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Positive solutions of a fractional thermostat model

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Abstract
We study the existence of positive solutions of a nonlinear fractional heat equation with nonlocal boundary conditions depending on a positive parameter. Our results extend the second-order thermostat model to the non-integer case. We base our analysis on the known Guo-Krasnosel'skii fixed point theorem on cones.

1 Introduction

Fractional calculus has been studied for centuries mainly as a pure theoretical mathematical discipline, but recently, there has been a lot of interest in its practical applications. In current research, fractional differential equations have arisen in mathematical models of systems and processes in various fields such as aerodynamics, acoustics, mechanics, electromagnetism, signal processing, control theory, robotics, population dynamics, finance, *etc.* [1–3]. For some recent results in fractional differential equations, see [4–12] and the references therein.

Infante and Webb [13] studied the nonlocal boundary value problem

$$-u'' = f(t,u), \quad t \in (0,1), \qquad u'(0) = 0, \qquad \beta u'(1) + u(\eta) = 0,$$

which models a thermostat insulated at $t = 0$ with the controller at $t = 1$ adding or discharging heat depending on the temperature detected by the sensor at $t = \eta$. Using fixed point index theory and some results on their work on Hammerstein integral equations [14, 15], they obtained results on the existence of positive solutions of the boundary value problem. In particular, they have shown that if $\beta \geq 1 - \eta$, then positive solutions exist under suitable conditions on f. This type of boundary value problem was earlier investigated by Guidotti and Merino [16] for the linear case with $\eta = 0$ where they have shown a loss of positivity as β decreases. In the present paper, we consider the following fractional analog of the thermostat model:

$$-{}^C D^\alpha u(t) = f\big(t, u(t)\big), \quad t \in [0,1], \tag{1}$$

where $1 < \alpha \leq 2$, ${}^C D^\alpha$ denotes the Caputo fractional derivative of order α and $f \in C([0,1] \times [0,\infty), [0,\infty))$ subject to the boundary conditions

$$u'(0) = 0, \qquad \beta\, {}^C D^{\alpha-1} u(1) + u(\eta) = 0, \tag{2}$$

where $\beta > 0$, $0 \leq \eta \leq 1$ are given constants.

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We point out that for $\alpha = 2$, we recover the second-order problem of [13]. We use the properties of the corresponding Green's function and the Guo-Krasnosel'skii fixed point theorem to show the existence of positive solutions of (1)-(2) under the condition that the nonlinearity f is either sublinear or superlinear.

2 Preliminaries

Here we present some necessary basic knowledge and definitions for fractional calculus theory that can be found in the literature [1–3].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is given by

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds$$

provided the integral exists.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}\, ds \quad \bigl(n-1 < \alpha < n, n = [\alpha]+1\bigr),$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 The Caputo derivative of order $\alpha > 0$ of a function $g \in AC^{n-1}[0,\infty)$ is given by

$${}^C D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)\, ds \quad \bigl(n-1 < \alpha < n, n = [\alpha]+1\bigr),$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 *Let $g \in L_1(0,1)$ and $\alpha, \beta > 0$.*

(i) *If $\alpha = n \in \mathbb{N}$, then $I^n g(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} g(s)\, ds$.*
(ii) *If $\alpha = n \in \mathbb{N}$, then ${}^C D^n g(t) = g^{(n)}(t)$.*
(iii) ${}^C D^\alpha I^\alpha g(t) = g(t)$.
(iv) $I^\alpha I^\beta g(t) = I^{\alpha+\beta} g(t)$.

Remark 2.1 In addition to the above properties, the Caputo derivative of a power function t^k, $k \in \mathbb{N}$, is given by

$${}^C D^\alpha t^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, & \text{for } k > n-1, \\ 0, & \text{for } k \le n-1, \end{cases}$$

where $n-1 < \alpha < n$, $n = [\alpha]+1$.

Lemma 2.2 *For $\alpha > 0$, the general solution of the fractional differential equation ${}^{C}D^{\alpha}u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ ($n-1 < \alpha < n$, $n = [\alpha] + 1$).

Lemma 2.3

$$I^{\alpha\,C}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1} \tag{3}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ ($n-1 < \alpha < n$, $n = [\alpha] + 1$).

We start by solving an auxiliary problem to get an expression for the Green's function of boundary value problem (1)-(2).

