Jiang *Boundary Value Problems* 2013, **2013**:207
http://www.boundaryvalueproblems.com/content/2013/1/207

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Solvability for p-Laplacian boundary value problem at resonance on the half-line

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Abstract
The existence of solutions for p-Laplacian boundary value problem at resonance on the half-line is investigated. Our analysis relies on constructing the suitable Banach space, defining appropriate operators and using the extension of Mawhin's continuation theorem. An example is given to illustrate our main result.
MSC: 70K30; 34B10; 34B15

Keywords: p-Laplacian; resonance; half-line; multi-point boundary value problem; continuation theorem

1 Introduction

A boundary value problem is said to be a resonance one if the corresponding homogeneous boundary value problem has a non-trivial solution. Resonance problems can be expressed as an abstract equation $Lx = Nx$, where L is a noninvertible operator. When L is linear, Mawhin's continuation theorem [1] is an effective tool in finding solutions for these problems, see [2–10] and references cited therein. But it does not work when L is nonlinear, for instance, p-Laplacian operator. In order to solve this problem, Ge and Ren [11] proved a continuation theorem for the abstract equation $Lx = Nx$ when L is a noninvertible nonlinear operator and used it to study the existence of solutions for the boundary value problems with a p-Laplacian:

$$\begin{cases} (\varphi_p(u'))' + f(t,u) = 0, & 0 < t < 1, \\ u(0) = 0 = G(u(\eta), u(1)), \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \eta < 1$. $\varphi_p(s)$ is nonlinear when $p \neq 2$.

As far as the boundary value problems on unbounded domain are concerned, there are many excellent results, see [12–15] and references cited therein.

To the best of our knowledge, there are few papers that study the p-Laplacian boundary value problem at resonance on the half-line. In this paper, we investigate the existence of solutions for the boundary value problem

$$\begin{cases} (\varphi_p(u'))' + f(t,u,u') = 0, & 0 < t < +\infty, \\ u(0) = 0, \qquad \varphi_p(u'(+\infty)) = \sum_{i=1}^{n} \alpha_i \varphi_p(u'(\xi_i)), \end{cases} \tag{1.1}$$

where $\alpha_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$.

In order to obtain our main results, we always suppose that the following conditions hold.

(H_1) $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$.
(H_2) $f : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(t,0,0) \not\equiv 0$, $t \in (0,\infty)$ and for any $r > 0$, there exists a nonnegative function $h_r(t) \in L^1[0,+\infty)$ such that

$$\left|f(t,x,y)\right| \leq h_r(t), \quad \text{a.e. } t \in [0,+\infty), x, y \in \mathbb{R}, \frac{|x|}{1+t} \leq r, |y| \leq r.$$

2 Preliminaries

For convenience, we introduce some notations and a theorem. For more details, see [11].

Definition 2.1 [11] Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Y$ is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y,
(ii) $\operatorname{Ker} M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$, where $\operatorname{dom} M$ denote the domain of the operator M.

Let $X_1 = \operatorname{Ker} M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. On the other hand, suppose that Y_1 is a subspace of Y, and that Y_2 is the complement of Y_1 in Y, *i.e.*, $Y = Y_1 \oplus Y_2$. Let $P : X \to X_1$ and $Q : Y \to Y_1$ be two projectors and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.2 [11] Suppose that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be M-compact in $\overline{\Omega}$ if there exist a vector subspace Y_1 of Y satisfying $\dim Y_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

(a) $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$,
(b) $QN_\lambda x = \theta, \lambda \in (0,1) \Leftrightarrow QNx = \theta$,
(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$,
(d) $M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda$.

