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Ground state homoclinic orbits of damped vibration problems

Guan-Wei Chen* and Jian Wang

*Correspondence: guanweic@163.com
School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, P.R. China

Abstract

In this paper, we consider a class of non-periodic damped vibration problems with superquadratic nonlinearities. We study the existence of nontrivial ground state homoclinic orbits for this class of damped vibration problems under some conditions weaker than those previously assumed. To the best of our knowledge, there has been no work focused on this case.

MSC: 49J40; 70H05

Keywords: non-periodic damped vibration problems; ground state homoclinic orbits; superquadratic nonlinearity

1 Introduction and main results

We shall study the existence of ground state homoclinic orbits for the following non-periodic damped vibration system:

$$\ddot{u}(t) + M\dot{u}(t) - L(t)u(t) + H_u\bigl(t, u(t)\bigr) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where M is an antisymmetric $N \times N$ constant matrix, $L(t) \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix, $H(t,u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $H_u(t,u)$ denotes its gradient with respect to the u variable. We say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. If $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

If $M = 0$ (zero matrix), then (1.1) reduces to the following second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + H_u\bigl(t, u(t)\bigr) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

This is a classical equation which can describe many mechanic systems such as a pendulum. In the past decades, the existence and multiplicity of periodic solutions and homoclinic orbits for (1.2) have been studied by many authors via variational methods; see [1–18] and the references therein.

The periodic assumptions are very important in the study of homoclinic orbits for (1.2) since periodicity is used to control the lack of compactness due to the fact that (1.2) is set on all $\mathbb{R}$. However, non-periodic problems are quite different from the ones described in periodic cases. Rabinowitz and Tanaka [11] introduced a type of coercive condition on the matrix $L(t)$,

$$l(t) := \inf_{|u|=1}\big(L(t)u,u\big) \to +\infty \quad \text{as } |t| \to \infty, \tag{1.3}$$

and obtained the existence of a homoclinic orbit for non-periodic (1.2) under the Ambrosetti-Rabinowitz (AR) superquadratic condition:

$$0 < \mu H(t,u) \leq \big(H_u(t,u),u\big), \quad \forall t \in \mathbb{R}, \forall u \in \mathbb{R}^N \setminus \{0\},$$

where $\mu > 2$ is a constant, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^N$ and the associated norm is denoted by $|\cdot|$.

We should mention that in the case where $M \neq 0$, *i.e.*, the damped vibration system (1.1), only a few authors have studied homoclinic orbits of (1.1); see [19–23]. Zhu [23] considered the *periodic* case of (1.1) (*i.e.*, $L(t)$ and $H(t,u)$ are T-periodic in t with $T > 0$) and obtained the existence of nontrivial homoclinic solutions of (1.1). The authors [19–22] considered the *non-periodic* case of (1.1): Zhang and Yuan [22] obtained the existence of at least one homoclinic orbit for (1.1) when H satisfies the *subquadratic* condition at infinity by using a standard minimizing argument; by a symmetric mountain pass theorem and a generalized mountain pass theorem, Wu and Zhang [21] obtained the existence and multiplicity of homoclinic orbits for (1.1) when H satisfies the local (AR) *superquadratic* growth condition:

$$0 < \mu H(t,u) \leq \big(H_u(t,u),u\big), \quad \forall t \in \mathbb{R}, \forall |u| \geq r, \tag{1.4}$$

where $\mu > 2$ and $r > 0$ are two constants. Notice that the authors [21, 22] all used condition (1.3). Recently, the author in [19, 20] obtained infinitely many homoclinic orbits for (1.1) when H satisfies the *subquadratic* [19] and *asymptotically quadratic* [20] condition at infinity by the following *weaker* conditions than (1.3):

(L_1) There is a constant $\beta > 1$ such that

$$\operatorname{meas}\big\{t \in \mathbb{R} : |t|^{-\beta}L(t) < bI_N\big\} < +\infty, \quad \forall b > 0;$$

(L_2) There is a constant $\gamma_0 \geq 0$ such that

$$l(t) := \inf_{|u|=1}\big(L(t)u,u\big) \geq -\gamma_0, \quad \forall t \in \mathbb{R},$$

which were firstly used in [15]. It is not hard to check that the matrix-valued function $L(t) := (t^4 \sin^2 t + 1)I_N$ satisfies (L_1) and (L_2), but does not satisfy (1.3).

We define an operator $\Gamma : H^1(\mathbb{R},\mathbb{R}^N) \to H^1(\mathbb{R},\mathbb{R}^N)$ by

$$(\Gamma u,v) := \int_{\mathbb{R}} \big(Mu(t),\dot{v}(t)\big)\,dt, \quad \forall u,v \in H^1\big(\mathbb{R},\mathbb{R}^N\big).$$

Since M is an antisymmetric $N \times N$ constant matrix, Γ is self-adjoint on $H^1(\mathbb{R},\mathbb{R}^N)$. Let χ denote the self-adjoint extension of the operator $-\frac{d^2}{dt^2} + L(t) + \Gamma$. We are interested in the *indefinite* case:

(J_1) $a := \sup(\sigma(\chi) \cap (-\infty, 0)) < 0 < b := \inf(\sigma(\chi) \cap (0, \infty))$.

