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Singular degenerate problems occurring in biosorption process

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Abstract

The boundary value problems for singular degenerate arbitrary order differential-operator equations with variable coefficients are considered. The uniform coercivity properties of ordinary and partial differential equations with small parameters are derived in abstract L_p spaces. It is shown that corresponding differential operators are positive and also are generators of analytic semigroups. In application, well-posedness of the Cauchy problem for an abstract parabolic equation and systems of parabolic equations are studied in mixed L_p spaces. These problems occur in fluid mechanics and environmental engineering.

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0 Introduction

Boundary value problems (BVPs) for differential-operator equations (DOEs) in H -valued (Hilbert space valued) function spaces have been studied extensively by many researchers (see, e.g., [1–14] and the references therein). A comprehensive introduction to DOEs and historical references may be found in [6] and [14]. The maximal regularity properties for DOEs have been studied, e.g., in [3, 10–19].

In this work, singular degenerate BVPs for arbitrary order DOEs with parameters are considered. This problem has numerous applications. The parameter-dependent BVPs occur in different situations of fluid mechanics and environmental engineering *etc.*

In Section 2, the BVP for the following singular degenerate ordinary DOE with a small parameter:

$$Lu = (-1)^m ta(x)u^{[2m]} + A(x)u + \sum_{k=0}^{2m-1} t^{\frac{k}{2m}} A_k(x)u^{[k]} = f(x),$$

is considered, where

$$D_x^{[i]} u = u^{[i]}(x) = \left[x^\gamma \frac{d}{dx} \right]^i u(x), \quad x \in (0, 1),$$

t is a small parameter, $a(x)$ is a complex-valued function, $\gamma > 1$, $A = A(x)$ is a principal, $A_1 = A_1(x)$ and $A_2 = A_2(x)$ are subordinate linear operators in a Banach space E .

Several conditions for the uniform coercivity and the resolvent estimates for this problem are given in abstract L_p -spaces. We prove that the problem has a unique solution $u \in W_{p,\gamma}^{[2]}(0, 1; E(A), E)$ for $f \in L_p(0, 1; E)$, $|\arg \lambda| \leq \varphi$, $0 \leq \varphi < \pi$ with sufficiently large $|\lambda|$ and the following uniform coercive estimate holds:

$$\sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} t^{\sigma_i} \|u^{[i]}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \leq C \|f\|_{L_p(0,1;E)},$$

where

$$\sigma_i = \frac{1}{2p(\gamma-1)} + \frac{i}{2}.$$

In Section 3, the partial DOE with small parameters

$$\sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u + A(x)u + \sum_{|\alpha:2m|<1} \prod_{k=1}^n t_k^{\frac{\alpha_k}{2m_k}} A_\alpha(x) D^{[\alpha]} u = f(x),$$

$$x = (x_1, x_2, \dots, x_n) \in G$$

is considered in a mixed $L_p(G; E)$ space, where a_k are complex-valued functions, A and A_α are linear operators in E , λ is a complex and t_k are positive parameters, G is an n -dimensional rectangular domain, $\mathbf{p} = (p_1, p_2, \dots, p_n)$. Here we prove that for $f \in L_{\mathbf{p}}(G; E)$, $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$, this problem has a unique solution u that belongs to the Sobolev space $W_{\mathbf{p}}^{[2]}(G; E(A), E)$ with a mixed \mathbf{p} norm and the following coercive uniform estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^{2m_k} |\lambda|^{1-\frac{i}{2m_k}} t_k^{\sigma_{ki}} \|D_{x_k}^{[i]} u\|_{L_{\mathbf{p}}(G;E)} + \|Au\|_{L_{\mathbf{p}}(G;E)} \leq C \|f\|_{L_{\mathbf{p}}(G;E)},$$

where

$$D_{x_k}^{[i]} = \left[x_k^{\alpha_k} \frac{\partial}{\partial x_k} \right]^i, \quad \alpha_k > 1, \quad \sigma_{ki} = \frac{1}{2m_k p(\alpha_k - 1)} + \frac{i}{2m_k}, \quad k = 1, 2, \dots, n.$$

In Section 4, the uniform well-posedness of the mixed problem for the following singular degenerate abstract parabolic equation:

$$\frac{\partial u}{\partial y} + \sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u + A(x)u = f(x, y)$$

is obtained. Particularly, the above problem occurs in atmospheric dispersion of pollutants and evolution models for phytoremediation of metals from soils. In application, particularly, by taking $E = R^3$, $A(x) = [a_{ij}(x)]$, $u = (u_1, u_2, u_3)$, $i, j = 1, 2, 3$, we consider the mixed problem for the system of the following parabolic equations with parameters:

$$\frac{\partial u_i}{\partial y} + \sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u_i + \sum_{j=1}^3 a_{ij}(x) u_j = f_i(x, y)$$

which arises in phytoremediation process, where $a_{ij}(x)$ are real-valued functions and $f_i(x, y)$ are data. The maximal regularity properties of this problem are studied. Note that the maximal regularity properties for undegenerate DOEs were investigated, *e.g.*, in [1–10, 14–16, 19, 20]. Regular degenerate DOEs in Banach spaces were treated in [11–13, 15, 17–19, 21]. It should be noted that contrary to these results, here high-order singular degenerated BVPs with small parameters are considered. Moreover, principal coefficients depend on space variables. The proofs are based on abstract harmonic analysis, operator theory, interpolation of Banach spaces, theory of semigroups of linear operators, microlocal analysis, embedding and trace theorems in vector-valued Sobolev-Lions spaces.

1 Notations and background

Let $\gamma = \gamma(x)$, $x = (x_1, x_2, \dots, x_n)$ be a positive measurable function on a domain $\Omega \subset \mathbb{R}^n$. $L_{p,\gamma}(\Omega; E)$ denotes the space of strongly measurable E -valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega; E)} = \left(\int \|f(x)\|_E^p \gamma(x) dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p = L_p(\Omega; E)$.

The Banach space E is called a UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L_p(\mathbb{R}, E)$, $p \in (1, \infty)$ (see, *e.g.*, [22]). UMD spaces include, *e.g.*, L_p , l_p spaces and Lorentz spaces L_{pq} , $p, q \in (1, \infty)$.

