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Infinitely many solutions for a boundary value problem with impulsive effects

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Abstract
In this paper we are interested in multiplicity results for a nonlinear Dirichlet boundary value problem subject to perturbations of impulsive terms. The study of the problem is based on the variational methods and critical point theory. Infinitely many solutions follow from a recent variational result.
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1 Introduction

The theory of impulsive differential equations provides a general framework for the mathematical modeling of many real world phenomena; see, for instance, [1–3] and [4]. Indeed, many dynamical systems have an impulsive dynamical behavior due to abrupt changes at certain instants during the evolution process. Impulsive differential equations are basic tools for studying these phenomena [5, 6].

There are some common techniques to approach these problems: the fixed point theorems [7, 8], the method of upper and lower solutions [9], or the topological degree theory [10–12]. On the other hand, in the last few years, some authors have studied the existence of solutions by variational methods; see [13–19].

Here, we use critical point theory to investigate the existence of infinitely many solutions for the following nonlinear impulsive differential problem:

$$\begin{cases} -u''(t) + a(t)u'(t) + b(t)u(t) = \lambda g(t, u(t)), & t \in [0, T], t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = \mu I_j(u(t_j)), & j = 1, 2, \ldots, n, \end{cases} \tag{$D_{\lambda,\mu}$}$$

where $\lambda \in]0, +\infty[$, $\mu \in]0, +\infty[$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, $a, b \in L^\infty([0, T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} a(t) \geq 0$ and $\operatorname{ess\,inf}_{t\in[0,T]} b(t) \geq 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1, 2, \ldots, n$.

We establish some multiplicity results for problem $(D_{\lambda,\mu})$ under an appropriate oscillation behavior of the primitive of the nonlinearity g and a suitable growth of the primitive of I_j at infinity, for all λ belonging to a precise interval and provided μ is small enough (Theorem 3.3, Theorem 3.4). It is worth noticing that, when the impulsive effects I_j, $j = 1, \ldots, n$, are sublinear at infinity, our results hold for all $\mu \geq 0$ (see Remark 3.1). Here, as an example of our results, we present the following special case of Theorem 3.3.

Theorem 1.1 *Let $g : \mathbb{R} \to [0, \infty)$ be a continuous function and put $G(\xi) = \int_0^\xi g(t)\,dt$ for every $\xi \in \mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = +\infty.$$

Then there is $\bar{\delta} > 0$, where $\bar{\delta} = \frac{4}{nTe^T}$, such that for each $\mu \in [0, \bar{\delta}[$, the problem

$$\begin{cases} -u''(t) + u'(t) + u(t) = g(t, u(t)), & t \in [0, T], t \neq t_j, \\ u(0) = u(T) = 0, \\ -\Delta u'(t_j) = \mu u(t_j), & j = 1, 2, \dots, n \end{cases}$$

admits infinitely many pairwise distinct classical solutions.

We explicitly observe that in Theorem 1.1 impulsive effects I_j, $j = 1, \dots, n$, (that is, $I_j(x) = x$ for all $x \in \mathbb{R}$) are linear, contrary to the usual assumption of sublinearity of impulses; see [14, 16, 20–22] and [23]. The rest of this paper is organized as follows. In Section 2, we introduce some notations and preliminary results. Moreover, the abstract critical point theorem (Theorem 2.1) is recalled. In Section 3, we obtain some existence results. In Section 4, we give some examples to illustrate our results.

2 Preliminaries

By a classical solution of $(D_{\lambda,\mu})$ we mean a function

$$u \in \{w \in C([0, T]) : w_{|[t_j, t_{j+1}]} \in H^2([t_j, t_{j+1}])\}$$

that satisfies the equation in $(D_{\lambda,\mu})$ a.e. on $[0, T] \setminus \{t_1, \dots, t_n\}$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j = 1, \dots, n$, exist, that satisfies the impulsive conditions $\Delta u'(t_j) = \mu I_j(u(t_j))$ and the boundary conditions $u(0) = u(T) = 0$. Clearly, if a, b and g are continuous, then a classical solution $u \in C^2([t_j, t_{j+1}])$, $j = 0, 1, \dots, n$, satisfies the equation in $(D_{\lambda,\mu})$ for all $t \in [0, T] \setminus \{t_1, \dots, t_n\}$.

We consider the following slightly different form of problem $(D_{\lambda,\mu})$:

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = \lambda f(t, u(t)), & t \in [0, T], t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), & j = 1, 2, \dots, n, \end{cases} \tag{$S_{\lambda,\mu}$}$$

where $p \in C^1([0, T],]0, +\infty[)$, and $q \in L^\infty([0, T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} q(t) \geq 0$.

It is easy to see that, by choosing

$$p(t) = e^{-\int_0^t a(\tau)\,d\tau}, \qquad q(t) = b(t)e^{-\int_0^t a(\tau)\,d\tau}, \qquad f(t, u) = g(t, u)e^{-\int_0^t a(\tau)\,d\tau},$$

the solutions of $(S_{\lambda,\mu})$ are solutions of $(D_{\lambda,\mu})$.

Let us introduce some notations. In the Sobolev space $H_0^1(0, T)$, consider the inner product

$$(u, v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt,$$

which induces the norm

$$\|u\| = \left(\int_0^T p(t)\big(u'(t)\big)^2\,dt + \int_0^T q(t)\big(u(t)\big)^2\,dt\right)^{1/2}.$$

The following lemmas are useful for proving our main result. Their proofs can be found in [24].