To state our main result, we still need the following assumptions:

(H_1) $|\nabla H(t,u)| \leq c(1 + |u|^{p-1})$ for some $c > 0$ and $p > 2$, $\forall t \in \mathbb{R}$ and $u \in \mathbb{R}^N$.
(H_2) $H(t,u) \geq \frac{1}{2}a|u|^2$, $\forall t \in \mathbb{R}$ and $u \in \mathbb{R}^N$.
(H_3) For some $\delta > 0$ and $\gamma \in (0, b)$,

$$\left|\nabla H(t,u)\right| \leq \gamma|u|, \qquad \left|H(t,u)\right| \leq \frac{1}{2}\left|\nabla H(t,u)\right| \cdot |u|, \quad \forall |u| < \delta, \forall t \in \mathbb{R}.$$

(H_4) $\frac{H(t,u)}{|u|^2} \to +\infty$ as $|u| \to +\infty$ and there exists $W_1(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$ such that

$$H(t,u) \geq -W_1(t), \quad \forall t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N. \tag{1.5}$$

(H_5) For all $t \in \mathbb{R}$ and $u, z \in \mathbb{R}^N$, there holds

$$\begin{aligned} &H(t, u+z) - H(t,u) - r\big(\nabla H(t,u), z\big) + \frac{(r-1)^2}{2}\big(\nabla H(t,u), u\big) \\ &\quad \geq -W_1(t), \quad \forall r \in [0,1]. \end{aligned}$$

Our main results read as follows.

Theorem 1.1 *If* (L_1)-(L_2), (J_1) *and* (H_1)-(H_5) *hold, then* (1.1) *has at least one nontrivial homoclinic orbit.*

Theorem 1.2 *Let* $\mathcal{M}$ *be the collection of solutions of* (1.1), *then there is a solution that minimizes the energy functional*

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} \big[\left|\dot{u}(t)\right|^2 + \big(Mu(t), \dot{u}(t)\big) + \big(L(t)u(t), u(t)\big)\big]\, dt - \int_{\mathbb{R}} H(t,u)\, dt, \quad u \in E$$

over $\mathcal{M}$, *where the space* E *is defined in Section* 2. *In addition, if*

$$\left|\nabla H(t,u)\right| = o\big(|u|\big) \quad \text{as } |u| \to 0$$

uniformly in t, *then there is a nontrivial homoclinic orbit that minimizes the energy functional over* $\mathcal{M} \setminus \{0\}$, *i.e., a ground state homoclinic orbit.*

Remark 1.1 Although the authors [21] have studied (1.1) with *superquadratic* nonlinearities, our superquadratic condition (H_4) is weaker than (1.4) in [21]. Moreover, we study the *ground state* homoclinic orbit of (1.1). To the best of our knowledge, there has been no result published concerning the *ground state* homoclinic orbit of (1.1).

Example 1.1
(1) $H(t,u) = |u|^p$,
(2) $H(t,u) = g(t)(|u|^p + (p-2)|u|^{p-\varepsilon}\sin^2(\frac{|u|^\varepsilon}{\varepsilon}))$,
where $p > 2$, $g(t) > 0$ is continuous and $0 < \varepsilon < p - 2$. It is easy to check that the above two functions satisfy assumptions (H_1)-(H_5) if we take $0 \leq W_1(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$, where $W_1(t)$ is the function in (H_4)-(H_5).

The rest of the present paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1), and we also give some preliminary lemmas, which are useful in the proofs of our main results. In Section 3, we give the detailed proofs of our main results.

2 Preliminary lemmas

In the following, we use $\|\cdot\|_{L^p}$ to denote the norm of $L^p(\mathbb{R},\mathbb{R}^N)$ for any $p\in[1,\infty]$. Let $W := H^1(\mathbb{R},\mathbb{R}^N)$ be a Hilbert space with the inner product and norm given respectively by

$$\langle u,v\rangle_W = \int_{\mathbb{R}} \big[\big(\dot{u}(t),\dot{v}(t)\big) + \big(u(t),v(t)\big)\big]\,dt, \qquad \|u\|_W = \langle u,u\rangle_E^{1/2}, \quad \forall u,v\in W.$$

It is well known that W is continuously embedded in $L^p(\mathbb{R},\mathbb{R}^N)$ for $p\in[2,\infty)$. We define an operator $\Gamma : W\to W$ by

$$(\Gamma u,v) := \int_{\mathbb{R}} \big(Mu(t),\dot{v}(t)\big)\,dt, \quad \forall u,v\in W.$$

Since M is an antisymmetric $N\times N$ constant matrix, Γ is self-adjoint on W. Moreover, we denote by χ the self-adjoint extension of the operator $-\frac{d^2}{dt^2}+L(t)+\Gamma$ with the domain $\mathcal{D}(\chi)\subset L^2(\mathbb{R},\mathbb{R}^N)$.

Let $E := \mathcal{D}(|\chi|^{1/2})$, the domain of $|\chi|^{1/2}$. We define respectively on E the inner product and the norm

$$\langle u,v\rangle_E := \big(|\chi|^{1/2}u,|\chi|^{1/2}v\big)_2 + (u,v)_2 \quad \text{and} \quad \|u\|_E = \langle u,u\rangle_E^{1/2},$$

where $(\cdot,\cdot)_2$ denotes the inner product in $L^2(\mathbb{R},\mathbb{R}^N)$.

By a similar proof of Lemma 3.1 in [15], we can prove that if conditions (L_1) and (L_2) hold, then

$$E \text{ is compactly embedded into } L^p\big(\mathbb{R},\mathbb{R}^N\big), \quad \forall p\in[1,+\infty]. \tag{2.1}$$

Therefore, it is easy to prove that the spectrum $\sigma(\chi)$ has a sequence of eigenvalues (counted with their multiplicities)

$$\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots\to\infty,$$

and the corresponding system of eigenfunctions $\{e_k : k\in\mathbb{N}\}$ ($\chi e_k=\lambda_k e_k$) forms an orthogonal basis in $L^2(\mathbb{R},\mathbb{R}^N)$.