Let C denote the set of complex numbers and

$$S_\varphi = \{\lambda; \lambda \in C, |\arg \lambda| \leq \varphi\} \cup \{0\}, \quad 0 \leq \varphi < \pi.$$

A linear operator A is said to be φ -positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and $\|(A + \lambda I)^{-1}\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$ for any $\lambda \in S_\varphi$, $0 \leq \varphi < \pi$, where I is the identity operator in E , $B(E)$ is the space of bounded linear operators in E . Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by A_λ . It is known [23, Section 1.15.1] that a positive operator A has well-defined fractional powers A^θ . Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the norm

$$\|u\|_{E(A^\theta)} = (\|u\|^p + \|A^\theta u\|^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty, 0 < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces continuously embedded in a locally convex space. By $(E_1, E_2)_{\theta, p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, we denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the K -method [23, Section 1.3.2].

Let $C(\Omega; E)$ denote the space of E -valued uniformly bounded continuous functions on the domain $\Omega \subset \mathbb{R}^n$.

Let N denote the set of natural numbers and $\{r_j\}$ be a sequence of independent symmetric $\{-1, 1\}$ -valued random variables on $[0, 1]$ (see [22]). A set $W_n \subset B(E_1, E_2)$ is called

uniform R -bounded with respect to h (see, e.g., [16]) if there is a constant C independent of $h \in Q \subset R$ such that for all $T_1(h), T_2(h), \dots, T_m(h) \in W_h$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy.$$

The smallest C for which the above estimate holds is called an R -bound of the collection W_h and is denoted by $R(W_h)$.

A φ -positive operator A is said to be R -positive in E if the set $L_A = \{\xi(A + \xi I)^{-1} : \xi \in S_\varphi\}$, $0 \leq \varphi < \pi$, is R -bounded.

Note that for Hilbert spaces H_1, H_2 , all norm-bounded sets are R -bounded (see, e.g., [16]). Therefore, in Hilbert spaces all positive operators are R -positive. If A is a generator of a contraction semigroup on L_q , $1 \leq q \leq \infty$, or A has the bounded imaginary powers with $\|A^{it}\|_{B(E)} \leq C e^{\nu|t|}$, $\nu < \frac{\pi}{2}$ in $E \in \text{UMD}$, then those operators are R -positive (e.g., see [16, Section 4.3]).

The operator $A(t)$, $t \in \sigma \subset R$ is said to be φ -positive in E uniformly with respect to $t \in \sigma$ if $D(A(t))$ is independent of t , $D(A(t))$ is dense in E and $\|(A(t) + \lambda)^{-1}\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, where M does not depend on t .

Let E_0 and E be two Banach spaces and E_0 be continuously and densely embedded into E . Let m be a positive integer. $W_{p,\gamma}^m(a, b; E_0, E)$ denotes an E_0 -valued function space defined by

$$W_{p,\gamma}^m(a, b; E_0, E) = \{u : u \in L_{p,\gamma}(a, b; E_0), u^{(m)} \in L_{p,\gamma}(a, b; E)\},$$

$$\|u\|_{W_{p,\gamma}^m} = \|u\|_{W_{p,\gamma}^m(a, b; E_0, E)} = \|u\|_{L_{p,\gamma}(a, b; E_0)} + \|u^{(m)}\|_{L_{p,\gamma}(a, b; E)} < \infty.$$

Let t be a positive parameter. We define a parameterized norm in $W_{p,\gamma}^m(a, b; E_0, E)$ as follows:

$$\|u\|_{W_{p,\gamma,t}^m} = \|u\|_{W_{p,\gamma,t}^m(a, b; E_0, E)} = \|u\|_{L_{p,\gamma}(a, b; E_0)} + \|tu^{(m)}\|_{L_{p,\gamma}(a, b; E)} < \infty.$$

Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$, $G = \prod_{k=1}^n (0, b_k)$, $L_{\mathbf{p}}(G; E)$ denote the space of all \mathbf{p} -summable E -valued functions with a mixed norm (see, e.g., [24, Section 8] for scalar case), i.e., the space of all measurable E -valued functions f defined on G , for which

$$\|f\|_{L_{\mathbf{p}}(G; E)} = \left(\int_0^{b_1} \left(\int_0^{b_2} \cdots \left(\int_0^{b_n} \|f(x)\|^{p_n} dx_n \right)^{\frac{p_{n-1}-1}{p_n}} dx_{n-1} \right) \cdots dx_1 \right)^{\frac{1}{p_1}} < \infty.$$

Let l_k be positive integers, $l = (l_1, l_2, \dots, l_n)$, t_k be positive parameters and $t = (t_1, t_2, \dots, t_n)$.

$$D_{x_k}^{[l]} u = \left[x_k^{\alpha_k} \frac{\partial}{\partial x_k} \right]^i u(x), \quad \alpha_k > 0, \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), k = 1, 2, \dots, n.$$

Consider the following weighted spaces of functions:

$$W_{\mathbf{p},\alpha}^{[l]}(G; E(A), E) = \{u : u \in L_{\mathbf{p}}(G; E(A)), D_k^{[l_k]} u \in L_{\mathbf{p}}(G; E)\},$$

with the mixed norm

$$\|u\|_{W_{p,\alpha}^{[l]}(G;E(A),E)} = \|u\|_{L_p(G;E_0)} + \sum_{k=1}^n \|D_k^{[l_k]} u\|_{L_p(G;E)}$$

and with the parameterized norm

$$\|u\|_{W_{p,\alpha,t}^{[l]}(G;E(A),E)} = \|u\|_{L_p(G;E_0)} + \sum_{k=1}^n \|t_k D_k^{[l_k]} u\|_{L_p(G;E)},$$

respectively.

Consider the BVP for DOE

$$(L + \lambda)u = (-1)^m t u^{[2m]}(x) + (A + \lambda)u(x) = f(x), \quad x \in (0, 1), \quad (1)$$

$$L_k u = \sum_{i=0}^{m_k} \left[\alpha_{ki} u^{[i]}(1) + \sum_{i=1}^N \delta_{ki} u^{[i]}(x_{ki}) \right] = 0,$$

where $u^{[i]} = [x^\gamma \frac{d}{dx}]^i u(x)$, α_{ki} , δ_{ki} are complex numbers and $x_{ki} \in (0, 1)$, $m_k \in \{0, 1, \dots, 2m - 1\}$; A is a possible unbounded operator in E .

In a similar way as in [17, Theorem 5.1], we obtain the following.

Theorem A₁ *Let the following conditions be satisfied:*

- (1) α_{ki} , δ_{kj} are complex numbers, $\alpha_{km_k} \neq 0$, t is a small positive parameter and $\sigma_i = \frac{1}{2mp(\gamma-1)} + \frac{i}{2m}$;
- (2) E is a UMD space, $\gamma > 1 + \frac{1+\sqrt{4p+1}}{2p}$, $1 < p < \infty$;
- (3) A is an R -positive operator in E .