Lemma 1 ([24, Proposition 2.1]) *Let $u \in H_0^1(0,T)$. Then*

$$\|u\|_\infty \le \frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\|, \tag{1}$$

where $p^* := \min_{t\in[0,T]} p(t)$.

Here, and in the sequel, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, namely:

(a) $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$;

(b) $x \to f(t,x)$ is continuous for almost every $t \in [0,T]$;

(c) for every $\rho > 0$, there exists a function $l_\rho \in L^1([0,T])$ such that

$$\sup_{|x|\le\rho} \big|f(t,x)\big| \le l_\rho(t)$$

for almost every $t \in [0,T]$.

Definition 1 A function $u \in H_0^1(0,T)$ is said to be a weak solution of $(S_{\lambda,\mu})$ if u satisfies

$$\begin{aligned}&\int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt\\ &\quad - \lambda\int_0^T f\big(t,u(t)\big)v(t)\,dt + \mu\sum_{j=1}^n p(t_j)I_j\big(u(t_j)\big)v(t_j) = 0\end{aligned} \tag{2}$$

for any $v \in H_0^1(0,T)$.

Lemma 2 ([24, Lemma 2.1]) *$u \in H_0^1(0,T)$ is a weak solution of $(S_{\lambda,\mu})$ if and only if u is a classical solution of $(S_{\lambda,\mu})$.*

Now, we define the functionals $\Phi, \Psi : H_0^1(0,T) \to \mathbb{R}$ in the following way:

$$\Phi(u) = \frac{1}{2}\|u\|^2 \quad \text{and} \quad \Psi(u) = \int_0^T F\big(t,u(t)\big)\,dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)\,dx, \tag{3}$$

for each $u \in H_0^1(0,T)$, where $F(t,\xi) = \int_0^\xi f(t,x)\,dx$ for each $(t,\xi) \in [0,T] \times \mathbb{R}$. Using the property of f and the continuity of I_j, $j = 1,2,\dots,n$, we have that $\Phi, \Psi \in C^1(H_0^1(0,T),\mathbb{R})$ and for any $v \in H_0^1(0,T)$, one has

$$\Phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt$$

and

$$\Psi'(u)(v) = \int_0^T f\big(t, u(t)\big)v(t)\,dt - \frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)I_j\big(u(t_j)\big)v(t_j).$$

So, arguing in a standard way, it is possible to prove that the critical points of the functional $E_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ are the weak solutions of problem $(S_{\lambda,\mu})$ and so they are classical solutions.

In the next section we shall prove our results applying the following infinitely many critical points theorem obtained in [25]. First, we recall the following definition.

Definition 2 Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ two Gâteaux differentiable functionals, $r \in]-\infty, +\infty]$. We say that functional $E := \Phi - \Psi$ satisfies the Palais-Smale condition cut off upper at r (in short $(\mathrm{PS})^{[r]}$-condition) if any sequence $\{u_n\}$, such that

(α) $\{E(u_n)\}$ is bounded,
(β) $\lim_{n\to+\infty} \|E'(u_n)\|_{X^*} = 0$,
(γ) $\Phi(u_n) < r$ for all $n \in \mathbb{N}$,

has a convergent subsequence.

When $r = +\infty$, the previous definition is the same as the classical definition of the Palais-Smale condition, while if $r < +\infty$, such a condition is more general than the classical one. We refer to [25] for more details.

Theorem 2.1 (see [25], Theorem 7.4) *Let X be a real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

(a) *If $\gamma < +\infty$ and for each $\lambda \in]0, \frac{1}{\gamma}[$, the functional $E_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for all $r \in \mathbb{R}$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*

(a$_1$) *E_λ has a global minimum*

or

(a$_2$) *there exists a sequence $\{u_n\}$ of critical points (local minima) of E_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ and for each $\lambda \in]0, \frac{1}{\delta}[$, the functional $E_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for all $r \in \mathbb{R}$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*

(b_1) *there exists a global minimum of* Φ *which is a local minimum of* E_λ

or

(b_2) *there exists a sequence of pairwise distinct critical points* (*local minima*) *of* E_λ *such that* $\lim_{n\to+\infty}\Phi(u_n)=\inf_X\Phi$.

We recall that Theorem 2.1 improves [26, Theorem 2.5] since no assumptions with respect to weak topology of X are made. In particular, the set $\overline{\Phi^{-1}(]-\infty,r[)}^{w}$ is not involved in the definition of φ and the sequential weak lower semicontinuity of E_λ is not required.

3 Main results

In this section, we present our main results. Put

$$k:=\frac{6p^*}{12\|p\|_\infty+T^2\|q\|_\infty}.$$

Moreover, let

$$A:=\liminf_{\xi\to+\infty}\frac{\int_0^T\max_{|x|\le\xi}F(t,x)\,dt}{\xi^2},\qquad B:=\limsup_{\xi\to+\infty}\frac{\int_{T/4}^{3T/4}F(t,\xi)\,dt}{\xi^2}.$$

Our first result is as follows.

Theorem 3.1 *Assume that*

(a_1) $F(t,\xi)\ge 0$ *for all* $(t,\xi)\in([0,\frac{T}{4}]\cup[\frac{3T}{4},T])\times\mathbb{R}$;
(a_2) $A<kB$.