Theorem 2.1 [11] *Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, and $\Omega \subset X$ an open and bounded nonempty set. Suppose that*

$$M : X \cap \operatorname{dom} M \to Y$$

is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ M-compact. In addition, if the following conditions hold:

(C_1) $Mx \neq N_\lambda x$, $\forall x \in \partial\Omega \cap \operatorname{dom} M$, $\lambda \in (0,1)$,
(C_2) $\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$,

then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$, *where $N = N_1$, $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(\theta) = \theta$.*

3 Main result

Let $X = \{u | u \in C^1[0,+\infty), u(0) = 0, \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t} < +\infty, \lim_{t\to+\infty} u'(t) \text{ exists}\}$ with norm $\|u\| = \max\{\|\frac{u}{1+t}\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|$. $Y = L^1[0,+\infty)$ with norm $\|y\|_1 = \int_0^{+\infty} |y(t)|\,dt$. Then $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ are Banach spaces.

Define operators $M : X \cap \operatorname{dom} M \to Y$ and $N_\lambda : X \to Y$ as follows:

$$Mu = \big(\varphi_p(u')\big)', \qquad N_\lambda u = -\lambda f\big(t, u, u'\big), \quad \lambda \in [0,1], t \in [0,+\infty),$$

where

$$\operatorname{dom} M = \Bigg\{u \in X \Bigg| \varphi_p\big(u'\big) \in AC[0,+\infty), \big(\varphi_p\big(u'\big)\big)' \in L^1[0,+\infty), \\ \varphi_p\big(u'(+\infty)\big) = \sum_{i=1}^n \alpha_i \varphi_p\big(u'(\xi_i)\big)\Bigg\}.$$

Then the boundary value problem (1.1) is equivalent to $Mu = Nu$.

Obviously,

$$\operatorname{Ker} M = \{at | a \in \mathbb{R}\}, \qquad \operatorname{Im} M = \Bigg\{y \Bigg| y \in Y, \sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} y(s)\,ds = 0\Bigg\}.$$

It is clear that $\operatorname{Ker} M$ is linearly homeomorphic to $\mathbb{R}$, and $\operatorname{Im} M \subset Y$ is closed. So, M is a quasi-linear operator.

Define $P : X \to X_1$, $Q : Y \to Y_1$ as

$$(Pu)(t) = u'(+\infty)t, \qquad (Qy)(t) = \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} y(s)\,ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-t},$$

where $X_1 = \operatorname{Ker} M$, $Y_1 = \operatorname{Im} Q = \{be^{-t} | b \in \mathbb{R}\}$. We can easily obtain that $P : X \to X_1$, $Q : Y \to Y_1$ are projectors. Set $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$.

Define an operator $R : X \times [0,1] \to X_2$:

$$R(u,\lambda)(t) = \int_0^t \varphi_q\Bigg[\int_\tau^{+\infty} \lambda\Bigg(f\big(s, u(s), u'(s)\big) - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r, u(r), u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\Bigg)\,ds \\ + \varphi_p\big(u'(+\infty)\big)\Bigg]\,d\tau - u'(+\infty)t,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi_q = \varphi_p^{-1}$. By (H_1) and (H_2), we get that $R : X \times [0,1] \to X_2$ is continuous.

Lemma 3.1 [15] *$V \subset X$ is compact if $\{\frac{u(t)}{1+t} | u \in V\}$ and $\{u'(t) | u \in V\}$ are both equicontinuous on any compact intervals of $[0,+\infty)$ and equiconvergent at infinity.*

Lemma 3.2 *$R : X \times [0,1] \to X_2$ is compact.*

Proof Let $\Omega \subset X$ be nonempty and bounded. There exists a constant $r > 0$ such that $\|u\| \le r$, $u \in \overline{\Omega}$. It follows from (H_2) that there exists a nonnegative function $h_r(t) \in L^1[0,+\infty)$

such that

$$\bigl|f\bigl(t,u(t),u'(t)\bigr)\bigr| \le h_r(t), \quad \text{a.e. } t\in[0,+\infty), u\in\overline{\Omega}.$$