Then problem (1) for $f \in L_p(0, 1; E)$ and $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$ has a unique solution $u \in W_{p,\gamma}^{[2m]}(0, 1; E(A), E)$. Moreover, the following uniform coercive estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u^{[i]}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \leq C \|f\|_{L_p(0,1;E)}.$$

By reasoning as in [17, Theorem 2.3], we obtain the following.

Theorem A₂ *Let the following conditions be satisfied:*

- (1) $\gamma(x) = x^{\frac{\gamma}{1-\gamma}}$, $\gamma > 1 + \frac{1+\sqrt{4p+1}}{2p}$, $p \in (1, \infty)$, $0 < t < 1$;
- (2) E is a UMD space and A is an R -positive operator in E ;
- (3) there exists a bounded linear extension operator from $W_{p,\gamma}^m(1, \infty; E(A), E)$ to $W_{p,\gamma}^m(R; E(A), E)$.

Then the embedding $D^j W_{p,\gamma}^m(1, \infty; E(A), E) \subset L_{p,\gamma}(1, \infty; E(A^{1-\frac{j}{m}-\mu}))$ is continuous and for $0 \leq \mu \leq 1 - \frac{j}{m}$, the uniform estimate

$$\left\| t^{\frac{j}{m}} u^{(j)} \right\|_{L_{p,\gamma}(1, \infty; E(A^{1-\frac{j}{m}-\mu}))} \leq h^\mu \|u\|_{W_{p,\gamma,t}^m(1, \infty; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_{p,\gamma}(1, \infty; E)}$$

holds for all $u \in W_{p,\gamma}^m(1, \infty; E(A), E)$ and $0 < h \leq h_0 < \infty$.

Let

$$D^{[\alpha]} = D_1^{[\alpha_1]} D_2^{[\alpha_2]} \cdots D_n^{[\alpha_n]}, \quad t = (t_1, t_2, \dots, t_n), \quad \psi(t) = \prod_{k=1}^n t_k^{\frac{\alpha_k}{2m_k}},$$

$$0 < t_k < 1, \quad G = \prod_{k=1}^n (0, b_k).$$

Theorem A₃ *Let the following conditions be satisfied:*

- (1) $\mathbf{p} = (p_1, p_2, \dots, p_n)$, $p_k \in (1, \infty)$, $\gamma_k(x) = x_k^{\gamma_k}$;
- (2) E is a UMD space and A is an R -positive operator in E ;
- (3) $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $l = (l_1, l_2, \dots, l_n)$ are n -tuples of a nonnegative integer such that

$$\varkappa = |\alpha : l| = \sum_{k=1}^n \frac{\alpha_k}{l_k} \leq 1, \quad 1 < p_k < \infty, \quad \gamma_k > 1 + \frac{1 + \sqrt{4p_k + 1}}{2p_k};$$

- (4) *there exists a bounded linear extension operator from $W_{\mathbf{p}, \gamma}^{[l]}(G; E(A), E)$ to $W_{\mathbf{p}, \gamma}^{[l]}(R^n; E(A), E)$.*

Then the embedding $D^\alpha W_{\mathbf{p}, \gamma}^{[l]}(G; E(A), E) \subset L_{\mathbf{p}}(G; E(A^{1-\varkappa-\mu}))$ is continuous. Moreover, there is a constant $h > 0$ such that for $u \in W_{\mathbf{p}, \gamma}^{[l]}(\Omega; E(A), E)$, the following uniform estimate holds:

$$\psi(t) \|D^{[\alpha]} u\|_{L_{\mathbf{p}}(G; E(A^{1-\varkappa-\mu}))} \leq h^\mu \|u\|_{W_{\mathbf{p}, \gamma, t}^{[l]}(G; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_{\mathbf{p}}(G; E)}.$$

2 Singular degenerate DOEs with parameter

Consider the BVP for the following differential-operator equation with parameter:

$$Lu = (-1)^m t a(x) u^{[2m]} + A_\lambda(x) u + \sum_{k=0}^{2m-1} t^{\frac{k}{2m}} A_k(x) u^{[k]} = f(x), \quad (2)$$

$$L_k u = \sum_{i=0}^{m_k} \left[\alpha_{ki} u^{[i]}(1) + \sum_{i=1}^N \delta_{ki} u^{[i]}(x_{ki}) \right] = 0, \quad k = 1, 2, \dots, m,$$

on the domain $(0, 1)$, where t is a positive parameter and λ is a complex parameter; α_{ki} , δ_{ki} are complex numbers and $x_{ki} \in (0, 1)$, $m_k \in \{0, 1, \dots, 2m_k - 1\}$, $a(x)$ is a complex-valued function on $[0, 1]$; $A(x)$ and $A_k(x)$ are linear operators in a Banach space E and $A_\lambda(x) = A(x) + \lambda$. Note that

$$\int_0^1 z^{-\gamma} dz = \infty.$$

A function $u \in W_{\mathbf{p}, \gamma}^{[2m]}(0, 1; E(A), E)$ satisfying equation (2) a.e. on $(0, 1)$ is said to be the solution of equation (2) on $(0, 1)$.

Remark 1 Let

$$y = - \int_x^1 z^{-\gamma} dz. \quad (3)$$

Under the substitution (3), spaces $L_p(0, 1; E)$ and $W_{p,\gamma}^{[2m]}(0, 1; E(A), E)$ are mapped isomorphically onto weighted spaces

$$L_{p,\tilde{\gamma}}(-\infty, 0; E), \quad W_{p,\tilde{\gamma}}^2(-\infty, 0; E(A), E),$$

respectively, where

$$\tilde{\gamma} = \tilde{\gamma}(x(y)) = [1 - (\gamma - 1)y]^{\frac{\gamma}{1-\gamma}}.$$

Moreover, under the substitution (3), problem (2) is transformed into the following non-degenerate problem:

$$Lu = (-1)^m ta(y)u^{(2m)}(y) + A_\lambda(y)u(y) + \sum_{k=0}^{2m-1} t^{\frac{k}{2m}} A_k(y)u^{(k)}(y) = f(y), \quad (4)$$

$$L_k u = \sum_{i=0}^{m_k} \left[\alpha_{ki} u^{(i)}(0) + \sum_{i=1}^N \delta_{ki} u^{[i]}(y_{ki}) \right] = 0, \quad k = 1, 2, \dots, m \quad (5)$$

in the weighted space $L_{p,\tilde{\gamma}}(-\infty, 0; E)$, where $\tilde{a}(y) = a(\tilde{\gamma}(x(y)))$, $\tilde{A}(y) = A(\tilde{\gamma}(x(y)))$, $\tilde{A}_k(y) = A_k(\tilde{\gamma}(x(y)))$, $\tilde{\gamma}(y) = \gamma(\tilde{\gamma}(x(y)))$ are again denoted by $a(y)$, $A(y)$, $A_k(y)$, γ after the substitution (3), respectively.