Lemma 2.4 *Suppose $f \in C[0,1]$. A function $u \in C[0,1]$ is a solution of the boundary value problem*

$$-{}^{C}D^{\alpha}u(t) = f(t), \qquad u'(0) = 0, \qquad \beta\,{}^{C}D^{\alpha-1}u(1) + u(\eta) = 0, \quad t \in [0,1]$$

if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,s)f(s)\,ds,$$

where $G(t,s)$ is the Green's function (depending on α) given by

$$G(t,s) = \beta + H_{\eta}(s) - H_t(s) \tag{4}$$

and for $r \in [0,1]$, $H_r : [0,1] \to \mathbb{R}$ is defined as $H_r(s) = \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \le r$ and $H_r(s) = 0$ for $s > r$.

Proof Using (3) we have, for some constants $c_0, c_1 \in \mathbb{R}$,

$$u(t) = -I^{\alpha}f(t) + c_0 + c_1 t = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds + c_0 + c_1 t. \tag{5}$$

In view of Lemma 2.1, we obtain

$$u'(t) = -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)\,ds + c_1.$$

Since $u'(0) = 0$, we find that $c_1 = 0$.

It also follows that

$${}^{C}D^{\alpha-1}u(t) = -I^{1}u(t).$$

Using the boundary condition $\beta\,{}^{C}D^{\alpha-1}u(1) + u(\eta) = 0$, we get

$$c_0 = \beta\int_0^1 f(s)\,ds + \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds.$$

Finally, substituting the values of c_0 and c_1 in (5), we have

$$\begin{aligned} u(t) &= \beta \int_0^1 f(s)\,ds + \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds \\ &= \int_0^1 G(t,s) f(s)\,ds, \end{aligned}$$

where $G(t,s)$ is given by (4). This completes the proof. □

Remark 2.2 We observe that H_r is continuous on $[0,1]$ for any $r \in [0,1]$. Thus, $G(t,s)$ given by (4) is continuous on $[0,1] \times [0,1]$.

Remark 2.3 By taking $\alpha = 2$, we get

$$u(t) = \beta \int_0^1 f(s)\,ds + \int_0^1 (\eta - s) f(s)\,ds - \int_0^t (t-s) f(s)\,ds = \int_0^1 G(t,s) f(s)\,ds$$

and $G(t,s)$ in this case coincides with the one obtained in [13] for the boundary value problem

$$-u''(t) = f(t), \qquad u'(0) = 0, \qquad \beta u'(1) + u(\eta) = 0.$$

Remark 2.4 We observe that for each fixed point $s \in [0,1]$, $\frac{\partial G}{\partial t} = 0$ for $t \le s$ and $\frac{\partial G}{\partial t} < 0$ for $t > s$ and deduce that $G(t,s)$ is a decreasing function of t. It then follows that

$$\max_{t\in[0,1]} G(t,s) = G(0,s) = \begin{cases} \beta, & s > \eta, \\ \frac{\beta\Gamma(\alpha) + (\eta - s)^{\alpha-1}}{\Gamma(\alpha)}, & s \le \eta, \end{cases}$$

and

$$\min_{t\in[0,1]} G(t,s) = G(1,s) = \begin{cases} \frac{\beta\Gamma(\alpha) - (1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s > \eta, \\ \frac{\beta\Gamma(\alpha) + (\eta-s)^{\alpha-1} - (1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \le \eta. \end{cases}$$

Consequently, by looking at the behavior of $G(t,s)$ with respect to s, we get

$$\min_{t,s\in[0,1]} G(t,s) = \frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)}$$

and

$$\max_{t,s\in[0,1]} G(t,s) = \frac{\beta\Gamma(\alpha) + \eta^{\alpha-1}}{\Gamma(\alpha)}.$$

To establish the existence of positive solutions of problem (1)-(2), we will show that $G(t,s)$ satisfies the following property introduced by Lan and Webb in [17]:

(A) There exist a measurable function $\phi : [0,1] \to [0,\infty)$, a subinterval $[a,b] \subseteq [0,1]$ and a constant $\lambda \in [0,1]$ such that

$$|G(t,s)| \le \phi(s) \quad \forall t,s \in [0,1]$$

and

$$G(t,s) \geq \lambda\phi(s) \quad \forall t \in [a,b], \forall s \in [0,1].$$

Lemma 2.5 *If* $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$, *then* $G(t,s) > 0$ *for all* $t,s \in [0,1]$, *and* $G(t,s)$ *satisfies property* (A).