Then, for every $\lambda\in\Lambda:=]\frac{2p^*}{kTB},\frac{2p^*}{TA}[$ *and for every continuous function* $I_j:\mathbb{R}\to\mathbb{R}$, $j=1,\dots,n$, *whose potential* $\mathcal{I}_j(\xi):=\int_0^\xi I_j(x)\,dx$, $\xi\in\mathbb{R}$, *satisfies*

(i_1) $\sup_{\xi\ge0}\mathcal{I}_j(\xi)=0$;
(i_2) $I_\infty:=\limsup_{\xi\to+\infty}\frac{\sum_{j=1}^n\max_{|t|\le\xi}(-\mathcal{I}_j(t))}{\xi^2}<+\infty$,

there exists $\delta_{I,\lambda}>0$, *where*

$$\delta_{I,\lambda}:=\frac{1}{\|p\|_\infty I_\infty}\left(\frac{2p^*}{T}-\lambda A\right),$$

such that for every $\mu\in[0,\delta_{I,\lambda}[$, *problem* ($S_{\lambda,\mu}$) *has an unbounded sequence of weak solutions.*

Proof First, we observe that owing to (a_2) the interval Λ is non-empty. Moreover, for each $\lambda\in\Lambda$ and taking into account that $\lambda<\frac{2p^*}{TA}$, one has $\delta_{I,\lambda}>0$. Now, fix λ and μ as in the conclusion. Our aim is to apply Theorem 2.1. For this end, take $X=H_0^1(0,T)$ and Φ, Ψ as in (3).

We divide our proof into three steps in order to show Theorem 3.1. First, we prove that $E_\lambda=\Phi-\lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r\in\mathbb{R}$. So, fix $r\in\mathbb{R}$ and let $\{u_n\}\subseteq X$ be a sequence such that $\{E_\lambda(u_n)\}$ is bounded, $\lim_{n\to+\infty}\|E'_\lambda(u_n)\|_{X^*}=0$ and $\Phi(u_n)<r$ for all $n\in\mathbb{N}$. From $\Phi(u_n)<r$, taking into account that Φ is coercive, $\{u_n\}$ is bounded in X.

Since the embedding of X in $C(0,T)$ is compact (see, for instance, [27, Theorem 8.8]) and X is reflexive, up to a subsequence, $\{u_n(x)\}$ is uniformly convergent to $u_0(x)$, and $\{u_n\}$ is weakly convergent to u_0 in X. The uniform convergence of $\{u_n\}$, taking also into account Lebesgue's theorem, ensures that

$$\lim_{n\to+\infty}\left[\int_0^T f\big(t,u_n(t)\big)\big(u_n(t)-u_0(t)\big)\,dt-\frac{\mu}{\lambda}\sum_{j=1}^n p(t_j)I_j\big(u_n(t_j)\big)\big(u_n(t_j)-u_0(t_j)\big)\right]=0,$$

that is,

$$\lim_{n\to+\infty}\Psi'(u_n)(u_n-u_0)=0. \tag{4}$$

Now, from $\lim_{n\to+\infty}\|E'_\lambda(u_n)\|_{X^*}=0$, there is a sequence $\{\epsilon_n\}$, with $\epsilon_n\to 0^+$, such that

$$\big|E'_\lambda(u_n)(v)\big|\le\epsilon_n$$

for all $v\in X$ with $\|v\|\le 1$ and for all $n\in\mathbb{N}$. Setting $=\frac{u_n-u_0}{\|u_n-u_0\|}$, one has

$$\big|E'_\lambda(u_n)(u_n-u_0)\big|\le\epsilon_n\|u_n-u_0\| \tag{5}$$

for all $n\in\mathbb{N}$. Moreover, having in mind that $ab\le\frac{1}{2}a^2+\frac{1}{2}b^2$, one has

$$\begin{aligned}\Phi'(u_n)(u_n-u_0) &= \int_0^T p(t)u_n'(t)\big(u_n'(t)-u_0'(t)\big)\,dt+\int_0^T q(t)u_n(t)\big(u_n(t)-u_0(t)\big)\,dt\\ &= \|u_n\|^2-\left[\int_0^T p(t)u_n'(t)u_0'(t)\,dt+\int_0^T q(t)u_n(t)u_0(t)\,dt\right]\\ &\ge \|u_n\|^2-\frac{1}{2}\left[\int_0^T p(t)\big(u_n'(t)\big)^2\,dt+\int_0^T p(t)\big(u_0'(t)\big)^2\,dt\right.\\ &\quad\left.+\int_0^T q(t)\big(u_n(t)\big)^2\,dt+\int_0^T q(t)\big(u_0(t)\big)^2\,dt\right]\\ &= \frac{1}{2}\|u_n\|^2-\frac{1}{2}\|u_0\|^2,\end{aligned}$$

that is,

$$\Phi'(u_n)(u_n-u_0)\ge\frac{1}{2}\|u_n\|^2-\frac{1}{2}\|u_0\|^2. \tag{6}$$

From (5) and (6) one has

$$\begin{aligned}&\Phi'(u_n)(u_n-u_0)-\lambda\Psi'(u_n)(u_n-u_0)\le\epsilon_n\|u_n-u_0\|,\\ &\frac{1}{2}\|u_n\|^2-\frac{1}{2}\|u_0\|^2-\lambda\Psi'(u_n)(u_n-u_0)\le\epsilon_n\|u_n-u_0\|,\end{aligned}$$

and owing to (4), one has

$$\limsup_{n\to+\infty}\frac{1}{2}\|u_n\|^2\le\frac{1}{2}\|u_0\|^2.$$

Hence, [27, Proposition III.30] ensures that $\{u_n\}$ strongly converges to $u_0 \in X$ and our claim is proved.