Let us consider boundary value problem (4)-(5).

Theorem 1 *Let the following conditions be satisfied:*

- (1) $\alpha_{km_k} \neq 0$, $a(y)$ is a positive uniformly bounded continuous function on $(-\infty, 0)$;
- (2) E is a UMD space, $\gamma > 1 + \frac{1+\sqrt{4p+1}}{2p}$, $1 < p < \infty$, and $\sigma_i = \frac{1}{2mp(\gamma-1)} + \frac{i}{2m}$;
- (3) $A(y)$ is R -positive in E uniformly with respect to $y \in [-\infty, 0]$ and $A(y)A^{-1}(y_0) \in C((-\infty, 0); B(E))$, $y_0 \in [-\infty, 0]$;
- (4) for any $\varepsilon > 0$, there is a positive $C(\varepsilon)$ such that

$$\|A_k(y)u\| \leq \varepsilon \|u\|_{(E(A), E)_{\frac{k}{2m}, \infty}} + C(\varepsilon) \|u\| \quad \text{for } u \in (E(A), E)_{\frac{k}{2m}, \infty}, k = 0, 1, \dots, 2m-1.$$

Then problem (4)-(5) has a unique solution $u \in W_{p,\gamma}^2(-\infty, 0; E(A), E)$ for $f \in L_{p,\tilde{\gamma}}(-\infty, 0; E)$ and $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u^{(i)}\|_{p,\gamma} + \|Au\|_{p,\gamma} \leq C \|f\|_{p,\gamma}. \quad (6)$$

Proof Let G_1, G_2, \dots be bounded intervals in $(-\infty, 0)$ and $\varphi_1, \varphi_2, \dots$ correspond to a partition of unit that functions φ_j are smooth on $(-\infty, 0)$, $\sigma_j = \text{supp } \varphi_j \subset G_j$ and $\sum_{j=1}^{\infty} \varphi_j(y) = 1$. Then, for all $u \in W_{p,\gamma}^{2m}(-\infty, 0; E(A), E)$, we have $u(y) = \sum_{j=1}^{\infty} u_j(y)$, where $u_j(y) = u(y)\varphi_j(y)$. For $u \in W_{p,\gamma}^2(-\infty, 0; E(A), E)$, from (4) we obtain

$$(L + \lambda)u_j = (-1)^m ta(y)u_j^{(2m)}(y) + A_\lambda(y)u_j(y) = f_j(y), \quad L_k u_j = 0, \quad (7)$$

where

$$f_j = f\varphi_j - (-1)^m a \sum_{i=0}^{2m-1} C_{2m}^i u^{(i)} \varphi_j^{(2m-i)} + \sum_{k=1}^{2m-1} \sum_{i=0}^k C_k^i t^{\frac{k}{2m}} u^{(i)} \varphi_j^{(k-i)} A_k. \quad (8)$$

Since a is uniformly bounded on $(-\infty, 0)$ for all small $\rho > 0$, there is a large $r_0 > 0$ such that $|a(y) - a(-\infty)| \leq \delta$ for all $|y| \geq r_0$. Let

$$G_0 = R^n \setminus \sigma_{r_0}(0), \quad \sigma_{r_0}(0) = \{y \in (-\infty, 0), |y| \leq r_0\}.$$

Cover $\sigma_{r_0}(0)$ by finitely many intervals $G_j = \sigma_{r_j}(y_{0j})$ such that

$$|a(y) - a(y_{0j})| \leq \delta \quad \text{for } |y - y_{0j}| \leq r_j, j = 1, 2, \dots$$

Define coefficients of local operators, i.e.,

$$a^0(y) = \begin{cases} a(y), & y \notin \sigma_{r_0}(0) \\ a(r_0^2 \frac{y}{|y|^2}), & y \in \sigma_{r_0}(0) \end{cases},$$

$$a^j(y) = \begin{cases} a(y), & y \in \sigma_{r_j}(y_{0j}) \\ a(y_{0j} + r_0^2 \frac{y-y_{0j}}{|y-y_{0j}|^2}), & y \notin \sigma_{r_j}(y_{0j}) \end{cases}$$

and

$$A^0(y)A^{-1}(y_{0j}) = \begin{cases} A(y)A^{-1}(y_{0j}), & y \notin \sigma_{r_0}(0) \\ A(r_0^2 \frac{y}{|y|^2})A^{-1}(y_{0j}), & y \in \sigma_{r_0}(0) \end{cases},$$

$$A^j(y)A^{-1}(y_{0j}) = \begin{cases} A(y)A^{-1}(y_{0j}), & y \in \sigma_{r_j}(y_{0j}) \\ A(y_{0j} + r_0^2 \frac{y-y_{0j}}{|y-y_{0j}|^2})A^{-1}(y_{0j}), & y \notin \sigma_{r_j}(y_{0j}) \end{cases}$$

for each $j = 1, 2, \dots$. Then, for all $y \in (-\infty, 0)$ and $j = 0, 1, 2, \dots$, we get

$$|a^j(x) - a^j(y_{0j})| \leq \delta \quad \text{and} \quad \|A^j(y)A^{-1}(y_{0j}) - A^j(y_{0j})A^{-1}(y_{0j})\|_{B(E)} < \delta.$$

Freezing coefficients in (7) obtain that

$$(-1)^m a(y_{0j}) t u_j^{(2m)}(y) + A_\lambda(y_{0j}) u_j(y) = F_j(y), \quad L_k u_j = 0, \quad (9)$$

where

$$F_j = f_j + [A(y_{0j}) - A(y)] u_j + (-1)^m t [a(y) - a(y_{0j})] u_j^{(2m)}, \quad j = 1, 2, \dots \quad (10)$$

Since functions $u_j(x)$ have compact supports in (9), if we extend $u_j(x)$ on the outsides of $\sigma_j = \text{supp } \varphi_j$, we obtain BVPs with constant coefficients

$$-ta(y_{0j}) u_j^{(2m)} + A_\lambda(y_{0j}) u_j = F_j, \quad L_k u_j = 0, \quad j = 1, 2, \dots \quad (11)$$