Proof If $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$, then $G(t,s) > 0$ for all $t,s \in [0,1]$. We choose $[a,b] = [0,1]$, and we have

$$\left|G(t,s)\right| = G(t,s) \leq \frac{\beta\Gamma(\alpha) + \eta^{\alpha-1}}{\Gamma(\alpha)} := \phi(s)$$

and

$$G(t,s) \geq \lambda\phi(s) \quad \forall s,t \in [0,1],$$

where

$$\lambda = \frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha) + \eta^{\alpha-1}}. \tag{6}$$

□

Lemma 2.6 *If* $\beta\Gamma(\alpha) = (1-\eta)^{\alpha-1}$, *then* $G(t,s) \geq 0$ *for all* $t,s \in [0,1]$, *and* $G(t,s)$ *satisfies property* (A).

Proof We choose $[a,b] = [0,b]$ with $\eta \leq b < 1$. Following the arguments in the previous lemma, we have

$$\left|G(t,s)\right| \leq \frac{\beta\Gamma(\alpha) + \eta^{\alpha-1}}{\Gamma(\alpha)} := \phi(s) \quad \forall t,s \in [0,1].$$

Also, by taking

$$\lambda = \frac{\beta\Gamma(\alpha) - (b-\eta)^{\alpha-1}}{\beta\Gamma(\alpha) + \eta^{\alpha-1}},$$

we obtain

$$G(t,s) \geq \lambda\phi(s) \quad \forall t \in [0,b], \forall s \in [0,1].$$

□

Lemma 2.7 *If* $\beta\Gamma(\alpha) < (1-\eta)^{\alpha-1}$, *then* $G(t,s)$ *changes sign on* $[0,1] \times [0,1]$, *and* $G(t,s)$ *satisfies property* (A).

Proof We choose $[a,b] = [0,b]$ with $\eta \leq b < 1$ such that $\beta\Gamma(\alpha) > (b-\eta)^{\alpha-1}$. We have

$$\left|G(t,s)\right| \leq \max\left\{\frac{\beta\Gamma(\alpha) + \eta^{\alpha-1}}{\Gamma(\alpha)}, \frac{(1-\eta)^{\alpha-1} - \beta\Gamma(\alpha)}{\Gamma(\alpha)}\right\} := \phi(s) \quad \forall t,s \in [0,1]$$

and

$$G(t,s) \geq \lambda\phi(s) \quad \forall t \in [0,b], \forall s \in [0,1],$$

where

$$\lambda = \min\left\{ \frac{\beta\Gamma(\alpha) - (b-\eta)^{\alpha-1}}{\beta\Gamma(\alpha) + \eta^{\alpha-1}}, \frac{\beta\Gamma(\alpha) - (b-\eta)^{\alpha-1}}{(1-\eta)^{\alpha-1} - \beta\Gamma(\alpha)} \right\}.$$

For the main results, we use the known Guo-Krasnosel'skii fixed point theorem [18]. □

Theorem 2.1 *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(i) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then the operator P has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

We set

$$f_0 = \lim_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad f_0^* = \lim_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$
$$f_\infty = \lim_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad f_\infty^* = \lim_{u\to \infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}.$$

We now state the main result of this paper.

Theorem 3.1 *Let $f(s,u(s)) \in C([0,1] \times [0,\infty), [0,\infty))$. Assume that one of the following conditions is satisfied*:

(i) (*Sublinear case*) $f_0 = \infty$ *and* $f_\infty = 0$.

(ii) (*Superlinear case*) $f_0^* = 0$ *and* $f_\infty^* = \infty$.

If $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$, then problem (1)-(2) *admits at least one positive solution.*

Theorem 3.2 *Let $f(s,u(s)) \in C([0,1] \times [-\infty,+\infty), [0,\infty))$. Assume that one of the following conditions is satisfied*:

(i) (*Sublinear case*) $f_0 = \infty$ *and* $f_\infty = 0$.

(ii) (*Superlinear case*) $f_0^* = 0$ *and* $f_\infty^* = \infty$.

If $\beta\Gamma(\alpha) \leq (1-\eta)^{\alpha-1}$, then problem (1)-(2) *admits a solution which is positive on an interval* $[0,b] \subset [0,1]$.

Proof of Theorem 3.1 Let $C[0,1]$ be the Banach space of all continuous real-valued functions on $[0,1]$ endowed with the usual supremum norm $\|\cdot\|$.

We define the operator $T : C[0,1] \to C[0,1]$ as

$$Tu(t) = \int_0^1 G(t,s) f\bigl(s,u(s)\bigr)\, ds,$$

where $G(t,s)$ is defined by (4).

It is clear from Lemma 2.4 that the fixed points of the operator T coincide with the solutions of problem (1)-(2).

We now define the cone

$$P = \left\{ u \middle| u \in C[0,1], u(t) \geq 0, \min_{t\in[0,1]} u(t) \geq \lambda \|u\| \right\},$$

where λ is given by (6).

First, we show that $T(P) \subset P$.

