{aligned}$$

Thus, for all $r \geq r_0 \geq 0$, we obtain

$$\begin{aligned} u(r) &= u(r_0) + \int_{r_0}^{r} \Phi_{p'}\left(M_1^{-1}(t)\int_0^t M_1(s)a(s)f\big(u(s),v(s)\big)\,ds\right)dt \\ &\leq u(r_0) + C\int_{r_0}^{r} \Phi_{p'}\left(M_1^{-1}(t)\int_0^t M_1(s)a(s)\big(1+U(s)+V(s)\big)^{m-1}\,ds\right)dt \\ &\leq u(r_0) + C\Phi_{p'}\left(\big(1+U(r)+V(r)\big)^{m-1}\right)\int_{r_0}^{r} \Phi_{p'}\left(M_1^{-1}(t)\int_0^t M_1(s)a(s)\,ds\right)dt \\ &\leq u(r_0) + C\big(1+U(r)+V(r)\big)\int_{r_0}^{r} \Phi_{p'}\big(a(t)\big)\,dt, \end{aligned}$$

where C is a positive constant. Because of $m = \min\{p,q\}$, the last inequality above is valid for $0 < m-1 < p-1$. Noticing that (2.2), we choose $r_0 > 0$ such that

$$\max\left\{\int_{r_0}^{\infty} \Phi_{p'}\big(a(r)\big)\,dr, \int_{r_0}^{\infty} \Phi_{q'}\big(b(r)\big)\,dr\right\} < \frac{1}{4C}. \tag{2.6}$$

It follows that $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$, and we can find $r_1 > r_0$ such that

$$\overline{U}(r) = \max_{r_0 \leq t \leq r} u(t), \qquad \overline{V}(r) = \max_{r_0 \leq t \leq r} v(t), \quad r \geq r_1. \tag{2.7}$$

Thus, we have

$$\overline{U}(r) \leq u(r_0) + C\big(1+\overline{U}(r)+\overline{V}(r)\big)\int_{r_0}^{r} \Phi_{p'}\big(a(t)\big)\,dt, \quad r \geq r_1.$$

From (2.6), we can get

$$\overline{U}(r) \leq u(r_0) + \frac{(1+\overline{U}(r)+\overline{V}(r))}{4}, \quad r \geq r_1,$$

that is,

$$\overline{U}(r) \le C_1 + \frac{(\overline{U}(r) + \overline{V}(r))}{4}, \quad r \ge r_1,$$

where $C_1 = C + \frac{1}{4} + u(r_0)$, $r \ge r_1$. Similarly,

$$\overline{V}(r) \le C_2 + \frac{(\overline{U}(r) + \overline{V}(r))}{4}, \quad r \ge r_1,$$

then we can get

$$\overline{U}(r) + \overline{V}(r) \le 2(C_1 + C_2), \quad r \ge r_1, \tag{2.8}$$

which means that U and V are bounded and so u and v are bounded, which is a contradiction. It follows that (1.1) has no positive entire radial large solutions. □

Remark 1 In fact, through a slight change of the proofs of Theorem 1, we can obtain the same result as that of problem (1.1). That is, if $p, q > 2$ and f, g satisfy

$$\max\left\{\sup_{s+t\ge 1}\frac{f(s,t)}{s+t}, \sup_{s+t\ge 1}\frac{g(s,t)}{s+t}\right\} < +\infty,$$

and a, b satisfy the decay conditions (2.2), then problem (1.1) still has no positive entire radial large solution.

Secondly, we give existence results of positive entire solutions of system (1.1).

Theorem 2 *Suppose that*

$$F(\infty) = \infty.$$

Then system (1.1) *has infinitely many positive entire solutions* $(u, v) \in C^2[0, +\infty)$. *Moreover, the following hold*:

(i) *If a and b satisfy* $H_p a(\infty) = H_q b(\infty) = \infty$, *then all entire positive solutions of* (1.1) *are large*.

(ii) *If a and b satisfy* $H_p a(\infty) < \infty$, $H_q b(\infty) < \infty$, *then all entire positive solutions of* (1.1) *are bounded*.