Let $\|\cdot\|_{G_j,p,\gamma}$ denote E -valued weighted L_p -norms with respect to domains G_j . Let φ_j be such that $0 \in \sigma_j$. Then, by virtue of Theorem A₁, we obtain that problem (11) has a unique solution u_j and for $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$, the following estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u_j^{(i)}\|_{G_j,p,\gamma_1} + \|Au_j\|_{G_j,p,\gamma_1} \leq C \|F_j\|_{G_j,p,\gamma_1}. \quad (12)$$

Theorem A₂ implies that for all $\varepsilon > 0$, there is a continuous function $C(\varepsilon)$ such that

$$\|F_j\|_{G_j,p,\gamma} \leq \varepsilon \|u_j\|_{W_{p,\gamma,t}^{2m}(G_j;E(A),E)} + C(\varepsilon) \|f_j\|_{G_j,p,\gamma}, \quad j = 1, 2, \dots \quad (13)$$

Consequently, by using Theorem A₂, from (12)-(13) we get

$$\begin{aligned} \sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u_j^{(i)}\|_{G_j,p,\gamma} + \|Au_j\|_{G_j,p,\gamma} \\ \leq C \|f_j\|_{G_j,p,\gamma} + \varepsilon \|u_j\|_{W_{p,\gamma}^{2m}} + C(\varepsilon) \|u_j\|_{G_j,p,\gamma}. \end{aligned} \quad (14)$$

Choosing $\varepsilon < 1$, from (14) we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u_j^{(i)}\|_{G_j,p,\gamma} + \|Au_j\|_{G_j,p,\gamma} \leq C [\|f_j\|_{G_j,p,\gamma} + \|u_j\|_{G_j,p,\gamma}]. \quad (15)$$

Then, by using the equality $u(y) = \sum_{j=1}^{\infty} u_j(y)$ and by virtue of (15) for $u \in W_{p,\gamma}^{2m}(-\infty, 0; E(A), E)$, we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u^{(i)}\|_{p,\gamma} + \|Au_j\|_{p,\gamma} \leq C [\|(L + \lambda)u\|_{p,\gamma} + \|u\|_{p,\gamma}]. \quad (16)$$

Let $u \in W_{p,\gamma}^{2m}(-\infty, 0; E(A), E)$ be a solution of problem (4)-(5). For $|\arg \lambda| \leq \varphi$, we have

$$\|u\|_{p,\gamma} = \|(L + \lambda)u - Lu\|_{p,\gamma} \leq \frac{1}{\lambda} [\|(L + \lambda)u\|_{p,\gamma} + \|u\|_{W_{p,\gamma}^{2m}}]. \quad (17)$$

By Theorem A₂, by virtue of (16) and (17) for sufficiently large $|\lambda|$, we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u^{(i)}\|_{p,\gamma} + \|Au\|_{p,\gamma} \leq C \|(L + \lambda)u\|_{p,\gamma}. \quad (18)$$

Consider the operator $O_{t\lambda}$ in $L_{p,\gamma}(-\infty, 0; E)$ generated by problem (4)-(5), i.e.,

$$\begin{aligned} D(O_\lambda) &= W_{p,\gamma}^{2m}(-\infty, 0; E(A), E, L_k), \\ O_{t\lambda} u &= (-1)^m \tau a u^{(2m)} + A_\lambda u + \sum_{k=0}^{2m-1} t^{\frac{k}{2m}} A_k(y) u^{(k)}. \end{aligned}$$

Estimate (18) implies that problem (4)-(5) has only a unique solution and the operator O_λ has an invertible operator in its rank space. We need to show that this rank space

coincides with the space $X = L_{p,\gamma}(-\infty, 0; E)$. We consider the smooth functions $g_j = g_j(y)$ with respect to the partition of the unit $\varphi_j = \varphi_j(y)$ on $(-\infty, 0)$ that equals one on $\text{supp } \varphi_j$, where $\text{supp } g_j \subset G_j$ and $|g_j(y)| < 1$. Let us construct for all j the function u_j that is defined on the regions $\Omega_j = (-\infty, 0) \cap G_j$ and satisfies problem (4)-(5). Problem (4)-(5) can be expressed in the form

$$\begin{aligned} & (-1)^m t a(y_{0j}) u_j^{(2)} + A_\lambda(y_{0j}) u_j \\ & = g_j \{ F_j + [A(y_{0j}) - A(y)] u_j + (-1)^m t [a(y) - a(y_{0j})] u_j^{(2)} \}, \\ & L_k u_j = 0, \quad j = 1, 2, \dots \end{aligned} \quad (19)$$

Consider operators $O_{j\lambda}$ in $L_{p,\gamma}(G_j; E)$ generated by BVPs (19). By virtue of Theorem A₁ for all $f \in L_{p,\gamma}(G_j; E)$, for $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$, we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \left\| \frac{d^i}{dy^i} O_{j\lambda}^{-1} f \right\|_{p,\gamma} + \|A O_{j\lambda}^{-1} f\|_{p,\gamma} \leq C \|f\|_{p,\gamma}. \quad (20)$$

Extending u_j zero on the outside of $\text{supp } \varphi_j$ in equalities (20) and passing substitutions $u_j = O_{j\lambda}^{-1} v_j$, we obtain operator equations with respect to v_j

$$v_j = K_{j\lambda} v_j + g_j f, \quad j = 1, 2, \dots \quad (21)$$

By virtue of Theorem A₂, by estimate (20), in view of the smoothness of the coefficients of $K_{j\lambda}$, for $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$, we have $\|K_{j\lambda}\| < \varepsilon$, where ε is sufficiently small. Consequently, equations (21) have unique solutions

$$v_j = [I - K_{j\lambda}]^{-1} g_j f.$$

Moreover,

$$\|v_j\|_{p,\gamma} = \|[I - K_{j\lambda}]^{-1} g_j f\|_{p,\gamma} \leq \|f\|_{p,\gamma}.$$

Whence, $[I - K_{j\lambda}]^{-1} g_j$ are bounded linear operators from X to $L_{p,\gamma}(G_j; E)$. Thus, we obtain that the functions

$$u_j = U_{j\lambda} f = O_{j\lambda}^{-1} [I - K_{j\lambda}]^{-1} g_j f$$

are the solutions of equations (21). Consider the linear operator $(U_t + \lambda)$ in X such that

$$(U_t + \lambda)f = \sum_{j=1}^{\infty} \varphi_j(y) U_{j\lambda} f.$$

It is clear from the constructions $U_{j\lambda}$ and estimate (20) that operators $U_{j\lambda}$ are bounded linear from X to $W_{p,\gamma}^{2m}(-\infty, 0; E(A), E)$ and