It follows from the continuity and the non-negativity of the functions G and f on their domains of definition that if $u \in P$, then $Tu \in C[0,1]$ and $Tu(t) \geq 0$ for all $t \in [0,1]$.

For a fixed $u \in P$ and for all $t \in [0,1]$, the fact that $G(t,s)$ satisfies property (A) leads to the following inequalities:

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s) f\big(s,u(s)\big)\,ds \\
&\geq \lambda \int_0^1 \phi(s) f\big(s,u(s)\big)\,ds \\
&\geq \lambda \int_0^1 \max_{t\in[0,1]} G(t,s) f\big(s,u(s)\big)\,ds \\
&\geq \lambda \max_{t\in[0,1]} \int_0^1 G(t,s) f\big(s,u(s)\big)\,ds \\
&= \lambda \|Tu\|.
\end{aligned}$$

Hence, $T(P) \subset P$.

We now show that $T : P \to P$ is completely continuous.

In view of the continuity of the functions G and f, the operator $T : P \to P$ is continuous.

Let $\Omega \subset P$ be bounded, that is, there exists a positive constant $M > 0$ such that $\|u\|_\infty \leq M$ for all $u \in \Omega$. Define

$$L = \max_{0\leq t\leq 1, 0\leq u\leq M} \big|f(t,u)\big| + 1.$$

Then for all $u \in \Omega$, we have

$$\big|Tu(t)\big| \leq \int_0^1 G(t,s) f\big(s,u(s)\big)\,ds \leq L \int_0^1 G(t,s)\,ds$$

for all $t \in [0,1]$. That is, the set $T(\Omega)$ is bounded.

For each $u \in \Omega$ and $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
\big|Tu(t_2) - Tu(t_1)\big| &= \left| -\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,u(s)\big)\,ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,u(s)\big)\,ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big) \big|f\big(s,u(s)\big)\big|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \big|f\big(s,u(s)\big)\big|\,ds \\
&\leq \frac{L}{\Gamma(\alpha)} \left(\int_0^{t_1} \big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big)\,ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds \right)
\end{aligned}$$

$$\begin{aligned} &= \frac{L}{\alpha\Gamma(\alpha)}\left(-(t_2-t_1)^\alpha + t_2^\alpha - t_1^\alpha + (t_2-t_1)^\alpha\right) \\ &= \frac{L}{\Gamma(\alpha+1)}\left(t_2^\alpha - t_1^\alpha\right). \end{aligned}$$

Clearly, the right-hand side of the above inequalities tends to 0 as $t_1 \to t_2$ and therefore the set $T(\Omega)$ is equicontinuous. It follows from the Arzela-Ascoli theorem that the operator $T : P \to P$ is completely continuous.

We now consider the two cases.

(i) Sublinear case ($f_0 = \infty$ and $f_\infty = 0$).

Since $f_0 = \infty$, there exists $\rho_1 > 0$ such that $f(t,u) \geq \delta_1 u$ for all $0 < u \leq \rho_1$, where δ_1 satisfies

$$\delta_1\left(\frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right) \geq 1. \tag{7}$$

We take $u \in P$ such that $\|u\| = \rho_1$, then we have the following inequalities:

$$\begin{aligned} Tu &= \int_0^1 G(t,s)f\left(s,u(s)\right)ds \\ &\geq \delta_1 \int_0^1 G(t,s)u(s)\,ds \\ &\geq \delta_1\|u\|\left(\frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right) \\ &\geq \|u\|. \end{aligned}$$

Let $\Omega_1 = \{u \in C[0,1] \mid \|u\| < \rho_1\}$. Hence, we have $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$.

Since $f(t,\cdot)$ is a continuous function on $[0,\infty)$, we can define the function:

$$\tilde{f}(t,u) = \max_{z\in[0,u]}\left\{f(t,z)\right\}.$$

It is clear that $\tilde{f}(t,u)$ is non-decreasing on $(0,\infty)$ and since $f_\infty = 0$, we have (see [19])

$$\lim_{u\to\infty}\left\{\max_{t\in[0,1]}\frac{\tilde{f}(t,u)}{u}\right\} = 0.$$

Therefore, there exists $\rho_2 > \rho_1 > 0$ such that $\tilde{f}(t,u) \leq \delta_2 u$ for all $u \geq \rho_2$, where δ_2 satisfies

$$\delta_2\left(\frac{\beta\Gamma(\alpha) + \eta^{\alpha-1}}{\Gamma(\alpha)}\right) \leq 1. \tag{8}$$

Define $\Omega_2 = \{u \in C[0,1] \mid \|u\| < \rho_2\}$ and let $u \in P$ such that $\|u\| = \rho_2$. Then

$$\begin{aligned} Tu &= \int_0^1 G(t,s)f\left(s,u(s)\right)ds \\ &\leq \int_0^1 G(t,s)\tilde{f}\left(s,\|u\|\right)ds \end{aligned}$$

$$\begin{aligned}&\le \delta_2\|u\|\left(\frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\\&\le \|u\|.\end{aligned}$$

Hence, we have $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_2$.