By (J_1), we may let

$$k_1 := \sharp\{j:\lambda_j<0\}, \qquad E^- := \operatorname{span}\{e_1,\dots,e_{k_1}\}, \qquad E^+ := \operatorname{cl}_E\big(\operatorname{span}\{e_{k_1+1},\dots\}\big).$$

Then one has the orthogonal decomposition

$$E = E^-\oplus E^+$$

with respect to the inner product $\langle\cdot,\cdot\rangle_E$. Now, we introduce respectively on E the following new inner product and norm:

$$\langle u,v\rangle := \big(|\chi|^{1/2}u,|\chi|^{1/2}v\big)_2, \qquad \|u\| = \langle u,u\rangle^{1/2}, \tag{2.2}$$

where $u, v \in E = E^- \oplus E^+$ with $u = u^- + u^+$ and $v = v^- + v^+$. Clearly, the norms $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent (see [4]), and the decomposition $E = E^- \oplus E^+$ is also orthogonal with respect to both inner products $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)_2$. Hence, by (J_1), E with equivalent norms, besides, we have

$$-\|u^-\|^2 = (\chi u^-, u^-)_2 \le a\|u^-\|_{L^2}^2, \quad \forall u^- \in E^- \tag{2.3}$$

and

$$\|u^+\|^2 = (\chi u^+, u^+)_2 \ge b\|u^+\|_{L^2}^2, \quad \forall u^+ \in E^+, \tag{2.4}$$

where a and b are defined in (J_1).

For problem (1.1), we consider the following functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} \bigl[|\dot{u}(t)|^2 + \bigl(Mu(t), \dot{u}(t)\bigr) + \bigl(L(t)u(t), u(t)\bigr)\bigr]\,dt - \int_{\mathbb{R}} H(t,u)\,dt, \quad u \in E.$$

Then I can be rewritten as

$$I(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}} H(t,u)\,dt, \quad u = u^- + u^+ \in E.$$

Let $\Psi(u) := \int_{\mathbb{R}} H(t,u)\,dt$. In view of the assumptions of H, we know $I, \Psi \in C^1(E, \mathbb{R})$ and the derivatives are given by

$$\Psi'(u)v = \int_{\mathbb{R}} \bigl(H_u(t,u), v\bigr)\,dt, \qquad I'(u)v = \langle u^+, v^+\rangle - \langle u^-, v^-\rangle - I'(u)v$$

for any $u, v \in E = E^- \oplus E^+$ with $u = u^- + u^+$ and $v = v^- + v^+$. By the discussion of [24], the (weak) solutions of system (1.1) are the critical points of the C^1 functional $I : E \to \mathbb{R}$. Moreover, it is easy to verify that if $u \not\equiv 0$ is a solution of (1.1), then $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$ (see Lemma 3.1 in [25]).

The following abstract critical point theorem plays an important role in proving our main result. Let E be a Hilbert space with the norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N^\perp$, $N \subset E$ is a closed and separable subspace. There exists a norm $|v|_\omega$ satisfying $|v|_\omega \le \|v\|$ for all $v \in N$ and inducing a topology equivalent to the weak topology of N on a bounded subset of N. For $u = v + w \in E = N \oplus N^\perp$ with $v \in N$, $w \in N^\perp$, we define $|u|_\omega^2 = |v|_\omega^2 + \|w\|^2$. Particularly, if $u_n = v_n + w_n$ is $\|\cdot\|$-bounded and $u_n \xrightarrow{|\cdot|_\omega} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E (*cf.* [26]).

Let $E := E^- \oplus E^+$, $z_0 \in E^+$ with $\|z_0\| = 1$. Let $N := E^- \oplus \mathbb{R}z_0$ and $E_1^+ := N^\perp = (E^- \oplus \mathbb{R}z_0)^\perp$. For $R > 0$, let

$$Q := \bigl\{u := u^- + sz_0 : s \in \mathbb{R}^+, u^- \in E^-, \|u\| < R\bigr\}$$

with $p_0 = s_0 z_0 \in Q$, $s_0 > 0$. We define

$$D := \bigl\{u := sz_0 + w^+ : s \in \mathbb{R}, w^+ \in E_1^+, \|sz_0 + w^+\| = s_0\bigr\}.$$

For $I \in C^1(E, \mathbb{R})$, define

$$\Gamma := \left\{ h : \begin{array}{l} h : [0,1] \times \bar{Q} \mapsto E \text{ is } |\cdot|_\omega\text{-continuous;} \\ h(0,u) = u \text{ and } I(h(s,u)) \leq I(u) \text{ for all } u \in \bar{Q}; \\ \text{For any } (s_0, u_0) \in [0,1] \times \bar{Q}, \text{there is a } |\cdot|_\omega\text{-neighborhood} \\ U_{(s_0,u_0)} \text{ s.t. } \{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{\text{fin}} \end{array} \right\},$$

where E_{fin} denotes various finite-dimensional subspaces of E, $\Gamma \neq 0$ since $id \in \Gamma$.

The variant weak linking theorem is as follows.

Lemma 2.1 ([26]) *The family of C^1-functionals $\{I_\lambda\}$ has the form*

$$I_\lambda(u) := \lambda K(u) - J(u), \quad \forall \lambda \in [1, \lambda_0],$$

where $\lambda_0 > 1$. Assume that

(a) $K(u) \geq 0$, $\forall u \in E$, $I_1 = I$;
(b) $|J(u)| + K(u) \to \infty$ *as* $\|u\| \to \infty$;
(c) I_λ *is* $|\cdot|_\omega$*-upper semicontinuous,* I'_λ *is weakly sequentially continuous on E. Moreover,* I_λ *maps bounded sets to bounded sets*;
(d) $\sup_{\partial Q} I_\lambda < \inf_D I_\lambda$, $\forall \lambda \in [1, \lambda_0]$.