Second, we wish to prove that

$$\gamma < +\infty.$$

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|x|\le\xi_n} F(t,x)\,dt}{\xi_n^2} = A.$$

Put $r_n = \frac{2p^*}{T}\xi_n^2$ for all $n \in \mathbb{N}$. By Lemma 1, for all $u \in X$, one has

$$\max_{t\in[0,T]} \bigl|v(t)\bigr| \le \xi_n$$

for all $v \in X$ such that $\|v\|^2 < 2r_n$. Hence, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v)}{r_n} \\
&\le \frac{T}{2p^*} \frac{\int_0^T \max_{|x|\le\xi_n} F(t,x)\,dt}{\xi_n^2} + \frac{\mu}{\lambda}\frac{T\|p\|_\infty}{2p^*} \sum_{j=1}^{n} \frac{\max_{|x|\le\xi_n}(-\mathcal{I}_j(x))}{\xi_n^2}.
\end{aligned}$$

So, from assumptions (a_2) and (i_2),

$$\gamma \le \liminf_{\xi\to+\infty} \left[\frac{T}{2p^*} \frac{\int_0^T \max_{|x|\le\xi} F(t,x)\,dt}{\xi^2} + \frac{\mu}{\lambda}\frac{T\|p\|_\infty}{2p^*} \sum_{j=1}^{n} \frac{\max_{|x|\le\xi}(-\mathcal{I}_j(x))}{\xi^2} \right] < +\infty.$$

Assumption $0 < \mu < \delta_{I,\lambda}$ immediately yields

$$\gamma \le \frac{T}{2p^*}A + \frac{\mu}{\lambda}\frac{T\|p\|_\infty}{2p^*} I_\infty < \frac{T}{2p^*}A + \frac{1 - \lambda\frac{T}{2p^*}A}{\lambda} = \frac{1}{\lambda},$$

that is, $\lambda < \frac{1}{\gamma}$. The previous inequality assures that conclusion (a) of Theorem 2.1 can be used, for which either $\Phi - \lambda\Psi$ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of problem $(S_{\lambda,\mu})$ such that $\lim_{n\to\infty} \|u_n\| = +\infty$.

The final step is to verify that the functional $\Phi - \lambda\Psi$ has no global minimum. From $\limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2} = B$, and taking into account that $\lambda > \frac{2p^*}{kTB}$, there is $h \in \mathbb{R}$ such that

$$\limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2} > h > \frac{2p^*}{\lambda kT}. \tag{7}$$

So, there exists a sequence of positive numbers η_n such that $\eta_n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_{T/4}^{3T/4} F(t,\eta_n)\,dt}{\eta_n^2} > h.$$

It follows that there is $\nu \in \mathbb{N}$ such that for all $n > \nu$, one has

$$\frac{\int_{T/4}^{3T/4} F(t,\eta_n)\,dt}{\eta_n^2} > h.$$

Now, consider a function $v_n \in X$ defined by setting

$$v_n(x) = \begin{cases} \frac{4\eta_n x}{T}, & x \in [0, \frac{T}{4}], \\ \eta_n, & x \in]\frac{T}{4}, \frac{3T}{4}], \\ \frac{4\eta_n}{T}(T-x), & x \in]\frac{3T}{4}, T]. \end{cases}$$

Clearly, one has

$$\Phi(v_n) \le \left(\frac{4}{T}\|p\|_\infty + \frac{T}{3}\|q\|_\infty\right)\eta_n^2 = \frac{2p^*}{kT}\eta_n^2.$$

Moreover, bearing in mind (a_1) and (i_1),

$$\begin{aligned} \Phi(v_n) - \lambda\Psi(v_n) &\le \frac{2p^*}{kT}\eta_n^2 - \lambda\int_{T/4}^{3T/4} F(t,\eta_n)\,dt \\ &< \eta_n^2\left(\frac{2p^*}{kT} - \lambda h\right). \end{aligned} \tag{8}$$

Putting together (7) and (8), we get that the functional $\Phi - \lambda\Psi$ is unbounded from below and so it has no global minimum.

Therefore, Theorem 2.1 assures that there is a sequence $\{u_n\} \subseteq X$ of critical points of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty}\|u_n\| = +\infty$ and, taking into account the considerations made in Section 2, the theorem is completely proved. □

Remark 3.1 Assume that $f : [0,1] \times \mathbb{R} \to [0,\infty)$. Clearly, condition (a_1) holds, and condition (a_2) assumes the following simpler form:

(a_2')

$$\liminf_{\xi\to+\infty} \frac{\int_0^T F(t,\xi)\,dt}{\xi^2} < k \limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2}.$$

In particular, if $\liminf_{\xi\to+\infty} \frac{\int_0^T F(t,\xi)\,dt}{\xi^2} = 0$ and $\limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2} = +\infty$, then (a_2') holds and problem $(S_{\lambda,\mu})$ has an unbounded sequence of weak solutions in X for every pair $(\lambda,\mu) \in]0,+\infty[\times [0, \frac{2}{TI_\infty}[$.

Moreover, under the assumption $I_\infty = 0$, Theorem 3.1 guarantees the existence of infinitely many solutions to problem $(S_{\lambda,\mu})$ for every $\mu \ge 0$.

As an example, we point out below a special case of Theorem 3.1.