Thus, by the first part of the Guo-Krasnosel'skii fixed point theorem, we conclude that (1)-(2) has at least one positive solution.

(ii) Superlinear case ($f_0^* = 0$ and $f_\infty^* = \infty$).

Let $\delta_2 > 0$ be given as in (8).

Since $f_0^* = 0$, there exists a constant $r_1 > 0$ such that $f(t,u) \le \delta_2 u$ for $0 \le u \le r_1$. Take $u \in P$ such that $\|u\| = r_1$. Then we have

$$\begin{aligned}Tu &= \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds\\&\le \delta_2\int_0^1 G(t,s)u(s)\,ds\\&\le \delta_2\|u\|\left(\frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\\&\le \|u\|.\end{aligned}$$

If we let $\Omega_1 = \{u \in C[0,1] \mid \|u\| < r_1\}$, we see that $\|Tu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$.

Now, since $f_\infty^* = \infty$, there exists $r > 0$ such that $f(t,u) \ge \delta_1 u$ for all $u \ge r$, where δ_1 is as in (7).

Define $\Omega_2 = \{u \in C[0,1] \mid \|u\| < r_2\}$, where $r_2 = \max(2r_1, \frac{r}{\lambda})$. Then $u \in P$ and $\|u\| = r_2$ imply that

$$\min u(t) \ge \lambda\|u\| = \lambda r_2 \ge r,$$

and so we obtain

$$\begin{aligned}Tu &= \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds\\&\ge \delta_1\int_0^1 G(t,s)u(s)\,ds\\&\ge \delta_1\|u\|\left(\frac{\beta\Gamma(\alpha)-(1-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right)\\&\ge \|u\|.\end{aligned}$$

This shows that $\|Tu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_2$. We conclude by the second part of the Guo-Krasnosel'skii fixed point theorem that (1)-(2) has at least one positive solution $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Remark 3.1 To prove Theorem 3.2, we use the cone

$$P = \Bigl\{u \bigm| u \in C[0,1], \min_{t\in[0,b]} u(t) \ge \lambda\|u\|\Bigr\},$$

where b and λ are defined in Lemma 2.6 for the case where $\beta\Gamma(\alpha) = (1-\eta)^{\alpha-1}$, and in Lemma 2.7 for the case where $\beta\Gamma(\alpha) < (1-\eta)^{\alpha-1}$. We skip the rest of the proof as it is similar to the proof of Theorem 3.1.

Example 3.1 Consider the fractional boundary value problem:

$$\begin{cases} -{}^{C}D^{\frac{3}{2}}u(t) = t^2 e^{-u(t)} + \sqrt{u(t)}, \quad t \in [0,1], \\ u'(0) = 0, \qquad \frac{4}{5}{}^{C}D^{\frac{1}{2}}u(1) + u(\frac{3}{4}) = 0, \end{cases} \tag{9}$$

which is problem (1)-(2) with $\alpha = \frac{3}{2}$, $\beta = \frac{4}{5}$, $\eta = \frac{3}{4}$ and $f(t,u(t)) = t^2 e^{-u(t)} + \sqrt{u(t)}$.

First, we note that $u = 0$ is not a solution of (9).

Clearly, $f_0 = \infty$ and $f_\infty = 0$, and we also have $\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1} = \frac{2\sqrt{\pi}}{5} - \frac{1}{2} \approx 0.20898 > 0$.

We take

$$\lambda = \frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha) + \eta^{\alpha-1}} = \frac{\frac{2\sqrt{\pi}}{5} - \frac{1}{2}}{\frac{2\sqrt{\pi}}{5} + \frac{\sqrt{3}}{2}} = \frac{4\sqrt{\pi} - 5}{4\sqrt{\pi} + 5\sqrt{3}} \approx 0.13269$$

and consider the cone $P = \{u \mid u \in C[0,1], u(t) \geq 0, \min_{t\in[0,1]} u(t) \geq \lambda\|u\|\}$.

By the first part of Theorem 3.1, we conclude that the boundary value problem (9) has a positive solution in the cone P.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors, JJN and JP, contributed equally and read and approved the final version of the manuscript.

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