Then, for almost all $\lambda \in [1, \lambda_0]$, there exists a sequence $\{u_n\}$ such that

$$\sup_n \|u_n\| < \infty, \qquad I'_\lambda(u_n) \to 0, \qquad I_\lambda(u_n) \to c_\lambda,$$

where $c_\lambda := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}} I_\lambda(h(t,u)) \in [\inf_D I_\lambda, \sup_{\bar{Q}} I_\lambda]$.

In order to apply Lemma 2.1, we shall prove a few lemmas. We pick λ_0 such that $1 < \lambda_0 < \min[2, b/\gamma]$. For $1 \leq \lambda \leq \lambda_0$, we consider

$$I_\lambda(u) := \frac{\lambda}{2}\|u^+\|^2 - \left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} H(t, u(t))\,dt\right) := \lambda K(u) - J(u).$$

It is easy to see that I_λ satisfies condition (a) in Lemma 2.1. To see (c), if $u_n \stackrel{|\cdot|_\omega}{\to} u$ and $I_\lambda(u_n) \geq c$, then $u_n^+ \to u^+$ and $u_n^- \rightharpoonup u^-$ in E, $u_n \to u$ a.e. on $\mathbb{R}$, going to a subsequence if necessary. It follows from the weak lower semicontinuity of the norm, Fatou's lemma and the fact $H(t,u) + W_1(t) \geq 0$ for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^N$ by (1.5) in (H_4) that

$$\begin{aligned} c &\leq \limsup_{n\to\infty} I_\lambda(u_n) \\ &= \limsup_{n\to\infty}\left[\frac{\lambda}{2}\|u_n^+\|^2 - \left(\frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}}\big(H(t,u_n) + W_1(t)\big)\,dt\right) + \int_{\mathbb{R}} W_1(t)\,dt\right] \\ &\leq \frac{\lambda}{2}\|u^+\|^2 - \liminf_{n\to\infty}\left[\frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}}\big(H(t,u_n) + W_1(t)\big)\,dt\right] + \int_{\mathbb{R}} W_1(t)\,dt \\ &\leq \frac{\lambda}{2}\|u^+\|^2 - \left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} H(t,u)\,dt\right) = I_\lambda(u). \end{aligned}$$

Thus we get $I_\lambda(u) \geq c$. It implies that I_λ is $|\cdot|_\omega$-upper semicontinuous. I'_λ is weakly sequentially continuous on E due to [27].

Lemma 2.2 *Under assumptions of Theorem* 1.1, *then*

$$J(u) + K(u) \to \infty \quad \textit{as } \|u\| \to \infty.$$

Proof By the definition of $I(u)$ and (H_4), we have

$$\begin{aligned} J(u) + K(u) &= \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} H(t, u(t))\, dt \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t)\, dt \to +\infty \quad \text{as } \|u\| \to \infty, \end{aligned}$$

which is due to $W_1(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$. □

Therefore, Lemma 2.2 implies that condition (b) holds. To continue the discussion, we still need to verify condition (d), that is, the following two lemmas.

Lemma 2.3 *Under assumptions of Theorem* 1.1, *there are two positive constants* $\epsilon, \rho > 0$ *such that*

$$I_\lambda(u) \geq \epsilon, \quad u \in E^+, \|u\| = \rho, \lambda \in [1, \lambda_0].$$

Proof By (H_1), (H_3), (2.4) and the Sobolev embedding theorem, for all $u \in E^+$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} H(t, u(t))\, dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\{t\in\mathbb{R}:|u|<\delta\}} H(t, u(t))\, dt - \int_{\{t\in\mathbb{R}:|u|\geq\delta\}} H(t, u(t))\, dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\gamma \int_{\{t\in\mathbb{R}:|u|<\delta\}} |u|^2\, dt - c\int_{\{t\in\mathbb{R}:|u|\geq\delta\}} \left(|u|^p + |u|\right) dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\gamma}{b}\frac{1}{2}\|u\|^2 - C\|u\|^p \\ &= \frac{1}{2}\|u\|^2\left(1 - \frac{\gamma}{b} - 2C\|u\|^{p-2}\right), \quad 0 \leq \gamma < b, \end{aligned}$$

where C is a positive constant. It implies the conclusion if we take $\|u\|$ sufficiently small. □

Lemma 2.4 *Under assumptions of Theorem* 1.1, *then there is an* $R > 0$ *such that*

$$I_\lambda(u) \leq 0, \quad u \in \partial Q_R, \lambda \in [1, \lambda_0],$$

where $Q_R := \{u := v + sz_0 : s \geq 0, v \in E^-, z_0 \in E^+ \textit{ with } \|z_0\| = 1, \|u\| \leq R\}$.