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \left\| \frac{d^i}{dy^i} U_{j\lambda}^{-1} f \right\|_{p,\gamma} + \|A U_{j\lambda}^{-1} f\|_{p,\gamma} \leq C \|f\|_{p,\gamma}. \quad (22)$$

Therefore, $(U_t + \lambda)$ is a bounded linear operator from X to X . Let \mathbf{L}_t denote the operator in $L_{p,\gamma}$ generated by BVP (4)-(5). Then the act of $(\mathbf{L}_t + \lambda)$ to $u = \sum_{j=1}^{\infty} \varphi_j U_{jt\lambda} f$ gives $(\mathbf{L}_t + \lambda)u = f + \sum_{j=1}^{\infty} \Phi_{jt\lambda} f$, where $\Phi_{jt\lambda}$ is a linear combination of $U_{j\lambda}$ and $\frac{d}{dy} U_{j\lambda}$. By virtue of Theorem A₂, estimate (22) and in view of the expression $\Phi_{jt\lambda}$, we obtain that operators $\Phi_{jt\lambda}$ are bounded linear from X to $L_{p,\gamma}(G_j; E)$ and $\|\Phi_{jt\lambda}\| < \varepsilon$. Therefore, there exists a bounded linear invertible operator $(I + \sum_{j=1}^{\infty} \Phi_{jt\lambda})^{-1}$. Whence, we obtain that for all $f \in X$, BVP (4)-(5) has a unique solution

$$\begin{aligned} u(y) &= (\mathbf{L}_t + \lambda)^{-1} f = (U_t + \lambda) \left(I + \sum_{j=1}^{\infty} \Phi_{jt\lambda} \right)^{-1} f \\ &= \sum_{j=1}^{\infty} \varphi_j(y) O_{jt\lambda}^{-1} [I - K_{jt\lambda}]^{-1} \left(I + \sum_{j=1}^{\infty} \Phi_{jt\lambda} \right)^{-1} f. \end{aligned} \quad (23)$$

Then, by using the above representation and in view of Theorem A₁, we obtain the assertion of Theorem 1. \square

Result 1 Theorem 1 implies that the operator \mathbf{L}_t has a resolvent $(\mathbf{L}_t + \lambda)^{-1}$ for $|\arg \lambda| \leq \varphi$ and the following estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|D^i(\mathbf{L}_t + \lambda)^{-1}\|_{B(X)} + \|A(\mathbf{L}_t + \lambda)^{-1}\|_{B(X)} \leq C.$$

Let \mathbf{G}_t denote the operator in $L_p(0, 1; E)$ generated by BVP (2). By virtue of Theorem 1 and Remark 1, we obtain the following result.

Result 2 Let all conditions of Theorem 1 be satisfied. Then

- (a) problem (2) has a unique solution $u \in W_{p,\gamma}^{[2m]}(0, 1; E(A), E)$ for $f \in L_p(0, 1; E)$ and sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|u^{[i]}\|_p + \|Au\|_p \leq C \|f\|_p;$$

- (b) \mathbf{G}_t has a resolvent operator $(\mathbf{G}_t + \lambda)^{-1}$ for $|\arg \lambda| \leq \varphi$ and

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\sigma_i} \|D^{[i]}(\mathbf{G}_t + \lambda)^{-1}\|_{B(L_p(0,1;E))} + \|A(\mathbf{G}_t + \lambda)^{-1}\|_{B(L_p(0,1;E))} \leq C.$$

Theorem 2 Let all conditions of Theorem 1 hold. Then the operator \mathbf{L}_t is uniformly R -positive in $L_p(0, 1; E)$, also \mathbf{L}_t is a generator of an analytic semigroup.

Proof By virtue of Theorem 1, we obtain that for $f \in L_p(0, 1; E)$, BVP (4)-(5) has a unique solution expressed in the form

$$\begin{aligned} u(y) &= (\mathbf{L}_t + \lambda)^{-1} f = (U_t + \lambda) \left(I + \sum_{j=1}^{\infty} \Phi_{jt\lambda} \right)^{-1} f \\ &= \sum_{j=1}^{\infty} \varphi_j(y) O_{jt\lambda}^{-1} [I - K_{jt\lambda}]^{-1} \left(I + \sum_{j=1}^{\infty} \Phi_{jt\lambda} \right)^{-1} f, \end{aligned}$$

where $O_{j\lambda t} = O_{jt} + \lambda$ are local operators generated by problems (7)-(8) and $K_{jt\lambda}$, $\Phi_{jt\lambda}$ are uniformly bounded operators in $L_p(0, 1; E)$. In a similar way as in [1, 11, 17], we obtain that operators O_{jt} are R -positive. Then, by using the above representation and by virtue of Kahane's contraction principle, the product and additional properties of the collection of R -bounded operators (see, e.g., [16], Lemma 3.5, Proposition 3.4), we obtain the assertions. \square

3 Singular degenerate anisotropic equation with parameters

Consider the following degenerate BVP with parameters:

$$\sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u + \sum_{|\beta: 2m| < 1} \prod_{k=1}^n t_k^{\frac{\alpha_k}{2m_k}} A_\beta(x) D^{[\beta]} u + A(x) + \lambda u = f(x), \quad (24)$$

$$L_{kj} u = \sum_{i=0}^{m_{kj}} \alpha_{kji} u_{x_k}^{[i]}(G_{kb}) = 0, \quad j = 1, 2, \dots, m_k, k = 1, 2, \dots, n,$$

where $A(x)$ and $A_\alpha(x)$ are linear operators in a Banach space E ,

$$G = \{x = (x_1, x_2, \dots, x_n), 0 < x_k < b_k, k = 1, 2, \dots, n\},$$

$$\beta = (\beta_1, \beta_2, \dots, \beta_n), \quad m = (m_1, m_2, \dots, m_n), \quad |\beta : 2m| = \sum_{k=1}^n \frac{\beta_k}{2m_k},$$

$$\gamma_k > 1, \quad G_{kb} = (x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n),$$

$$G_{kb} = (x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n), \quad 0 \leq m_{kj} \leq 2m_k - 1,$$

a_k are complex-valued functions on G , α_{kji} are complex numbers, t_k are positive and λ is a complex parameter.

Note that BVP (24) is degenerated with different speeds on different directions in general.

The main result of this section is the following.

