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Solvability of boundary value problem with p-Laplacian at resonance

Weihua Jiang*

*Correspondence:
weihuajiang@hebust.edu.cn
College of Sciences, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, P.R. China

Abstract
By generalizing the extension of the continuous theorem of Ge and Ren and constructing suitable Banach spaces and operators, we investigate the existence of solutions for p-Laplacian boundary value problems at resonance. An example is given to illustrate our results.

MSC: 34B15

Keywords: continuous theorem; resonance; p-Laplacian; boundary value problem

1 Introduction

In this paper, we will study the boundary value problem

$$\begin{cases} (\varphi_p(u''))'(t) = f(t, u(t), u'(t), u''(t)), \\ u(0) = u''(0) = 0, \qquad u'(1) = \int_0^1 k(t)u'(t)\,dt, \end{cases} \tag{1.1}$$

and

$$\begin{cases} (\varphi_p(u''))'(t) = f(t, u(t), u'(t), u''(t)), \\ u''(0) = 0, \qquad u'(0) = \int_0^1 g(t)u'(t)\,dt, \qquad u'(1) = \int_0^1 h(t)u'(t)\,dt, \end{cases} \tag{1.2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\int_0^1 k(t)\,dt = 1$, $\int_0^1 g(t)\,dt = 1$, $\int_0^1 h(t)\,dt = 1$.

A boundary value problem is said to be a resonance one if the corresponding homogeneous boundary value problem has a non-trivial solution. Mawhin's continuous theorem [1] is an effective tool to solve this kind of problems when the differential operator is linear, see [2–10] and references cited therein. But it does not work for nonlinear cases such as boundary value problems with a p-Laplacian, which attracted the attention of mathematicians in recent years [11–15]. Ge and Ren extended Mawhin's continuous theorem [15] and many authors used their results to solve boundary value problems with a p-Laplacian, see [16, 17]. In this new theorem, two projectors P and Q must be constructed. But it is difficult to give the projector Q in many boundary value problems with a p-Laplacian. In this paper, we generalize the extension of the continuous theorem and show that the p-Laplacian problem is solvable when Q is not a projector. And we will use this new theorem to discuss problems (1.1) and (1.2), respectively.

In this paper, we will always suppose that

(H_1) $k(t), g(t), h(t) \in L^1[0,1]$ are nonnegative and $\|k\|_1 = \|g\|_1 = \|h\|_1 = 1$, where $\|k\|_1 := \int_0^1 |k(t)|\,dt$.
(H_2) $f(t,u,v,w)$ is continuous in $[0,1] \times \mathbb{R}^3$.

2 Preliminaries

Definition 2.1 [15] Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Y$ is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y,
(ii) $\operatorname{Ker} M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $\operatorname{dom} M$ denote the domain of the operator M.

Let $X_1 = \operatorname{Ker} M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. Let $P : X \to X_1$ be a projector and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.2 Suppose $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous and bounded operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} \cap \operatorname{dom} M : Mx = N_\lambda x\}$. N_λ is said to be M-quasi-compact in $\overline{\Omega}$ if there exists a vector subspace Y_1 of Y satisfying $\dim Y_1 = \dim X_1$ and two operators Q, R with $Q : Y \to Y_1$, $QY = Y_1$, being continuous, bounded, and satisfying $Q(I-Q) = 0$, $R : \overline{\Omega} \times [0,1] \to X_2 \cap \operatorname{dom} M$ continuous and compact such that for $\lambda \in [0,1]$,

(a) $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$,
(b) $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$,
(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$,
(d) $M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda$.

Theorem 2.1 *Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, and let $\Omega \subset X$ be an open and bounded nonempty set. Suppose*

$$M : X \cap \operatorname{dom} M \to Y$$

is a quasi-linear operator and that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is M-quasi-compact. In addition, if the following conditions hold:

(C_1) $Mx \neq N_\lambda x$, $\forall x \in \partial\Omega \cap \operatorname{dom} M$, $\lambda \in (0,1)$,
(C_2) $\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$,

then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$, *where $N = N_1$, $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(\theta) = \theta$.*

Proof The proof is similar to the one of Lemma 2.1 and Theorem 2.1 in [15]. □

We can easily get the following inequalities.