Proof Suppose by contradiction that there exist $R_n \to \infty$, $\lambda_n \in [1, \lambda_0]$ and $u_n = v_n + s_n z_0 \in \partial Q_R$ such that $I_{\lambda_n}(u_n) > 0$. If $s_n = 0$, then by (H_2) and (2.3), we have

$$I_{\lambda_n}(v_n) = -\frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}} H(t, v_n)\, dt \leq -\frac{1}{2}\|v_n\|^2 - \frac{1}{2}a\|v_n\|_{L^2}^2 \leq 0.$$

Therefore, $s_n \neq 0$ and $\|u_n\|^2 = \|v_n\|^2 + s_n^2 = R_n^2$. Let $\tilde{u}_n = \frac{u_n}{\|u_n\|} = \tilde{s}_n z_0 + \tilde{v}_n$, then

$$\|\tilde{u}_n\|^2 = \|\tilde{v}_n\|^2 + \tilde{s}_n^2 = 1.$$

It follows from $I_{\lambda_n}(u_n) > 0$ and the definition of I that

$$\begin{aligned} 0 < \frac{I_{\lambda_n}(u_n)}{\|u_n\|^2} &= \frac{1}{2}\left(\lambda_n \tilde{s}_n^2 - \|\tilde{v}_n\|^2\right) - \int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt \\ &= \frac{1}{2}\left[(\lambda_n + 1)\tilde{s}_n^2 - 1\right] - \int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt. \end{aligned} \tag{2.5}$$

There are renamed subsequences such that $\tilde{s}_n \to \tilde{s}$, $\lambda_n \to \lambda$, and there is a renamed subsequence such that $\tilde{u}_n = \frac{u_n}{\|u_n\|} = \tilde{s}_n z_0 + \tilde{v}_n \rightharpoonup \tilde{u}$ in E and $\tilde{u}_n \to \tilde{u}$ a.e. on $\mathbb{R}$.

We claim that

$$\liminf_{n\to\infty} \int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt \geq 0. \tag{2.6}$$

Case 1. If $\tilde{u} \not\equiv 0$. Let Ω_0 be the subset of $\mathbb{R}$ where $\tilde{u} \neq 0$, then for all $t \in \Omega_0$ we have $|u_n| = |\tilde{u}_n| \cdot \|u_n\| \to \infty$. It follows from ($H_4$) and $W_1(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$ that

$$\int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt \geq \int_{\Omega_0} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt - \int_{\mathbb{R}\setminus\Omega_0} \frac{W_1(t)}{\|u_n\|^2}\,dt \to +\infty \quad \text{as } n \to \infty.$$

Case 2. If $\tilde{u} \equiv 0$, then by (H_4) and $W_1(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$, we have

$$\int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt = \int_{\mathbb{R}} \frac{H(t,u_n)}{\|u_n\|^2}\,dt \geq -\int_{\mathbb{R}} \frac{W_1(t)}{\|u_n\|^2}\,dt \to 0 \quad \text{as } n \to \infty.$$

Therefore, Cases 1 and 2 imply that (2.6) holds. Therefore, by (2.5), (2.6) and the facts $\tilde{s}_n \to \tilde{s}$, $\lambda_n \to \lambda$, we have

$$(\lambda + 1)\tilde{s}^2 - 1 \geq 0,$$

that is, $\tilde{s}^2 \geq \frac{1}{1+\lambda} \geq \frac{1}{1+\lambda_0} > 0$. Thus, $\tilde{u} \neq 0$. It follows from (H_4) that

$$\int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2\,dt \to +\infty \quad \text{as } n \to \infty,$$

which contradicts (2.5). The proof is finished. □

Therefore, Lemmas 2.3 and 2.4 imply that condition (d) of Lemma 2.1 holds. Applying Lemma 2.1, we soon obtain the following fact.

Lemma 2.5 *Under assumptions of Theorem* 1.1, *for almost all* $\lambda \in [1, \lambda_0]$, *there exists a sequence* $\{u_n\}$ *such that*

$$\sup_n \|u_n\| < \infty, \qquad I'_\lambda(u_n) \to 0, \qquad I_\lambda(u_n) \to c_\lambda,$$

where the definition of c_λ *is given in Lemma* 2.1.

Lemma 2.6 *Under assumptions of Theorem* 1.1, *for almost all* $\lambda \in [1, \lambda_0]$, *there exists a* $u_\lambda \in E$ *such that*

$$I'_\lambda(u_\lambda) = 0, \qquad I_\lambda(u_\lambda) = c_\lambda.$$

Proof Let $\{u_n\}$ be the sequence obtained in Lemma 2.5. Since $\{u_n\}$ is bounded, we can assume $u_n \rightharpoonup u_\lambda$ in E and $u_n \to u_\lambda$ a.e. on $\mathbb{R}$. By (H_1), (H_3), (2.1) and Theorem A.4 in [27], we have

$$\int_{\mathbb{R}} \frac{1}{2}\big(\nabla H(t,u_n), u_n\big)\,dt \to \int_{\mathbb{R}} \frac{1}{2}\big(\nabla H(t,u_\lambda), u_\lambda\big)\,dt \tag{2.7}$$

and

$$\int_{\mathbb{R}} H(t,u_n)\,dt \to \int_{\mathbb{R}} H(t,u_\lambda)\,dt. \tag{2.8}$$

By Lemma 2.5 and the fact that I'_λ is weakly sequentially continuous, we have

$$I'_\lambda(u_\lambda)\varphi = \lim_{n\to\infty} I'_\lambda(u_n)\varphi = 0, \quad \forall \varphi \in E.$$

That is, $I'_\lambda(u_\lambda) = 0$. By Lemma 2.5, we have

$$I_\lambda(u_n) - \frac{1}{2}I'_\lambda(u_n)u_n = \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t,u_n), u_n\big) - H(t,u_n)\right) dt \to c_\lambda.$$

It follows from (2.7), (2.8) and the fact $I'_\lambda(u_\lambda) = 0$ that

$$I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{2}I'_\lambda(u_\lambda)u_\lambda = \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t,u_\lambda), u_\lambda\big) - H(t,u_\lambda)\right) dt = c_\lambda.$$

The proof is finished. □

Applying Lemma 2.6, we soon obtain the following fact.