Theorem 3 Assume the following conditions hold:

- (1) E is a UMD space, $A(x)$ is R -positive in E uniformly with respect to $x \in G$ and $A(x)A^{-1}(x_0) \in C(\bar{G}; B(E))$, $x_0 \in [-\infty, 0]$;
- (2) for any $\varepsilon > 0$, there is a positive $C(\varepsilon)$ such that

$$\|A_\beta(x)u\| \leq \varepsilon \|u\|_{(E(A), E)_{1-|\beta: 2m|, \infty}} + C(\varepsilon) \|u\| \quad \text{for } u \in (E(A), E)_{1-|\alpha: 2m|, \infty};$$

- (3) $\gamma_k > 1 + \frac{1+\sqrt{4p+1}}{2p}$, $\sigma_{ki} = \frac{1}{2m_k p(\alpha_k - 1)} + \frac{i}{2m_k}$, $i = 0, 1, \dots, 2m_k$, $p_k \in (1, \infty)$;

- (4) a_k are continuous positive functions on \bar{G} .

Then, for $f \in L_p(G; E)$, $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$, problem (24) has a unique solution u that belongs to $W_{p, \alpha}^{[2m]}(G; E(A), E)$ and the following coercive uniform estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^{2m_k} |\lambda|^{1 - \frac{i}{2m_k}} t_k^{\sigma_{ki}} \|D_{x_k}^{[i]} u\|_{L_p(G; E)} + \|Au\|_{L_p(G; E)} \leq C \|f\|_{L_p(G; E)}. \quad (25)$$

Proof Consider the BVP

$$\begin{aligned} (-1)^{m_1} a_1(x_1) t_1 D_{x_1}^{[2m_1]} u(x_1) + A(x) u(x_1) + \lambda u(x_1) &= f(x_1), \\ x_1 \in (0, b_1), \quad L_{1k} u &= 0, \quad k = 1, 2, \dots, m_1, \end{aligned} \quad (26)$$

where L_{1k} are boundary conditions of type (24) on $(0, b_1)$. By virtue of Result 2, problem (26) has a unique solution $u \in W_{p_1, \alpha_1}^{[2]}(0, b_1; E(A), E)$ for all $f \in L_{p_1}(0, b_1; E)$, $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$. Moreover, the following coercive uniform estimate holds:

$$\sum_{i=0}^{2m_1} |\lambda|^{1-\frac{i}{2m_1}} t_1^{\sigma_{1i}} \|u^{[i]}\|_{L_{p_1}(0, b_1; E)} + \|Au\|_{L_{p_1}(0, b_1; E)} \leq C \|f\|_{L_{p_1}(0, b_1; E)}.$$

Let us now consider in $L_{p_1, p_2}(G_2; E)$ the BVP on the domain G_2

$$\begin{aligned} \sum_{k=1}^2 (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u + \lambda u &= f(x_1, x_2), \\ L_{kj} u &= 0, \quad j = 1, 2, \dots, m_k, k = 1, 2, \end{aligned} \quad (27)$$

where $G_2 = (0, b_1) \times (0, b_2)$. Since $L_{p_2}(0, b_2; L_{p_1}(0, b_1; E)) = L_{p_1, p_2}(G_2; E)$, then problem (27) can be expressed in the following way:

$$(-1)^{m_2} t_2 a_2 D_{x_2}^{[2m_2]} u(x_2) + (B + \lambda) u(x_2) = f(x_2), \quad L_{2k} u = 0,$$

where B is the differential operator in $L_{p_1}(0, b_1; E)$ generated by problem (26), i.e.,

$$D(B) = W_{p_1, \alpha_1}^{[2m_1]}(0, b_1; L_{1k}), \quad Bu = a_1(x_1) t_2 D^2 u(x_1) + du(x_1).$$

By virtue of [22], $L_{p_1}(0, b_1; E) \in \text{UMD}$ for $p_1 \in (1, \infty)$ provided $E \in \text{UMD}$. Moreover, by virtue of Theorem 2, the operator B is R -positive in $L_{p_1}(0, b_1; E)$. Hence, by Result 2, we get that problem (27) has a unique solution

$$u \in W_{p_1, p_2, \tilde{\alpha}}^{[2]}(G_2; E(A), E), \quad \tilde{\alpha} = (\alpha_1, \alpha_2)$$

for $f \in L_{p_1, p_2}(G_2; E)$, $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$, and coercive uniform estimate (25) holds. By continuing this process for $k = n$, we obtain that the following problem:

$$\begin{aligned} \sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u + A(x) u + \lambda u &= f(x), \\ L_{kj} u &= 0, \quad j = 1, 2, \dots, m_k, k = 1, 2, \dots, n, \end{aligned} \quad (28)$$

for $f \in L_{\mathbf{p}}(G; E)$, $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$, has a unique solution $u \in W_{\mathbf{p}, \alpha}^{[2m]}(G; E(A), E)$ and the following coercive uniform estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t_k^{\sigma_{ki}} \|D_{x_k}^{[i]} u\|_{L_{\mathbf{p}}(G; E)} + \|Au\|_{L_{\mathbf{p}}(G; E)} \leq C \|f\|_{L_{\mathbf{p}}(G; E)}. \quad (29)$$

Moreover, by virtue of embedding Theorem A₃, we have the Ehrling-Nirenberg-Gagliardo type estimate

$$\|L_1 u\|_{L_p(G;E)} \leq h^\mu \|u\|_{W_{\mathbf{p},\alpha}^{[2]}(G;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(G;E)}, \quad h \in (0,1).$$

Let Q_t denote the operator generated by problem (28) and

$$L_1 u = \sum_{|\beta:2m|<1} \prod_{k=1}^n t_k^{\frac{\beta_k}{2m_k}} A_\beta(x) D^{[\beta]} u.$$

By using estimate (29), we obtain that there is a $\delta \in (0,1)$ such that

$$\|L_1(Q_t + \lambda)^{-1}\|_{L(L_p(G;E))} < \delta.$$

Then, by using perturbation elements, we obtain the assertion. \square

From Theorem 2 and Theorem 3, we obtain the following result.

Result 3 Let all conditions of Theorem 3 hold for $\varphi > \frac{\pi}{2}$ and $A_\alpha = 0$. Then the operator Q_t is uniformly R -positive in $L_p(G;E)$, it also is a generator of an analytic semigroup.