Corollary 3.1 *Let $f : \mathbb{R} \to [0,\infty)$ be a continuous function, put $F(\xi) = \int_0^\xi f(t)\,dt$ for every $\xi \in \mathbb{R}$, and let $q \in C^0([0,T])$. Assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < \frac{k}{2} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}.$$

Then, for each $\lambda \in]\frac{4p^*}{kT^2}\frac{1}{\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}}, \frac{2p^*}{kT^2}\frac{1}{\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}}[$, *and for each continuous function* $I_j : \mathbb{R} \to [0,\infty)$ *such that* $\lim_{\xi\to+\infty}\frac{I_j(x)}{\xi} = 0$, $j = 1,\dots,n$, *the problem*

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = \lambda f(u(t)), & t \in [0,T], t \neq t_j, \\ u(0) = u(T) = 0, \\ -\Delta u'(t_j) = I_j(u(t_j)), & j = 1,2,\dots,n, \end{cases}$$

admits infinitely many pairwise distinct classical solutions.

Replacing the condition at infinity of the potential F by a similar one at zero, and arguing as in the proof of Theorem 3.1 but using conclusion (b) of Theorem 2.1 instead of (a), one establishes the following result. Put

$$A^* := \liminf_{\xi\to 0^+}\frac{\int_0^T \max_{|x|\le\xi} F(t,x)\,dt}{\xi^2}, \qquad B^* := \limsup_{\xi\to 0^+}\frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2}.$$

Theorem 3.2 *Assume that*

(a_1) $F(t,\xi) \ge 0$ *for all* $(t,\xi) \in ([0,\frac{T}{4}] \cup [\frac{3T}{4},T]) \times \mathbb{R}$;
(b_2) $A^* < kB^*$.

Then, for every $\lambda \in \Lambda^* :=]\frac{2p^*}{kTB^*}, \frac{2p^*}{TA^*}[$ *and for every continuous function* $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\dots,n$, *whose potential* $\mathcal{I}_j(\xi) := \int_0^\xi I_j(x)\,dx$, $\xi \in \mathbb{R}$, *satisfies*

(i_1) $\sup_{\xi\ge 0}\mathcal{I}_j(\xi) = 0$;
(j_2) $I_0 := \limsup_{\xi\to 0^+}\frac{\sum_{j=1}^n \max_{|t|\le\xi}(-\mathcal{I}_j(t))}{\xi^2} < +\infty$,

there exists $\delta^*_{I,\lambda} > 0$, *where*

$$\delta^*_{I,\lambda} := \frac{1}{I_0\|p\|_\infty}\left(\frac{2p^*}{T} - \lambda A^*\right)$$

such that for every $\mu \in [0,\delta^*_{I,\lambda}[$, *problem* $(S_{\lambda,\mu})$ *has a sequence of non-zero weak solutions, which strongly converges to* 0.

Proof We take X, Φ and Ψ as in the proof of Theorem 3.1. Fix $\lambda^* \in \Lambda^*$, let I_j be a function that satisfies assumptions (i_1) and (j_2) and take $0 \le \mu^* < \delta^*_{I,\lambda}$. Arguing as in the proof of Theorem 3.1, one has $\delta = \liminf_{r\to 0^+}\varphi(r) < +\infty$. Now, arguing again as in the proof of Theorem 3.1, there is a sequence of positive numbers $\{\eta_n\}$ such that $\eta_n \to 0^+$ and $\frac{\int_{T/4}^{3T/4} F(t,\eta_n)\,dt}{\eta_n^2} > h$ for all $n > \nu$ and for some $\nu \in \mathbb{N}$. By choosing v_n as in the proof of Theorem 3.1, the sequence $\{v_n\}$ strongly converges to 0 in X and $\Phi(v_n) - \lambda^*\Psi(v_n) < 0$ for each $n > \nu$. Therefore, taking into account that $(\Phi - \lambda^*\Psi)(0) = 0$, 0 is not a local minimum of $\Phi - \lambda^*\Psi$. The part (b) of Theorem 2.1 ensures that there exists a sequence $\{u_n\}$ in X of critical points of $\Phi - \lambda^*\Psi$ such that $\lim_{n\to+\infty}\|u_n\| = 0$ and the proof is complete. □

Let $A(t)$ be a primitive of $a(t)$, $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function and put

$$G(t,\xi) = \int_0^\xi g(t,x)\,dx, \qquad \tilde{k} := \frac{6}{e^{\|a\|_1}(12 + T^2\|b\|_\infty)}.$$

Moreover, let

$$\alpha := \liminf_{\xi \to +\infty} \frac{\int_0^T e^{-A(t)} \max_{|x| \le \xi} G(t,x)\,dt}{\xi^2}, \qquad \beta := \limsup_{\xi \to +\infty} \frac{\int_{T/4}^{3T/4} e^{-A(t)} G(t,\xi)\,dt}{\xi^2}.$$

In virtue of Theorems 3.1 and 3.2, we obtain the following results for problem $(D_{\lambda,\mu})$.

Theorem 3.3 *Assume that*

(c_1) $G(t,\xi) \ge 0$ *for all* $(t,\xi) \in ([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]) \times \mathbb{R}$;
(c_2) $\alpha < \tilde{k}\beta$.