Lemma 2.7 *Under assumptions of Theorem* 1.1, *for every* $\lambda \in [1, \lambda_0]$, *there are sequences* $\{u_n\} \subset E$ *and* $\lambda_n \in [1, \lambda_0]$ *with* $\lambda_n \to \lambda$ *such that*

$$I'_{\lambda_n}(u_n) = 0, \qquad I_{\lambda_n}(u_n) = c_{\lambda_n}.$$

Lemma 2.8 *Under assumptions of Theorem* 1.1, *then*

$$\int_{\mathbb{R}} \left[H(t,u) - H(t,rw) + r^2\big(\nabla H(t,u), w\big) - \frac{1+r^2}{2}\big(\nabla H(t,u), u\big)\right] dt \le C,$$

where $u \in E$, $w \in E^+$, $0 \le r \le 1$ *and the constant* $C := \int_{\mathbb{R}} |W_1(t)|\,dt$ *does not depend on* u, w, r.

Proof This follows from (H_5) if we take $u = u$ and $z = rw - u$. □

Lemma 2.9 *The sequences given in Lemma* 2.7 *are bounded.*

Proof Write $u_n = u_n^+ + u_n^-$, where $u_n^\pm \in E^\pm$. Suppose that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty.$$

Let $v_n := \frac{u_n}{\|u_n\|}$, then $v_n^+ = \frac{u_n^+}{\|u_n\|}$, $v_n^- = \frac{u_n^-}{\|u_n\|}$, $\|v_n\|^2 = \|v_n^+\|^2 + \|v_n^-\|^2 = 1$ and $\|v_n^+\| \le 1$. Thus $v_n^+ \rightharpoonup v^+$ in E and $v_n^+ \to v^+$ a.e. on $\mathbb{R}$, after passing to a subsequence.

Case 1. If $v^+ \not\equiv 0$. Let Ω_1 be the subset of $\mathbb{R}$ where $v^+ \neq 0$. Then $v \neq 0$ and $|u_n| = |v_n| \cdot \|u_n\| \to \infty$ on Ω_1. It follows from (H_4) and $W_1(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$ that

$$\int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2} |v_n|^2 \, dt \ge \int_{\Omega_1} \frac{H(t,u_n)}{|u_n|^2} |v_n|^2 \, dt - \int_{\mathbb{R}\setminus\Omega_1} \frac{W_1(t)}{\|u_n\|^2} \, dt \to +\infty \quad \text{as } n \to \infty,$$

which together with Lemmas 2.3 and 2.7 and $v_n^\pm \to v^\pm$ in $L^q(\mathbb{R}, \mathbb{R}^N)$ for all $1 \le q \le \infty$ (by (2.1)) implies that

$$0 \le \frac{c_{\lambda_n}}{\|u_n\|^2} = \frac{I_{\lambda_n}(u_n)}{\|u_n\|^2} = \frac{\lambda_n}{2} \|v_n^+\|^2 - \frac{1}{2} \|v_n^-\|^2 - \int_{\mathbb{R}} \frac{H(t,u_n)}{|u_n|^2} |v_n|^2 \, dt \to -\infty \quad \text{as } n \to \infty.$$

It is a contradiction.

Case 2. If $v^+ \equiv 0$. We claim that there is a constant C independent of u_n and λ_n such that

$$I_{\lambda_n}(ru_n^+) - I_{\lambda_n}(u_n) \le C, \quad \forall r \in [0,1]. \tag{2.9}$$

Since

$$\frac{1}{2} I'_{\lambda_n}(u_n)\varphi = \frac{1}{2}\lambda_n \langle u_n^+, \varphi^+ \rangle - \frac{1}{2} \langle u_n^-, \varphi^- \rangle - \frac{1}{2} \int_{\mathbb{R}} \big(\nabla H(t,u_n), \varphi\big) \, dt = 0, \quad \forall \varphi \in E,$$

it follows from the definition of I that

$$\begin{aligned} I_{\lambda_n}(ru_n^+) - I_{\lambda_n}(u_n) &= \frac{1}{2}\lambda_n (r^2 - 1) \|u_n^+\|^2 + \frac{1}{2} \|u_n^-\|^2 + \int_{\mathbb{R}} \big(H(t,u_n) - H(t, ru_n^+)\big) \, dt \\ &\quad + \frac{1}{2}\lambda_n \langle u_n^+, \varphi^+ \rangle - \frac{1}{2} \langle u_n^-, \varphi^- \rangle - \frac{1}{2} \int_{\mathbb{R}} \big(\nabla H(t,u_n), \varphi\big) \, dt. \end{aligned} \tag{2.10}$$

Take $\varphi := (r^2+1)u_n^- - (r^2-1)u_n^+ = (r^2+1)u_n - 2r^2 u_n^+$ in (2.10), then it follows from Lemma 2.8 that

$$\begin{aligned} I_{\lambda_n}(ru_n^+) - I_{\lambda_n}(u_n) &= -\frac{r^2}{2} \|u_n^-\|^2 + \int_{\mathbb{R}} \bigg[H(t,u_n) - H(t, ru_n^+) + r^2 \big(\nabla H(t,u_n), u_n^+\big) \\ &\quad - \frac{1+r^2}{2} \big(\nabla H(t,u_n), u_n\big) \bigg] \, dt \\ &\le C. \end{aligned}$$

Thus (2.9) holds.