For any $T>0$, $t_1,t_2\in[0,T]$, $u\in\overline{\Omega}$, $\lambda\in[0,1]$, we have

$$\begin{aligned}
&\left|\frac{R(u,\lambda)(t_1)}{1+t_1}-\frac{R(u,\lambda)(t_2)}{1+t_2}\right| \\
&\quad\le \left|\frac{1}{1+t_1}\int_0^{t_1}\varphi_q\biggl[\int_\tau^{+\infty}\lambda\biggl(f\bigl(s,u(s),u'(s)\bigr)-\frac{\sum_{i=1}^n\alpha_i\int_{\xi_i}^{+\infty}f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}e^{-s}\biggr)\,ds\right. \\
&\qquad+\varphi_p\bigl(u'(+\infty)\bigr)\biggr]\,d\tau \\
&\qquad-\frac{1}{1+t_2}\int_0^{t_2}\varphi_q\biggl[\int_\tau^{+\infty}\lambda\biggl(f\bigl(s,u(s),u'(s)\bigr)-\frac{\sum_{i=1}^n\alpha_i\int_{\xi_i}^{+\infty}f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}e^{-s}\biggr)\,ds \\
&\qquad\left.+\varphi_p\bigl(u'(+\infty)\bigr)\biggr]\,d\tau\right|+\left|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\right|\bigl|u'(+\infty)\bigr| \\
&\quad\le \left|\frac{1}{1+t_1}\int_{t_2}^{t_1}\varphi_q\biggl[\int_\tau^{+\infty}\lambda\biggl(f\bigl(s,u(s),u'(s)\bigr)-\frac{\sum_{i=1}^n\alpha_i\int_{\xi_i}^{+\infty}f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}e^{-s}\biggr)\,ds\right. \\
&\qquad\left.+\varphi_p\bigl(u'(+\infty)\bigr)\biggr]\,d\tau\right|+\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right| \\
&\qquad\times\left|\int_0^{t_2}\varphi_q\biggl[\int_\tau^{+\infty}\lambda\biggl(f\bigl(s,u(s),u'(s)\bigr)-\frac{\sum_{i=1}^n\alpha_i\int_{\xi_i}^{+\infty}f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}e^{-s}\biggr)\,ds\right. \\
&\qquad\left.+\varphi_p\bigl(u'(+\infty)\bigr)\biggr]\,d\tau\right|+\left|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\right|r \\
&\quad\le \varphi_q\biggl[\|h_r\|_1\biggl(1+\frac{1}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}\biggr)+\varphi_p(r)\biggr]\biggl[|t_1-t_2|+T\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|\biggr] \\
&\qquad+\left|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\right|r.
\end{aligned}$$

Since $\{t,\frac{1}{1+t},\frac{t}{1+t}\}$ are equicontinuous on $[0,T]$, we get that $\{\frac{R(u,\lambda)(t)}{1+t},u\in\overline{\Omega}\}$ are equicontinuous on $[0,T]$.

$$\begin{aligned}
&\bigl|R(u,\lambda)'(t_1)-R(u,\lambda)'(t_2)\bigr| \\
&\quad=\left|\varphi_q\biggl[\int_{t_1}^{+\infty}\lambda\biggl(f\bigl(s,u(s),u'(s)\bigr)-\frac{\sum_{i=1}^n\alpha_i\int_{\xi_i}^{+\infty}f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}e^{-s}\biggr)\,ds\right. \\
&\qquad+\varphi_p\bigl(u'(+\infty)\bigr)\biggr] \\
&\qquad-\varphi_q\biggl[\int_{t_2}^{+\infty}\lambda\biggl(f\bigl(s,u(s),u'(s)\bigr)-\frac{\sum_{i=1}^n\alpha_i\int_{\xi_i}^{+\infty}f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n\alpha_i e^{-\xi_i}}e^{-s}\biggr)\,ds \\
&\qquad\left.+\varphi_p\bigl(u'(+\infty)\bigr)\biggr]\right|.
\end{aligned}$$

Let

$$g(t,u) = \int_t^{+\infty} \lambda\left(f\big(s,u(s),u'(s)\big) - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\right) ds + \varphi_p\big(u'(+\infty)\big).$$