Then, for every $\lambda \in \Lambda :=]\frac{2}{\tilde{k}Te^{\|a\|_1}\beta}, \frac{2}{Te^{\|a\|_1}\alpha}[$ *and for every continuous function* $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \dots, n$, *whose potential* $\mathcal{I}_j(\xi) := \int_0^\xi I_j(x)\,dx$, $\xi \in \mathbb{R}$, *satisfies*

(i_1) $\sup_{\xi \ge 0} \mathcal{I}_j(\xi) = 0$,
(i_2) $I_\infty := \limsup_{\xi \to +\infty} \frac{\sum_{j=1}^n \max_{|t| \le \xi}(-\mathcal{I}_j(t))}{\xi^2} < +\infty$,

there exists $\delta_{I,\lambda} > 0$, *where*

$$\delta_{I,\lambda} := \frac{1}{I_\infty}\left(\frac{2}{Te^{\|a\|_1}} - \lambda\alpha\right),$$

such that for each $\mu \in [0, \delta_{I,\lambda}[$, *problem* $(D_{\lambda,\mu})$ *has an unbounded sequence of weak solutions.*

Proof As seen in Section 2, we put $p(t) = e^{-A(t)}$, $q(t) = b(t)e^{-A(t)}$ and $f(t,u) = g(t,u)e^{-A(t)}$, $t \in [0,T]$. Clearly, one has $F(t,u) = e^{-A(t)}G(t,u)$, $A = \alpha$, $B = \beta$, $p^* = \frac{1}{e^{\|a\|_1}}$, $k \ge \tilde{k}$. Hence, from Theorem 3.1 the conclusion is achieved. □

Remark 3.2 Theorem 1.1 in Introduction is an immediate consequence of Theorem 3.3. In fact, it is enough to observe that (c_1) is verified and one has $\alpha = 0$ and $\beta = +\infty$, for which $1 \in \Lambda =]0, +\infty[$. Moreover, from $I_\infty = \lim_{\xi \to +\infty} \frac{\sum_{j=1}^n \xi^2/2}{\xi^2} = \frac{n}{2} < +\infty$, one has $\bar{\delta} = \delta_{I,\lambda} = \frac{4}{nTe^T}$ and the conclusion is achieved.

Replacing the condition at infinity of the potential G by a similar one at zero, one establishes the following result. Put

$$\alpha^* := \liminf_{\xi \to 0^+} \frac{\int_0^T e^{-A(t)} \max_{|x| \le \xi} G(t,x)\,dt}{\xi^2}, \qquad \beta^* := \limsup_{\xi \to 0^+} \frac{\int_{T/4}^{3T/4} e^{-A(t)} G(t,\xi)\,dt}{\xi^2}.$$

Theorem 3.4 *Assume*

(c_1) $G(t,\xi) \ge 0$ *for all* $(t,\xi) \in ([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]) \times \mathbb{R}$;
(c_2') $\alpha^* < \tilde{k}\beta^*$.

Then, for every $\lambda \in \Lambda' :=]\frac{2}{\tilde{k}Te^{\|a\|_1}\beta^*}, \frac{2}{Te^{\|a\|_1}\alpha^*}[$ *and for every continuous function* $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, n$, *whose potential* $\mathcal{I}_j(\xi) := \int_0^\xi I_j(x)\,dx$, $\xi \in \mathbb{R}$, *satisfies*

(i_1) $\sup_{\xi \ge 0} \mathcal{I}_j(\xi) = 0$,

(j_2) $I_0 := \limsup_{\xi\to 0^+} \frac{\sum_{j=1}^{n} \max_{|t|\leq\xi}(-\mathcal{I}_j(t))}{\xi^2} < +\infty$,

there exists $\delta^*_{I,\lambda} > 0$, *where*

$$\delta^*_{I,\lambda} := \frac{1}{I_0}\left(\frac{2}{Te^{\|a\|_1}} - \lambda\alpha^*\right),$$

such that for each $\mu \in [0, \delta^*_{I,\lambda}[$, *problem* $(D_{\lambda,\mu})$ *has a sequence of non-zero weak solutions, which strongly converges to* 0.

Proof The conclusion follows from Theorem 3.2 by arguing as in the proof of Theorem 3.3. □

Remark 3.3 We point out that in Theorem 3.3 (as in Theorem 3.4) the assumption $\operatorname{ess\,inf}_{[0,T]} a \geq 0$ can be deleted provided that we assume the constant $\tilde{k} := \frac{6\min_{[0,T]} e^{-A(t)}}{12+T^2\|be^{-A}\|_\infty}$ and the interval $\Lambda =]\frac{2\min_{[0,T]} e^{-A(t)}}{\tilde{k}T\beta}, \frac{2\min_{[0,T]} e^{-A(t)}}{T\alpha}[$.

Finally, we observe that the existence of infinitely many solutions to problem $(D_{\lambda,\mu})$ can be obtained from Theorem 3.3 and Theorem 3.4 even under small perturbations of the nonlinearity. As an example, we point out the following consequence of Theorem 3.3.

Corollary 3.2 *Let* $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be an* L^1*-Carathéodory function satisfying* (c_1) *and* (c_2) *of Theorem* 3.3.

Then, for every $\lambda \in \Lambda =]\frac{2}{\tilde{k}Te^{\|a\|_1}\beta}, \frac{2}{Te^{\|a\|_1}\alpha}[$, *for every nonnegative* L^1*-Carathéodory function* $h : [0,T] \times \mathbb{R} \to \mathbb{R}$, *whose potential* $h(t,\xi) = \int_0^\xi h(t,x)\,dx$ *satisfies*

$$H_\infty = \limsup_{\xi\to+\infty} \frac{\int_0^T H(t,\xi)\,dt}{\xi^2} < +\infty,$$

and for every continuous function $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \ldots, n$, *whose potential* $\mathcal{I}_j(\xi) := \int_0^\xi I_j(x)\,dx$, $\xi \in \mathbb{R}$, *satisfies* (i_1) *and* (i_2) *of Theorem* 3.3, *there exist* $\gamma^*_{H,\lambda} > 0$ *and* $\delta^*_{I,\lambda} > 0$, *where*

$$\gamma^*_{H,\lambda} := \frac{1}{H_\infty}\left(\frac{2p^*}{T} - \lambda\alpha\right),$$

$$\delta^*_{I,\lambda} := \frac{1}{I_\infty}\left(\frac{2p^*}{T} - \lambda\alpha\right)$$

such that for all $\gamma \in [0, \gamma^*_{H,\lambda}[$ *and for all* $\mu \in [0, \delta^*_{I,\lambda}[$, *the problem*

$$\begin{cases} -u''(t) + a(t)u'(t) + b(t)u(t) = \lambda g(t,u(t)) + \gamma h(t,u(t)), & t \in [0,T] \setminus \{t_j\}, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), & j = 1, 2, \ldots, n, \end{cases} \qquad (D_{\lambda,\gamma,\mu})$$

has an unbounded sequence of weak solutions.