Let $C_0 > 0$ be a fixed constant and take

$$r_n := \frac{C_0}{\|u_n\|} \to 0 \quad \text{as } n \to \infty.$$

Therefore, (2.9) implies that

$$I_{\lambda_n}\big(r_n u_n^+\big) - I_{\lambda_n}(u_n) \le C.$$

It follows from $v_n^+ = \frac{u_n^+}{\|u_n\|}$ and Lemma 2.7 that

$$I_{\lambda_n}\big(C_0 v_n^+\big) \le C'. \tag{2.11}$$

Note that Lemmas 2.3 and 2.7 and (H_4) imply that

$$\begin{aligned} 0 \le \frac{c_{\lambda_n}}{\|u_n\|^2} = \frac{I_{\lambda_n}(u_n)}{\|u_n\|^2} &= \frac{\lambda_n}{2}\big\|v_n^+\big\|^2 - \frac{1}{2}\big\|v_n^-\big\|^2 - \frac{\int_{\mathbb{R}} H(t,u_n)}{\|u_n\|^2}\,dt \\ &\le \frac{\lambda_0}{2}\big\|v_n^+\big\|^2 - \frac{1}{2}\big\|v_n^-\big\|^2 + \frac{\int_{\mathbb{R}} W_1(t)\,dt}{\|u_n\|^2}. \end{aligned}$$

It follows from the fact $\frac{\int_{\mathbb{R}} W_1(t)\,dt}{\|u_n\|^2} \to 0$ as $n \to \infty$ due to $W_1(t) \in L^1(\mathbb{R},\mathbb{R}^+)$ that

$$\frac{\lambda_0}{2}\big\|v_n^+\big\|^2 - \frac{1}{2}\big\|v_n^-\big\|^2 + \varepsilon \ge 0, \quad \forall \varepsilon > 0 \tag{2.12}$$

for all sufficiently large n. We take $\varepsilon = \frac{1}{4}$, by (2.12) and $\|v_n\|^2 = \|v_n^+\|^2 + \|v_n^-\|^2 = 1$, we have

$$\big\|v_n^+\big\|^2 \ge \frac{1}{2(1+\lambda_0)} \tag{2.13}$$

for all sufficiently large n. By (H_1) and (H_3), we have

$$\begin{aligned} &\int_{\mathbb{R}} H\big(t, C_0 v_n^+\big)\,dt \\ &\quad\le \frac{1}{2}\gamma C_0^2 \int_{\{t\in\mathbb{R}:|C_0 v_n^+|<\delta\}} \big|v_n^+\big|^2\,dt + \frac{1}{2}c\int_{\{t\in\mathbb{R}:|C_0 v_n^+|\ge\delta\}} \big(C_0\big|v_n^+\big| + C_0^p\big|v_n^+\big|^p\big)\,dt \\ &\quad\le \frac{1}{2}\gamma C_0^2 \int_{\{t\in\mathbb{R}:|C_0 v_n^+|<\delta\}} \big|v_n^+\big|^2\,dt + C_1 C_0^p \int_{\{t\in\mathbb{R}:|C_0 v_n^+|\ge\delta\}} \big|v_n^+\big|^p\,dt. \end{aligned} \tag{2.14}$$

For all sufficiently large n, by (2.13) and (2.14), it follows from $\lambda_n \to \lambda$ and $v_n^+ \to v^+ \equiv 0$ in $L^q(\mathbb{R},\mathbb{R}^N)$ for all $1 \le q \le \infty$ (by (2.1)) that

$$\begin{aligned} I_{\lambda_n}\big(C_0 v_n^+\big) &= \frac{1}{2}\lambda_n C_0^2\big\|v_n^+\big\|^2 - \int_{\mathbb{R}} H\big(t, C_0 v_n^+\big)\,dt \\ &\ge \frac{1}{2}\lambda_n C_0^2 \frac{1}{2(1+\lambda_0)} - \frac{1}{2}\gamma C_0^2 \int_{\{t\in\mathbb{R}:|C_0 v_n^+|<\delta\}} \big|v_n^+\big|^2\,dt \\ &\quad - C_1 C_0^p \int_{\{t\in\mathbb{R}:|C_0 v_n^+|\ge\delta\}} \big|v_n^+\big|^p\,dt \\ &\to \frac{\lambda C_0^2}{4(1+\lambda_0)} \quad \text{as } n \to \infty. \end{aligned}$$

This implies that $I_{\lambda_n}(C_0 v_n^+) \to \infty$ as $C_0 \to \infty$, contrary to (2.11).

Therefore, $\{u_n\}$ are bounded. The proof is finished. □

3 Proofs of the main results

Proof of Theorem 1.1 From Lemma 2.7, there are sequences $1 < \lambda_n \to 1$ and $\{u_n\} \subset E$ such that $I'_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) = c_{\lambda_n}$. By Lemma 2.9, we know that $\{u_n\}$ is bounded in E. Thus we can assume $u_n \rightharpoonup u$ in E, $u_n \to u$ a.e. on $\mathbb{R}$. Therefore,

$$I'_{\lambda_n}(u_n)\varphi = \lambda_n\langle u_n^+, \varphi\rangle - \langle u_n^-, \varphi\rangle - \int_{\mathbb{R}} \big(\nabla H(t, u_n), \varphi\big)\, dt = 0, \quad \forall \varphi \in E.$$

Hence, in the limit,

$$I'(u)\varphi = \langle u^+, \varphi\rangle - \langle u^-, \varphi\rangle - \int_{\mathbb{R}} \big(\nabla H(t, u), \varphi\big)\, dt = 0, \quad \forall \varphi \in E.$$

Thus $I'(u) = 0$. Note that

$$I_{\lambda_n}(u_n) - \frac{1}{2}I'_{\lambda_n}(u_n)u_n = \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t, u_n), u_n\big) - H(t, u_n)\right) dt = c_{\lambda_n} \geq c_1. \tag{3.1}$$

Similar to (2.7) and (2.8), we know

$$\int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t, u_n), u_n\big) - H(t, u_n)\right) dt \to \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t, u), u\big) - H(t, u)\right) dt \quad \text{as } n \to \infty.$$