Proof We start by showing that (1.1) has positive radial solutions. To this end, we fix $c, d > \beta$ and show that the system

$$\begin{cases} (|u'|^{p-2}u')' + \frac{N-1}{r}(|u'|^{p-2}u') + m_1(r)|u'|^{p-1} = a(r)f(u,v), \\ (|v'|^{q-2}v')' + \frac{N-1}{r}(|v'|^{q-2}v') + m_2(r)|v'|^{q-1} = b(r)g(u,v), \\ u', v' \ge 0 \quad \text{on } [0,\infty), \\ u(0) = d > 0, \qquad v(0) = c > 0, \end{cases} \tag{2.9}$$

has solutions (u,v). Thus, $U(x)=u(|x|)$, $V(x)=v(|x|)$ are positive solutions of system (1.1). Integrating (2.9), we have

$$u(r)=d+\int_0^r \Phi_{p'}\left(M_1^{-1}(t)\int_0^t M_1(s)a(s)f(u,v)\,ds\right)dt,\quad r\ge 0,$$
$$v(r)=c+\int_0^r \Phi_{q'}\left(M_2^{-1}(t)\int_0^t M_2(s)b(s)g(u,v)\,ds\right)dt,\quad r\ge 0.$$

Let $\{u_k\}_{k\ge 0}$ and $\{v_k\}_{k\ge 0}$ be sequences of positive continuous functions defined on $[0,\infty)$ by

$$\begin{cases} u_0(r)=d, \qquad v_0(r)=c, \\ u_{k+1}(r)=d+\int_0^r \Phi_{p'}(M_1^{-1}(t)\int_0^t M_1(s)a(s)f(u_k,v_k)\,ds)\,dt, \quad r\ge 0, \\ v_{k+1}(r)=c+\int_0^r \Phi_{q'}(M_2^{-1}(t)\int_0^t M_2(s)b(s)g(u_k,v_k)\,ds)\,dt, \quad r\ge 0. \end{cases}$$

Obviously, $u_k(r)\ge c$, $v_k(r)\ge d$, $u_0\le u_1$, $v_0\le v_1$ for all $r\ge 0$. And the monotonicity of f and g yields $u_1(r)\le u_2(r)$, $v_1(r)\le v_2(r)$ for $r\ge 0$.

Repeating such arguments, we can deduce that

$$u_k(r)\le u_{k+1}(r),\qquad v_k(r)\le v_{k+1}(r),\quad \text{for } r\ge 0, k\ge 1,$$

and $\{u_k\}_{k\ge 0}$, $\{v_k\}_{k\ge 0}$ are nondecreasing sequences on $[0,\infty)$. Noticing that

$$\begin{aligned} u'_{k+1}(r) &= \Phi_{p'}\left(M_1^{-1}(r)\int_0^r M_1(s)a(s)f\bigl(u_k(r),v_k(r)\bigr)\,ds\right) \\ &\le \Phi_{p'}\bigl(f\bigl(u_k(r),v_k(r)\bigr)\bigr)\Phi_{p'}\left(M_1^{-1}(r)\int_0^r M_1(s)a(s)\,ds\right) \\ &\le \bigl[\Phi_{p'}\bigl(f(u_{k+1}+v_{k+1},u_{k+1}+v_{k+1})\bigr)+\Phi_{q'}\bigl(g(u_{k+1}+v_{k+1},u_{k+1}+v_{k+1})\bigr)\bigr] \\ &\quad\times \Phi_{p'}\left(M_1^{-1}(r)\int_0^r M_1(s)a(s)\,ds\right), \end{aligned}$$

and

$$\begin{aligned} v'_{k+1}(r) &= \Phi_{q'}\left(M_2^{-1}(r)\int_0^r M_2(s)b(s)g\bigl(u_k(r),v_k(r)\bigr)\,ds\right) \\ &\le \Phi_{q'}\bigl(g\bigl(u_k(r),v_k(r)\bigr)\bigr)\Phi_{q'}\left(M_2^{-1}(r)\int_0^r M_2(s)b(s)\,ds\right) \\ &\le \bigl[\Phi_{p'}\bigl(f(u_{k+1}+v_{k+1},u_{k+1}+v_{k+1})\bigr)+\Phi_{q'}\bigl(g(u_{k+1}+v_{k+1},u_{k+1}+v_{k+1})\bigr)\bigr] \\ &\quad\times \Phi_{q'}\left(M_2^{-1}(r)\int_0^r M_2(s)b(s)\,ds\right), \end{aligned}$$

it follows that

$$\begin{aligned} &\frac{u'_{k+1}(r)+v'_{k+1}(r)}{\Phi_{p'}(f(u_{k+1}+v_{k+1},u_{k+1}+v_{k+1}))+\Phi_{q'}(g(u_{k+1}+v_{k+1},u_{k+1}+v_{k+1}))} \\ &\quad\le \Phi_{p'}\left(M_1^{-1}(r)\int_0^r M_1(s)a(s)\,ds\right)+\Phi_{q'}\left(M_2^{-1}(r)\int_0^r M_2(s)b(s)\,ds\right). \end{aligned}$$