Proof It is enough to apply Theorem 3.3 to the following function:

$$\bar{g}(t,x) = g(t,x) + \frac{\bar{\gamma}}{\bar{\lambda}}h(t,x), \quad (t,x) \in [0,T] \times \mathbb{R},$$

where $\bar{\gamma}$ is fixed in $[0, \gamma^*_{H,\lambda}[$ and $\bar{\lambda}$ is fixed in Λ. In fact, one has

$$\begin{aligned}\bar{\alpha} &= \liminf_{\xi\to+\infty} \frac{\int_0^T e^{-A(t)} \max_{|x|\le\xi} \bar{G}(t,x)\,dt}{\xi^2} \le \alpha + \frac{\bar{\gamma}}{\bar{\lambda}}H_\infty < \alpha + \frac{\gamma^*_{H,\lambda}}{\bar{\lambda}}H_\infty \\ &= \alpha + \frac{2}{Te^{\|a\|_1}}\frac{1}{\bar{\lambda}} - \alpha = \frac{2}{Te^{\|a\|_1}}\frac{1}{\bar{\lambda}}\end{aligned} \tag{9}$$

and

$$\bar{\beta} = \limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} e^{-A(t)}\bar{G}(t,\xi)\,dt}{\xi^2} \ge \limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} e^{-A(t)}G(t,\xi)\,dt}{\xi^2} = \beta, \tag{10}$$

for which $\bar{\alpha} < \frac{2}{Te^{\|a\|_1}}\frac{1}{\bar{\lambda}} < \frac{2}{Te^{\|a\|_1}}\frac{\tilde{k}Te^{\|a\|_1}\beta}{2} = \tilde{k}\beta \le \tilde{k}\bar{\beta}$, that is, $\bar{\alpha} < \tilde{k}\bar{\beta}$. Moreover, from (9) one has $\bar{\lambda} < \frac{2}{Te^{\|a\|_1}}\frac{1}{\bar{\alpha}}$ and from (10) $\bar{\lambda} > \frac{2}{\tilde{k}Te^{\|a\|_1}}\frac{1}{\bar{\beta}}$. Hence, $\bar{\lambda} \in]\frac{2}{\tilde{k}Te^{\|a\|_1}}\frac{1}{\bar{\beta}}, \frac{2}{Te^{\|a\|_1}}\frac{1}{\bar{\alpha}}[$ and Theorem 3.3 ensures the conclusion. □

4 Applications

In many papers [13, 20, 22, 28] and [23], the authors obtain the existence of infinitely many solutions for problem $(D_{\lambda,\mu})$ while the impulsive term is supposed to be odd. The next examples provide problems that admit infinitely many solutions for which those other results cannot be applied.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} -(\frac{\sqrt{4t+1}}{t+1}u'(t))' + (1+\sqrt{t})u(t) = \lambda f(t,u(t)), & t \in [0,1], t \ne t_j, \\ u(0) = u(1) = 0, \\ \Delta u'(t_j) = \mu(\frac{2u}{u^2+1} - 1), & j = 1,2,\dots,n, \end{cases} \tag{11}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the function defined as follows:

$$f(t,u) = \begin{cases} \cos(\frac{\pi}{2}t)u\sin^2\ln(u) & \text{if } u > 0, \\ 0 & \text{if } u \le 0. \end{cases}$$

It is easy to see that conditions (a_1), (a_2), (i_1) and (i_2) of Theorem 3.1 hold. In particular, $k = \frac{3}{4\sqrt{3}+1}$ and

$$\begin{aligned}&\liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|x|\le\xi} F(t,x)\,dt}{\xi^2} = \frac{2-\sqrt{2}}{4\pi}, \\ &\limsup_{\xi\to+\infty} \frac{\int_{1/4}^{3/4} F(t,\xi)\,dt}{\xi^2} = \frac{(2+\sqrt{2})\sqrt{4-2\sqrt{2}}}{8\pi}.\end{aligned}$$

Then, for each $\lambda \in [36,42]$ and for every $\mu \ge 0$, problem (11) has an unbounded sequence of solutions in X.

Now, we give an application of Theorem 3.4.