It follows from $I'(u) = 0$, (3.1) and Lemma 2.3 that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2}I'(u)u = \int_{\mathbb{R}} \left[\frac{1}{2}\big(\nabla H(t, u), u\big) - H(t, u)\right] dt \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t, u_n), u_n\big) - H(t, u_n)\right) dt \geq c_1 \geq \epsilon > 0. \end{aligned}$$

Therefore, $u \neq 0$. □

Proof of Theorem 1.2 By Theorem 1.1, $\mathcal{M} \neq \emptyset$, where $\mathcal{M}$ is the collection of solutions of (1.1). Let

$$\alpha := \inf_{u\in\mathcal{M}} I(u).$$

If u is a solution of (1.1), then by Lemma 2.8 (take $r = 0$),

$$I(u) = I(u) - \frac{1}{2}I'(u)u = \int_{\mathbb{R}} \left[\frac{1}{2}\big(\nabla H(t, u), u\big) - H(t, u)\right] dt \geq -C = -\int_{\mathbb{R}} \big|W_1(t)\big|\, dt.$$

Thus $\alpha > -\infty$. Let $\{u_n\}$ be a sequence in $\mathcal{M}$ such that

$$I(u_n) \to \alpha. \tag{3.2}$$

By Lemma 2.9, the sequence $\{u_n\}$ is bounded in E. Therefore, $u_n \rightharpoonup u$ in E, $u_n \to u$ a.e. on $\mathbb{R}$ and $u_n \to u$ in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [1, +\infty]$ (by (2.1)), after passing to a subsequence. Therefore,

$$I'(u_n)\varphi = \langle u_n^+, \varphi\rangle - \langle u_n^-, \varphi\rangle - \int_{\mathbb{R}} \big(\nabla H(t, u_n), \varphi\big)\, dt = 0, \quad \forall \varphi \in E.$$

Hence, in the limit,

$$I'(u)\varphi = \langle u^+, \varphi\rangle - \langle u^-, \varphi\rangle - \int_{\mathbb{R}} \big(\nabla H(t,u), \varphi\big)\, dt = 0, \quad \forall \varphi \in E.$$

Thus $I'(u) = 0$. Similar to (2.7) and (2.8), we have

$$\begin{aligned} I(u_n) - \frac{1}{2}I'(u_n)u_n &= \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t,u_n), u_n\big) - H(t,u_n)\right) dt \\ &\to \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t,u), u\big) - H(t,u)\right) dt \quad \text{as } n \to \infty. \end{aligned}$$

It follows from $I'(u) = 0$ and (3.2) that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2}I'(u)u = \int_{\mathbb{R}} \left[\frac{1}{2}\big(\nabla H(t,u), u\big) - H(t,u)\right] dt \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \left(\frac{1}{2}\big(\nabla H(t,u_n), u_n\big) - H(t,u_n)\right) dt \\ &= \lim_{n\to\infty} I(u_n) = \alpha. \end{aligned}$$

Now suppose that

$$\big|\nabla H(t,u)\big| = o\big(|u|\big) \quad \text{as } |u| \to 0.$$

It follows from (H_1) that for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\big|\nabla H(t,u)\big| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}. \tag{3.3}$$

Let

$$\beta := \inf_{u\in M'} I(u),$$

where $M' := \mathcal{M} \setminus \{0\}$. Let $\{u_n\}$ be a sequence in $\mathcal{M} \setminus \{0\}$ such that

$$I(u_n) \to \beta. \tag{3.4}$$

Note that

$$0 = I'(u_n)u_n^+ = \big\|u_n^+\big\|^2 - \int_{\mathbb{R}} \big(\nabla H(t,u_n), u_n^+\big)\, dt,$$

which together with (3.3), Hölder's inequality and the Sobolev embedding theorem implies

$$\begin{aligned} \big\|u_n^+\big\|^2 &= \int_{\mathbb{R}} \big(\nabla H(t,u_n), u_n^+\big)\, dt \\ &\le \varepsilon \int_0^T |u_n| \cdot \big|u_n^+\big|\, dt + C_\varepsilon \int_0^T |u_n|^{p-1}\big|u_n^+\big|\, dt \\ &\le \varepsilon\|u_n\| \cdot \big\|u_n^+\big\| + C'_\varepsilon\|u_n\|_{L^p}^{p-1}\big\|u_n^+\big\| \end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon \|u_n\| \cdot \|u_n^+\| + C''_\varepsilon \|u_n\|_{L^p}^{p-2} \|u_n\| \cdot \|u_n^+\| \\
&\le \varepsilon \|u_n\|^2 + C''_\varepsilon \|u_n\|_{L^p}^{p-2} \|u_n\|^2. \qquad (3.5)
\end{aligned}
$$

Similarly, we have

$$
\|u_n^-\|^2 \le \varepsilon \|u_n\|^2 + C''_\varepsilon \|u_n\|_{L^p}^{p-2} \|u_n\|^2. \qquad (3.6)
$$

From (3.5) and (3.6), we get

$$
\|u_n\|^2 \le 2\varepsilon \|u_n\|^2 + 2C''_\varepsilon \|u_n\|_{L^p}^{p-2} \|u_n\|^2,
$$

which means $\|u_n\|_{L^p} \ge C$ for some constant $C > 0$. Since $u_n \to u$ in $L^p(\mathbb{R}, \mathbb{R}^N)$, we know $u \ne 0$. As before, $I(u_n) \to I(u) = \beta$ as $n \to \infty$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by G-WC and G-WC prepared the manuscript initially and JW performed a part of steps of the proofs in this research. All authors read and approved the final manuscript.

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