Lemma 2.1 *For any $u, v \geq 0$, we have*

(1) $\varphi_p(u+v) \leq \varphi_p(u) + \varphi_p(v)$, $1 < p \leq 2$.
(2) $\varphi_p(u+v) \leq 2^{p-2}(\varphi_p(u) + \varphi_p(v))$, $p \geq 2$.

In the following, we will always suppose that q satisfies $1/p + 1/q = 1$.

3 The existence of a solution for problem (1.1)

Let $X = C^2[0,1]$ with norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$, $Y = C[0,1] \times C[0,1]$ with norm $\|(y_1,y_2)\| = \max\{\|y_1\|_\infty, \|y_2\|_\infty\}$, where $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$. We know that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach spaces.

Define operators $M : X \cap \operatorname{dom} M \to Y$, $N_\lambda : X \to Y$ as follows:

$$Mu = \begin{bmatrix} (\varphi_p(u''))'(t) \\ T(\varphi_p(u''))'(t) \end{bmatrix}, \qquad N_\lambda u = \begin{bmatrix} \lambda f(t, u(t), u'(t), u''(t)) \\ 0 \end{bmatrix},$$

where $Ty = c$, $y \in C[0,1]$, c satisfying

$$\begin{aligned} &\int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s y(r) - c\, dr\right) ds\, dt = 0, \\ &\operatorname{dom} M = \left\{u \in X \mid \varphi_p\left(u''\right) \in C^1[0,1], u(0) = u''(0) = 0\right\}. \end{aligned} \tag{3.1}$$

Lemma 3.1 *For $y \in C[0,1]$, there is only one constant $c \in \mathbb{R}$ such that $Ty = c$ with $|c| \le \|y\|_\infty$ and that $T : C[0,1] \to \mathbb{R}$ is continuous.*

Proof For $y \in C[0,1]$, let

$$F(c) = \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left(y(r) - c\right) dr\right) ds\, dt.$$

Obviously, $F(c)$ is continuous and strictly decreasing in $\mathbb{R}$. Take $a = \min_{t\in[0,1]} y(t)$, $b = \max_{t\in[0,1]} y(t)$. It is easy to see that $F(a) \ge 0$, $F(b) \le 0$. Thus, there exists a unique constant $c \in [a,b]$ such that $F(c) = 0$, *i.e.* there is only one constant $c \in \mathbb{R}$ such that $Ty = c$ with $|c| \le \|y\|_\infty$.

For $y_1, y_2 \in C[0,1]$, assume $Ty_1 = c_1$, $Ty_2 = c_2$. By $k(t) \ge 0$, $\int_0^1 k(t)\, dt = 1$ and φ_q being strictly increasing, we obtain, if $c_2 - c_1 > \max_{t\in[0,1]}(y_2(t) - y_1(t))$, then

$$\begin{aligned} 0 &= \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left(y_2(r) - c_2\right) dr\right) ds\, dt \\ &= \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left[\left(y_1(r) - c_1\right) + \left(y_2(r) - y_1(r) - (c_2 - c_1)\right) dr\right]\right) ds\, dt \\ &< \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left(y_1(r) - c_1\right) dr\right) ds\, dt = 0. \end{aligned}$$

This is a contradiction. On the other hand, if $c_2 - c_1 < \min_{t\in[0,1]}(y_2(t) - y_1(t))$, then

$$\begin{aligned} 0 &= \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left(y_2(r) - c_2\right) dr\right) ds\, dt \\ &= \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left[\left(y_1(r) - c_1\right) + \left(y_2(r) - y_1(r) - (c_2 - c_1)\right) dr\right]\right) ds\, dt \\ &> \int_0^1 k(t) \int_t^1 \varphi_q\left(\int_0^s \left(y_1(r) - c_1\right) dr\right) ds\, dt = 0. \end{aligned}$$