Then we can get

$$\int_0^r \frac{u'_{k+1}(r) + v'_{k+1}(r)}{\Phi_{p'}(f(u_{k+1} + v_{k+1}, u_{k+1} + v_{k+1})) + \Phi_{q'}(g(u_{k+1} + v_{k+1}, u_{k+1} + v_{k+1}))}\,dr \\ \leq H_p a(r) + H_q b(r),$$

that is,

$$F\bigl(u_k(r) + v_k(r)\bigr) - F(b + c) \leq H_p a(r) + H_q b(r), \quad r \geq 0. \tag{2.10}$$

It follows from F^{-1} is increasing on $[0, \infty)$ and (2.10) that

$$u_k(r) + v_k(r) \leq F^{-1}\bigl(H_p a(r) + H_q b(r) + F(b + c)\bigr), \quad r \geq 0. \tag{2.11}$$

And from $F(\infty) = \infty$, we know that $F^{-1}(\infty) = \infty$. By (2.11), the sequences u_k and v_k are bounded and increasing on $[0, c_0]$ for any $c_0 > 0$. Thus, u_k and v_k have subsequences converging uniformly to u and v on $[0, c_0]$. Consequently, (u, v) is a positive solution of (2.9); therefore, (U, V) is an entire positive solution of (1.1). Noticing that $U(0) = c$, $V(0) = d$ and $(c, d) \in (0, \infty) \times (0, \infty)$ are chosen arbitrarily, we can obtain that system (1.1) has infinitely many positive entire solutions.

(i) If $H_p a(\infty) = H_q b(\infty) = \infty$, since $u(r) \geq c + \Phi_{p'}(f(c, d))H_p a(r)$, $v(r) \geq d + \Phi_{q'}(g(c, d))H_q b(r)$ for $r \geq 0$, we have

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty,$$

which yields (U, V) is the positive entire large solution of (1.1).

(ii) If $H_p a(\infty) < \infty$, $H_q b(\infty) < \infty$, then

$$u(r) + v(r) \leq F^{-1}\bigl(F(b + c) + H_p a(\infty) + H_q b(\infty)\bigr) < \infty,$$

which implies that (U, V) is the positive entire bounded solution of system (1.1).

Thus, the proof of Theorem 2 is finished. □

Theorem 3 *If* $F(\infty) < \infty$, $H_p a(\infty) < \infty$, $H_q b(\infty) < \infty$, *and there exist* $c > \beta$, $d > \beta$ *such that*

$$H_p a(\infty) + H_q b(\infty) < F(\infty) - F(c + d), \tag{2.12}$$

then system (1.1) *has an entire positive radial bounded solution* $(u, v) \in C^{1+\theta}([0, \infty)) \times C^{1+\theta}([0, \infty))$ (*for* $0 < \theta < 1$) *satisfying*

$$c + \Phi_{p'}\bigl(f(c, d)\bigr)H_p a(r) \leq u(r) \leq F^{-1}\bigl(F(b + c) + H_p a(r) + H_q b(r)\bigr), \\ r \geq 0,$$
$$d + \Phi_{q'}\bigl(g(c, d)\bigr)H_q b(r) \leq v(r) \leq F^{-1}\bigl(F(b + c) + H_p a(r) + H_q b(r)\bigr), \\ r \geq 0.$$

Proof If the condition (2.12) holds, then we have

$$\begin{aligned} F\big(u_k + v_k(r)\big) &\le F(c+d) + H_p a(r) + H_q b(r) \\ &\le F(c+d) + H_p a(\infty) + H_q b(\infty) < F(\infty) < \infty. \end{aligned}$$

Since F^{-1} is strictly increasing on $[0,\infty)$, we have

$$u_k + v_k(r) \le F^{-1}\big(F(b+c) + H_p a(\infty) + H_q b(\infty)\big) < \infty.$$

The rest of the proof obviously holds from the proof of Theorem 2. The proof of Theorem 3 is now finished. □

Theorem 4

(i) *If*

$$H_p a(\infty) = H_q b(\infty) = \infty,$$

and

$$\lim_{s\to\infty} \frac{f(s,s) + g(s,s)}{s} = 0, \tag{2.13}$$

then system (1.1) *has infinitely many positive entire large solutions.*

(ii) *If* $H_p a(\infty) < \infty$, $H_q b(\infty) < \infty$, *and*

$$\sup_{s\ge 0}\big(f(s,s) + g(s,s)\big) < \infty,$$

then system (1.1) *has infinitely many positive entire bounded solutions.*

Proof (i) It follows from the proof of Theorem 2 that

$$\begin{aligned} u_k(r) \le u_{k+1}(r) &\le \Phi_{p'}\big(f\big(u_k(r), v_k(r)\big)\big) H_p a(r) \\ &\le \Phi_{p'}\big(f\big(u_k(r) + v_k(r), u_k(r) + v_k(r)\big)\big) H_p a(r) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} v_k(r) \le v_{k+1}(r) &\le \Phi_{q'}\big(g\big(u_k(r), v_k(r)\big)\big) H_q b(r) \\ &\le \Phi_{q'}\big(g\big(u_k(r) + v_k(r), u_k(r) + v_k(r)\big)\big) H_q b(r). \end{aligned} \tag{2.15}$$