Then

$$\big|g(t,u)\big| \leq \|h_r\|_1 \left(1 + \frac{1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}\right) + \varphi_p(r) := k, \quad t \in [0,T], u \in \overline{\Omega}. \tag{3.1}$$

For $t_1, t_2 \in [0,T]$, $t_1 < t_2$, $u \in \overline{\Omega}$, we have

$$\begin{aligned}\big|g(t_1,u) - g(t_2,u)\big| &= \left|\int_{t_1}^{t_2} \lambda\left(f\big(s,u(s),u'(s)\big) - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\right) ds\right| \\ &\leq \int_{t_1}^{t_2} h_r(s) + \frac{\|h_r\|_1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\,ds.\end{aligned}$$

It follows from the absolute continuity of integral that $\{g(t,u), u \in \overline{\Omega}\}$ are equicontinuous on $[0,T]$. Since $\varphi_q(x)$ is uniformly continuous on $[-k,k]$, by (3.1), we can obtain that $\{R(u,\lambda)'(t), u \in \overline{\Omega}\}$ are equicontinuous on $[0,T]$.

For $u \in \overline{\Omega}$, since

$$\begin{aligned}&\left|\int_\tau^{+\infty} \lambda\left(f\big(s,u(s),u'(s)\big) - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\right) ds\right| \\ &\quad\leq \int_\tau^{+\infty} h_r(s) + \frac{\|h_r\|_1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\,ds, \\ &\lim_{\tau\to+\infty} \int_\tau^{+\infty} h_r(s) + \frac{\|h_r\|_1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\,ds = 0,\end{aligned}$$

and $\varphi_q(x)$ is uniformly continuous on $[-r-r^{p-1}, r+r^{p-1}]$, for any $\varepsilon > 0$, there exists a constant $T_1 > 0$ such that if $\tau \geq T_1$, then

$$\begin{aligned}&\left|\varphi_q\left[\int_\tau^{+\infty} \lambda\left(f\big(s,u(s),u'(s)\big) - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\right) ds + \varphi_p\big(u'(+\infty)\big)\right]\right. \\ &\quad \left. - u'(+\infty)\right| < \frac{\varepsilon}{4}, \quad \forall u \in \overline{\Omega}.\end{aligned} \tag{3.2}$$

Since

$$\begin{aligned}&\left|\int_0^{T_1} \varphi_q\left[\int_\tau^{+\infty} \lambda\left(f\big(s,u(s),u'(s)\big) - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s}\right) ds\right.\right. \\ &\quad \left.\left. + \varphi_p\big(u'(+\infty)\big)\right] d\tau - u'(+\infty)T_1\right| \\ &\quad \leq \left\{\varphi_q\left[\|h_r\|_1 \left(1 + \frac{1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}\right) + \varphi_p(r)\right] + r\right\} T_1,\end{aligned} \tag{3.3}$$

there exists a constant $T > T_1$ such that if $t > T$, then

$$\frac{1}{1+t}\left\{\varphi_q\left[\|h_r\|_1\left(1+\frac{1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}\right)+\varphi_p(r)\right]+r\right\}T_1 < \frac{\varepsilon}{4}. \tag{3.4}$$