Example 4.2 Consider the Dirichlet problem

$$\begin{cases} -u''(t)+u'(t)+u(t)=\lambda g(t,u(t)), & t\in[0,\frac{1}{2}], t\neq t_1,\\ u(0)=u(1/2)=0, \\ \Delta u'(t_1)=\mu(-e^u(u^2+2u)), \end{cases} \tag{12}$$

where $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is the function defined as follows:

$$g(t,u)=\begin{cases} e^t u(\frac{5}{2}-2\sin(\ln|u|)-\cos(\ln|u|)) & \text{if } u\neq 0,\\ 0 & \text{if } u=0. \end{cases}$$

By a simple calculation, we get $k=\frac{24}{49\sqrt{e}}$ and

$$\liminf_{\xi\to 0^+}\frac{\int_0^T e^{-A(t)}\max_{|x|\leq\xi}G(t,x)\,dt}{\xi^2}=\frac{1}{8},$$

$$\limsup_{\xi\to 0^+}\frac{\int_{T/4}^{3T/4} e^{-A(t)}G(t,\xi)\,dt}{\xi^2}=\frac{9}{16}.$$

Then, from Theorem 3.4, for each $\lambda\in[15,19]$ and for every $\mu\in[0,\frac{11}{20}[$, problem (12) admits a sequence of pairwise distinct classical solutions strongly converging at 0. We observe that, in this case, as direct computations show, also zero is a solution of the problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed equally to this research work. All authors read and approved the final manuscript.

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References

1. Benchohra, M, Henderson, J, Ntouyas, S: Theory of Impulsive Differential Equations. Contemporary Mathematics and Its Applications, vol. 2. Hindawi Publishing Corporation, New York (2006)
2. Chen, L, Sun, J: Nonlinear boundary value problem for first order impulsive functional differential equations. J. Math. Anal. Appl. **318**, 726-741 (2006)
3. Chu, J, Nieto, JJ: Impulsive periodic solutions of first-order singular differential equations. Bull. Lond. Math. Soc. **40**(1), 143-150 (2008)
4. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
5. Agarwal, RP, Franco, D, O'Regan, D: Singular boundary value problems for first and second order impulsive differential equations. Aequ. Math. **69**, 83-96 (2005)
6. Baek, H: Extinction and permanence of a three-species Lotka-Volterra system with impulsive control strategies. Discrete Dyn. Nat. Soc. **2008**, Article ID 752403 (2008)
7. Lee, EL, Lee, YH: Multiple positive solutions of two point boundary value problems for second order impulsive differential equations. Appl. Math. Comput. **158**, 745-759 (2004)
8. Chen, J, Tisdell, CC, Yuan, R: On the solvability of periodic boundary value problems with impulse. J. Math. Anal. Appl. **331**, 902-912 (2007)

9. Luo, Z, Nieto, JJ: New result for the periodic boundary value problem for impulsive integro-differential equations. Nonlinear Anal. TMA **70**, 2248-2260 (2009)
10. Lee, YH, Liu, X: Study of singular boundary value problems for second order impulsive differential equation. J. Math. Anal. Appl. **331**, 159-176 (2007)
11. Mawhin, J: Topological degree and boundary value problems for nonlinear differential equations. In: Topological Methods for Ordinary Differential Equations. Lecture Notes in Math., vol. 1537, pp. 74-142. Springer, Berlin (1993)
12. Qian, D, Li, X: Periodic solutions for ordinary differential equations with sublinear impulsive effects. J. Math. Anal. Appl. **303**, 288-303 (2005)
13. Chen, H, Li, J: Variational approach to impulsive differential equations with Dirichlet boundary conditions. Bound. Value Probl. **2010**, Article ID 3254152 (2010)
14. Liu, Z, Chen, H, Zhou, T: Variational methods to the second-order impulsive differential equation with Dirichlet boundary value problem. Comput. Math. Appl. **61**, 1687-1699 (2011)
15. Nieto, JJ, O'Regan, D: Variational approach to impulsive differential equations. Nonlinear Anal., Real World Appl. **70**, 680-690 (2009)
16. Sun, J, Chen, H, Yang, L: The existence and multiplicity of solutions for an impulsive differential equation with two parameters via a variational method. Nonlinear Anal. **73**, 440-449 (2010)
17. Xiao, J, Nieto, JJ, Luo, Z: Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods. Commun. Nonlinear Sci. Numer. Simul. **17**, 426-432 (2012)
18. Zhang, Z, Yuan, R: An application of variational methods to Dirichlet boundary value problem with impulses. Nonlinear Anal., Real World Appl. **11**, 155-162 (2010)
19. Zhang, D, Dai, B: Infinitely many solutions for a class of nonlinear impulsive differential equations with periodic boundary conditions. Comput. Math. Appl. **61**, 3153-3160 (2011)
20. Bai, L, Dai, B: An application of variational methods to a class of Dirichlet boundary value problems with impulsive effects. J. Franklin Inst. **348**, 2607-2624 (2011)
21. Chen, P, Tang, XH: New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects. Math. Comput. Model. **55**, 723-739 (2012)
22. Sun, J, Chen, H: Multiplicity of solutions for class of impulsive differential equations with Dirichlet boundary conditions via variant fountain theorems. Nonlinear Anal., Real World Appl. **11**, 4062-4071 (2010)
23. Wang, W, Yang, X: Multiple solutions of boundary-value problems for impulsive differential equations. Math. Methods Appl. Sci. **34**, 1649-1657 (2011)
24. Bonanno, G, Di Bella, B, Henderson, J: Existence of solutions to second-order boundary-value problems with small perturbations of impulses. Electron. J. Differ. Equ. **126**, 1-14 (2013)
25. Bonanno, G: A critical point theorem via the Ekeland variational principle. Nonlinear Anal. TMA **75**(5), 2992-3007 (2012)
26. Ricceri, B: A general variational principle and some of its applications. J. Comput. Appl. Math. **113**, 401-410 (2000)
27. Brezis, H: Analyse fonctionnelle; théorie et applications. Masson, Paris (1983)
28. Zhou, J, Li, Y: Existence and multiplicity of solutions for some Dirichlet problems with impulse effects. Nonlinear Anal. TMA **71**, 2856-2865 (2009)

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