Choosing an arbitrary $R > 0$, from (2.14) and (2.15), we can get

$$\begin{aligned} u_k(R) + v_k(R) &\le c + d + \Phi_{p'}\big(f\big(u_k(r) + v_k(r), u_k(r) + v_k(r)\big)\big) H_p a(r) \\ &\quad + \Phi_{q'}\big(g\big(u_k(r) + v_k(r), u_k(r) + v_k(r)\big)\big) H_q b(r) \\ &\le c + d + \big[\Phi_{p'}\big(f\big(u_k(r) + v_k(r), u_k(r) + v_k(r)\big)\big) + \Phi_{q'}\big(g\big(u_k(r) \\ &\quad + v_k(r), u_k(r) + v_k(r)\big)\big)\big]\big(H_p a(r) + H_q b(r)\big), \quad k \ge 1, \end{aligned}$$

which implies

$$1 \le \frac{c+d}{u_k(R)+v_k(R)} + \frac{\Phi_{p'}(f(u_k(r)+v_k(r),u_k(r)+v_k(r))) + \Phi_{q'}(g(u_k(r)+v_k(r),u_k(r)+v_k(r)))}{u_k(R)+v_k(R)} \times \bigl(H_p a(r) + H_q b(r)\bigr), \quad k \ge 1. \tag{2.16}$$

Taking account of the monotonicity of $(u_k(R)+v_k(R))_{k\ge 1}$, there exists

$$L(R) = \lim_{k\to\infty} \bigl(u_k(R)+v_k(R)\bigr).$$

We claim that $L(R)$ is finite. Indeed, if not, we let $k \to \infty$ in (2.16) and the assumption (2.13) leads to a contradiction. Thus, $L(R)$ is finite. Since u_k, v_k are increasing functions, it follows that the map $L : (0,\infty) \to (0,\infty)$ is nondecreasing and

$$u_k(r)+v_k(r) \le u_k(R)+v_k(R) \le L(R), \quad r \in [0,R], k \ge 1.$$

Thus, the sequences $(u_k)_{k\ge 1}$ and $(v_k)_{k\ge 1}$ are bounded from above on bounded sets. Let

$$u(r) = \lim_{k\to\infty} u_k(r), \qquad v(r) = \lim_{k\to\infty} v_k(r), \quad r \ge 0,$$

then (u,v) is a positive solution of (2.9).

In order to conclude the proof, we need to show that (u,v) is a large solution of (2.9). By the proof of Theorem 2, we have

$$u(r) \ge c + \Phi_{p'}\bigl(f(c,d)\bigr)H_p a(r), \qquad v(r) \ge d + \Phi_{q'}\bigl(g(c,d)\bigr)H_q b(r), \quad r \ge 0.$$

And because f and g are positive functions and

$$H_p a(\infty) = H_q b(\infty) = \infty,$$

we can conclude that (u,v) is a large solution of (2.9) and so (U,V) is a positive entire large solution of (1.1). Thus, any large solution of (2.9) provides a positive entire large solution (U,V) of (1.1) with $U(0)=c$ and $V(0)=d$. Since $(c,d) \in (0,\infty)\times(0,\infty)$ was chosen arbitrarily, it follows that (1.1) has infinitely many positive entire large solutions.

(ii) If

$$\sup_{s\ge 0}\bigl(f(s,s)+g(s,s)\bigr) < \infty,$$

holds, then by (2.16), we have

$$L(R) = \lim_{k\to\infty} \bigl(u_k(R)+v_k(R)\bigr) < \infty.$$

Thus,

$$u_k(r)+v_k(r) \le u_k(R)+v_k(R) \le L(R), \quad r \in [0,R], k \ge 1.$$

Thus, the sequences $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ are bounded from above on bounded sets. Let

$$u(r) = \lim_{k\to\infty} u_k(r), \qquad v(r) = \lim_{k\to\infty} v_k(r), \quad r \geq 0,$$

then (u, v) is a positive solution of (2.9).

It follows from (2.14) and (2.15) that (u, v) is bounded, which implies that (1.1) has infinitely many positive entire bounded solutions. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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