For $t_2 > t_1 > T$, by (3.2), (3.3) and (3.4), we have

$$\begin{aligned}
&\left|\frac{R(u,\lambda)(t_1)}{1+t_1}-\frac{R(u,\lambda)(t_2)}{1+t_2}\right| \\
&\quad= \left|\frac{1}{1+t_1}\int_0^{t_1}\left\{\varphi_q\left[\int_\tau^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right.\right. \\
&\qquad\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right\}d\tau \\
&\qquad-\frac{1}{1+t_2}\int_0^{t_2}\left\{\varphi_q\left[\int_\tau^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right. \\
&\qquad\left.\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right\}d\tau\right| \\
&\quad\le \left|\frac{1}{1+t_1}\int_0^{T_1}\left\{\varphi_q\left[\int_\tau^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right.\right. \\
&\qquad\left.\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right\}d\tau\right| \\
&\qquad+\left|\frac{1}{1+t_1}\int_{T_1}^{t_1}\left\{\varphi_q\left[\int_\tau^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right.\right. \\
&\qquad\left.\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right\}d\tau\right| \\
&\qquad+\left|\frac{1}{1+t_2}\int_0^{T_1}\left\{\varphi_q\left[\int_\tau^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right.\right. \\
&\qquad\left.\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right\}d\tau\right| \\
&\qquad+\left|\frac{1}{1+t_2}\int_{T_1}^{t_2}\left\{\varphi_q\left[\int_\tau^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right.\right. \\
&\qquad\left.\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right\}d\tau\right| < \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\left|R(u,\lambda)'(t_1)-R(u,\lambda)'(t_2)\right| \\
&\quad\le \left|\varphi_q\left[\int_{t_1}^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right. \\
&\qquad\left.\left.+\varphi_p\big(u'(+\infty)\big)\right]-u'(+\infty)\right| \\
&\qquad+\left|\varphi_q\left[\int_{t_2}^{+\infty}\lambda\left(f\big(s,u(s),u'(s)\big)-\frac{\sum_{i=1}^n \alpha_i\int_{\xi_i}^{+\infty} f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}e^{-s}\right)ds\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \varphi_p\bigl(u'(+\infty)\bigr)\Bigr] - u'(+\infty)\Bigr| \\
&< \varepsilon.
\end{aligned}
$$

By Lemma 3.1, we get that $\{R(u,\lambda)|u \in \overline{\Omega}, \lambda \in [0,1]\}$ is compact. The proof is completed. □

In the spaces X and Y, the origin $\theta = 0$. In the following sections, we denote the origin by 0.

Lemma 3.3 *Let $\Omega \subset X$ be nonempty, open and bounded. Then N_λ is M-compact in $\overline{\Omega}$.*

Proof By (H_2), we know that $N_\lambda : \overline{\Omega} \to Y$ is continuous. Obviously, $\dim X_1 = \dim Y_1$. For $u \in \overline{\Omega}$, since $Q(I-Q)$ is a zero operator, we get $(I-Q)N_\lambda(u) \in \operatorname{Im} M$. For $y \in \operatorname{Im} M$, $y = Qy + (I-Q)y = (I-Q)y \in (I-Q)Y$. So, we have $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$. It is clear that

$$
QN_\lambda u = 0, \quad \lambda \in (0,1) \quad \Leftrightarrow \quad QNu = 0
$$

and $R(u,0) = 0$, $\forall u \in X$. $u \in \Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$ means that $N_\lambda u \in \operatorname{Im} M$ and $(\varphi_p(u'))' + \lambda f(t,u,u') = 0$, thus,

$$
\begin{aligned}
R(u,\lambda)(t) &= \int_0^t \varphi_q\biggl[\int_\tau^{+\infty} -\bigl(\varphi_p\bigl(u'\bigr)\bigr)'\,ds + \varphi_p\bigl(u'(+\infty)\bigr)\biggr]\,d\tau - u'(+\infty)t \\
&= u(t) - u'(+\infty)t = (I-P)u(t).
\end{aligned}
$$

For $u \in X$, we have

$$
\begin{aligned}
M\bigl[P + R(u,\lambda)\bigr](t) &= -\lambda f\bigl(t,u(t),u'(t)\bigr) + \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} \lambda f(r,u(r),u'(r))\,dr}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-t} \\
&= (I-Q)N_\lambda u(t).
\end{aligned}
$$

These, together with Lemma 3.2, mean that N_λ is M-compact in $\overline{\Omega}$. The proof is completed. □

In order to obtain our main results, we need the following additional conditions.