4 Singular degenerate parabolic DOE

Consider the following mixed problem for a parabolic DOE with parameter:

$$\begin{aligned} \frac{\partial u}{\partial y} + \sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u + (A(x) + d)u &= f(y, x), \quad y \in R_+, x \in G \\ L_{kj} u &= \sum_{i=0}^{m_{kj}} \alpha_{kji} u_{x_k}^{[i]}(G_{kb}) = 0, \quad j = 1, 2, \dots, m_k, \\ u(0, x) &= 0, \end{aligned} \quad (30)$$

where $t_k, a_k(x), G, G_{kb}, m_{kj}$ are defined as in Section 3, $d > 0$.

For $\tilde{\mathbf{p}} = (p_1, p_2, \dots, p_n, p_0)$, $\Delta_+ = R_+ \times G$, $L_{\tilde{\mathbf{p}}}(\Delta_+; E)$ will denote the space of all E -valued $\tilde{\mathbf{p}}$ -summable functions with a mixed norm. Analogously,

$$W_{\tilde{\mathbf{p}},\gamma}^{[1,2m]}(\Delta_+, E(A), E)$$

denotes the Sobolev space with a corresponding mixed norm (see [24] for a scalar case).

Let Q_t denote a differential operator generated by (28) for $\lambda = 0$.

Theorem 4 Let all conditions of Theorem 3 hold for $A_\beta = 0$ and $\varphi > \frac{\pi}{2}$. Then, for $f \in L_{\tilde{\mathbf{p}}}(\Delta_+; E)$ and sufficiently large $d > 0$, problem (30) has a unique solution belonging to $W_{\tilde{\mathbf{p}},\alpha}^{[1,2m]}(\Delta_+; E(A), E)$ and the following coercive estimate holds:

$$\left\| \frac{\partial u}{\partial y} \right\|_{L_{\tilde{\mathbf{p}}}(G_+; E)} + \sum_{k=1}^n t_k \|D_{x_k}^{[2m_k]} u\|_{L_{\tilde{\mathbf{p}}}(G_+; E)} + \|Au\|_{L_{\tilde{\mathbf{p}}}(G_+; E)} \leq C \|f\|_{L_{\tilde{\mathbf{p}}}(G_+; E)}.$$

Proof Problem (30) can be expressed as the following Cauchy problem:

$$\frac{du}{dy} + (Q_t + d)u(y) = f(y), \quad u(0) = 0. \quad (31)$$

Result 3 implies that the operator Q_t is R -positive in $F = L_p(G; E)$. By [23, Section 1.14], Q_t is a generator of an analytic semigroup in F . Then, by virtue of [20, Theorem 4.2], we obtain that for $f \in L_{p_0}(R_+; F)$ problem (31) has a unique solution belonging to $W_{p_0}^1(R_+; D(Q_t), F)$ and the following estimate holds:

$$\left\| \frac{du}{dy} \right\|_{L_{p_0}(R_+; F)} + \|Q_t u\|_{L_{p_0}(R_+; F)} \leq C \|f\|_{L_{p_0}(R_+; F)}.$$

Since $L_{p_0}(R_+; F) = L_{\bar{p}}(\Delta_+; E)$, by Theorem 3 we have

$$\|(Q_t + d)u\|_{L_{p_0}(R_+; F)} = D(Q_t).$$

The above estimate proves the hypothesis to be true. \square

5 Cauchy problem for infinite systems of degenerate parabolic equations with small parameters

Consider the infinity systems of BVP for the degenerate anisotropic parabolic equation:

$$\frac{\partial u_i}{\partial y} + \sum_{k=1}^n (-1)^{m_k} t_k a_k(x) D_{x_k}^{[2m_k]} u_i + \sum_{j=1}^N a_{ij}(x) u_j + du = f_i(x, y), \quad i = 1, 2, \dots, N, \quad (32)$$

$$L_{kj} u = \sum_{i=0}^{m_{kj}} \beta_{kij} D_{x_k}^{(v)} u_i(G_{kb}, y) = 0, \quad j = 1, 2, \dots, m_k,$$

$$u(0, x) = 0, \quad x \in G,$$

where N is finite or infinite natural number, $t_k, a_k(x), G, G_{kb}, G_+, m_{kj}, d$ are defined as in Sections 3 and 4, $a_{ij}(x)$ are real functions and

$$A(x) = \{a_{ij}(x)\}, \quad u = \{u_j\}, \quad Au = \left\{ \sum_{j=1}^N a_{ij}(x) u_j \right\}, \quad i, j = 1, 2, \dots, N,$$

$$l_q(A) = \left\{ u \in l_q, \|u\|_{l_q(A)} = \|Au\|_{l_q} = \sup_i \left(\sum_{j=1}^N |a_{ij}(x) u_j|^q \right)^{\frac{1}{q}} < \infty \right\}.$$

From Theorem 4 we obtain the following.

Theorem 5 Let $\gamma_k > 1 + \frac{1+\sqrt{4p+1}}{2p}$, $p_k, p_0 \in (1, \infty)$, $a_k, a_{ij} \in C(\bar{G})$, $a_k(x) > 0$, $a_{ij} \neq 0$ and $a_{ij} = a_{ji}$. Then for $f(x, y) = \{f_i(x, y)\}_1^N \in L_{\bar{p}}(G_+; l_q)$ and sufficiently large $d > 0$, problem (32) has a unique solution $u = \{u_i(x, y)\}_1^N$ that belongs to the space $W_{\bar{p}, \alpha}^{1, [2m]}(G_+, l_q(A), l_q)$ and the following coercive uniform estimate holds:

$$\left\| \frac{\partial u}{\partial y} \right\|_{L_{\bar{p}}(G_+; l_q)} + \sum_{k=1}^n t_k \|D_{x_k}^{[2m_k]} u\|_{L_{\bar{p}}(G_+; l_q)} + \|Au\|_{L_{\bar{p}}(G_+; l_q)} \leq C \|f\|_{L_{\bar{p}}(G_+; l_q)}.$$

Proof Let $E = l_q$ and A be infinite matrices such that

$$A = [a_{ij}], \quad i, j = 1, 2, \dots, N.$$

It is clear that the operator A is R -positive in l_q . Problem (32) can be rewritten as problem (30). Then, from Theorem 4, we obtain the assertion. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The results to proving the uniform coercivity properties of ordinary and partial differential equations with small parameters in abstract L_p spaces, the showing that corresponding differential operators are positive and also are generators of analytic semigroups and well-posedness of Cauchy problem for abstract parabolic equation and systems of parabolic equations are studied in mixed L_p spaces due to VS. The applications of these abstract problems to concrete mathematics and engineering problem belongs to AS. Both authors read and approved the final manuscript.

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