(H_3) There exist nonnegative functions $a(t)$, $b(t)$, $c(t)$ with $(1+t)^{p-1}a(t), b(t), c(t) \in Y$ and $\|(1+t)^{p-1}a(t)\|_1 + \|b(t)\|_1 < 1$ such that

$$
\bigl|f(t,x,y)\bigr| \le a(t)\bigl|\varphi_p(x)\bigr| + b(t)\bigl|\varphi_p(y)\bigr| + c(t), \quad \text{a.e. } t \in [0,+\infty).
$$

(H_4) There exists a constant $d_0 > 0$ such that if $|d| > d_0$, then one of the following inequalities holds:

$$
\begin{aligned}
&df(t,x,d) < 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}; \\
&df(t,x,d) > 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}.
\end{aligned}
$$

Lemma 3.4 *Assume that* (H_3) *and* (H_4) *hold. The set*

$$\Omega_1 = \{u \mid u \in \operatorname{dom} M, Mu = N_\lambda u, \lambda \in [0,1]\}$$

is bounded in X.

Proof If $u \in \Omega_1$, then $QN_\lambda u = 0$, *i.e.*, $\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(r, u(r), u'(r))\,dr = 0$. By (H_4), there exists $t_0 \in [0, +\infty)$ such that $|u'(t_0)| \le d_0$. It follows from $Mu = N_\lambda u$ that

$$\varphi_p(u'(t)) = -\int_{t_0}^t \lambda f(s, u(s), u'(s))\,ds + \varphi_p(u'(t_0)).$$

Considering (H_3), we have

$$\begin{aligned} |\varphi_p(u'(t))| &\le \int_0^{+\infty} \bigl[a(t)|\varphi_p(u(t))| + b(t)|\varphi_p(u'(t))| + c(t)\bigr]\,dt + \varphi_p(d_0) \\ &\le \|a(t)(1+t)^{p-1}\|_1 \varphi_p\left(\left\|\frac{u}{1+t}\right\|_\infty\right) + \|b\|_1 \varphi_p(\|u'\|_\infty) + \|c\|_1 + \varphi_p(d_0). \end{aligned} \tag{3.5}$$

Since $u(t) = \int_0^t u'(s)\,ds$, we get

$$\left|\frac{u(t)}{1+t}\right| \le \frac{t}{1+t}\|u'\|_\infty \le \|u'\|_\infty.$$

Thus,

$$\left\|\frac{u}{1+t}\right\|_\infty \le \|u'\|_\infty. \tag{3.6}$$

By (3.5), (3.6) and (H_3), we get

$$\|\varphi_p(u')\|_\infty \le \frac{\|c\|_1 + \varphi_p(d_0)}{1 - \|a(t)(1+t)^{p-1}\|_1 - \|b\|_1}.$$

So,

$$\|u'\|_\infty \le \varphi_q\left(\frac{\|c\|_1 + \varphi_p(d_0)}{1 - \|a(t)(1+t)^{p-1}\|_1 - \|b\|_1}\right).$$

This, together with (3.6), means that Ω_1 is bounded. The proof is completed. □

Lemma 3.5 *Assume that* (H_4) *holds. The set*

$$\Omega_2 = \{u \mid u \in \operatorname{Ker} M, QNu = 0\}$$

is bounded in X.

Proof $u \in \Omega_2$ means that $u = at$, $a \in \mathbb{R}$ and $QNu = 0$, *i.e.*,

$$\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(s, as, a)\,ds = 0.$$

By (H_4), we get that $|a| \le d_0$. So, Ω_2 is bounded. The proof is completed. □

Theorem 3.1 *Suppose that* (H_1)-(H_4) *hold. Then problem* (1.1) *has at least one solution.*

Proof Let $\Omega = \{u \in X \mid \|u\| < d_0'\}$, where $d_0' = \max\{d_0, \sup_{u\in\Omega_1}\|u\|, \sup_{u\in\Omega_2}\|u\|\} + 1$. It follows from the definition of Ω_1 and Ω_2 that $Mu \neq N_\lambda u$, $\lambda \in (0,1)$, $u \in \partial\Omega$ and $QNu \neq 0$, $u \in \partial\Omega \cap \operatorname{Ker} M$.

Define a homeomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} M$ as $J(ke^{-t}) = kt$. If $df(t,x,d) < 0$ for $|d| > d_0$, take the homotopy

$$H(u,\mu) = \mu u + (1-\mu)JQNu, \quad u \in \overline{\Omega} \cap \operatorname{Ker} M, \mu \in [0,1].$$

For $u \in \overline{\Omega} \cap \operatorname{Ker} M$, we have $u = kt$. Then

$$H(u,\mu) = \mu kt - (1-\mu)\frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(s,ks,k)\,ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} t.$$

Obviously, $H(u,1) \neq 0$, $u \in \partial\Omega \cap \operatorname{Ker} M$. For $\mu \in [0,1)$, $u = kt \in \partial\Omega \cap \operatorname{Ker} M$, if $H(u,\mu) = 0$, we have

$$\frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} kf(s,ks,k)\,ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} = \frac{\mu}{1-\mu}k^2 \geq 0.$$

A contradiction with $df(t,x,d) < 0$, $|d| > d_0$. If $df(t,x,d) > 0$, $|d| > d_0$, take

$$H(u,\mu) = \mu u - (1-\mu)JQNu, \quad u \in \overline{\Omega} \cap \operatorname{Ker} M, \mu \in [0,1],$$

and the contradiction follows analogously. So, we obtain $H(u,\mu) \neq 0$, $\mu \in [0,1]$, $u \in \partial\Omega \cap \operatorname{Ker} M$.

By the homotopy of degree, we get that

$$\begin{aligned}\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} M, 0\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} M, 0\bigr) = \deg(I, \Omega \cap \operatorname{Ker} M, 0) = 1.\end{aligned}$$

By Theorem 2.1, we can get that $Mu = Nu$ has at least one solution in $\overline{\Omega}$. The proof is completed. □

4 Example

Let us consider the following boundary value problem at resonance

$$\begin{cases} (|u'|^{-\frac{1}{2}}u')' + \frac{e^{-4t}}{\sqrt{1+t}}\sin\sqrt{|u|} + e^{-4t}|u'|^{-\frac{1}{2}}u' + \frac{1}{4}e^{-4t} = 0, & 0 < t < +\infty, \\ u(0) = 0, \qquad |u'(+\infty)|^{-\frac{1}{2}}u'(+\infty) = \sum_{i=1}^n \alpha_i |u'(\xi_i)|^{-\frac{1}{2}}u'(\xi_i), \end{cases} \tag{4.1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$.

Corresponding to problem (1.1), we have $p = \frac{3}{2}$, $f(t,x,y) = \frac{e^{-4t}}{\sqrt{1+t}}\sin\sqrt{|x|} + e^{-4t}|y|^{-\frac{1}{2}}y + \frac{1}{4}e^{-4t}$.

Take $a(t) = \frac{e^{-4t}}{\sqrt{1+t}}$, $b(t) = e^{-4t}$, $c(t) = \frac{1}{4}e^{-4t}$, $d_0 = 4$. By simple calculation, we can get that conditions (H_1)-(H_4) hold. By Theorem 3.1, we obtain that problem (4.1) has at least one solution.

Competing interests
The author declares that she has no competing interests.

Author's contributions
All results belong to WJ.

Acknowledgements
This work is supported by the National Science Foundation of China (11171088) and the Natural Science Foundation of Hebei Province (A2013208108). The author is grateful to anonymous referees for their constructive comments and suggestions, which led to the improvement of the original manuscript.

Received: 5 April 2013 Accepted: 23 August 2013 Published: 11 September 2013

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doi:10.1186/1687-2770-2013-207
Cite this article as: Jiang: Solvability for p-Laplacian boundary value problem at resonance on the half-line. *Boundary Value Problems* 2013 **2013**:207.