This is a contradiction, too. So, we have $\min_{t\in[0,1]}(y_2(t)-y_1(t)) \le c_2 - c_1 \le \max_{t\in[0,1]}(y_2(t)-y_1(t))$, *i.e.* $|c_2 - c_1| \le \|y_2 - y_1\|_\infty$. So, $T: C[0,1] \to \mathbb{R}$ is continuous. The proof is completed. $\square$

It is clear that $u \in \operatorname{dom} M$ is a solution if and only if it satisfies $Mu = Nu$, where $N = N_1$. For convenience, let $(a,b)^L := \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$.

Lemma 3.2 *M is a quasi-linear operator.*

Proof It is easy to see that $\operatorname{Ker} M = \{bt \mid b \in \mathbb{R}\} := X_1$.

For $u \in X \cap \operatorname{dom} M$, if $Mu = (y,c)^L$, then c satisfies (3.1). On the other hand, if $y \in C[0,1]$, $Ty = c$, take

$$u(t) = \int_0^t (t-s)\varphi_q\left(\int_0^s y(r)\,dr\right)ds.$$

By a simple calculation, we get $u \in X \cap \operatorname{dom} M$ and $Mu = (y,c)^L$. Thus

$$\operatorname{Im} M = \left\{(y,c)^L \mid y \in C[0,1], c \text{ satisfies (3.1)}\right\}.$$

By the continuity of T, we find that $\operatorname{Im} M \subset Y$ is closed. So, M is quasi-linear. The proof is completed. $\square$

Lemma 3.3 $T(c) = c$, $T(y+c) = T(y) + c$, $T(cy) = cT(y)$, $c \in \mathbb{R}$, $y \in C[0,1]$.

Proof The proof is simple. Therefore, we omit it. $\square$

Take a projector $P: X \to X_1$ and an operator $Q: Y \to Y_1$ as follows:

$$(Pu)(t) = u'(0)t, \qquad Q(y,y_1)^L = (0, Ty_1 - Ty)^L,$$

where $Y_1 = \{(0,c)^L \mid c \in \mathbb{R}\}$. Obviously, $QY = Y_1$, and $\dim Y_1 = \dim X_1$.

By the continuity and boundedness of T, we can easily see that Q is continuous and bounded in Y. It follows from Lemma 3.3 that $Q(I-Q)(y,y_1)^L = (0,0)^L$, $y, y_1 \in C[0,1]$.

Define an operator $R: X \times [0,1] \to X_2$ as

$$R(u,\lambda)(t) = \int_0^t (t-s)\varphi_q\left(\int_0^s \lambda f\bigl(r, u(r), u'(r), u''(r)\bigr)\,dr\right)ds,$$

where $\operatorname{Ker} M \oplus X_2 = X$. By (H_2) and the Arzela-Asscoli theorem, we can easily see that $R: \overline{\Omega} \times [0,1] \to X_2 \cap \operatorname{dom} M$ is continuous and compact, where $\Omega \subset X$ is an open bounded set.

Lemma 3.4 *Assume that* $\Omega \subset X$ *is an open bounded set. Then* N_λ *is M-quasi-compact in* $\overline{\Omega}$.

Proof It is clear that $\operatorname{Im} P = \operatorname{Ker} M$, $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$ and $R(\cdot,0) = 0$. For $u \in \overline{\Omega}$,

$$\begin{aligned}(I-Q)N_\lambda u &= \begin{bmatrix} \lambda f(t,u(t),u'(t),u''(t)) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -T[\lambda f(t,u(t),u'(t),u''(t))] \end{bmatrix} \\ &= \begin{bmatrix} \lambda f(t,u(t),u'(t),u''(t)) \\ T[\lambda f(t,u(t),u'(t),u''(t))] \end{bmatrix} \in \operatorname{Im} M.\end{aligned}$$

Since $\operatorname{Im} M \subset \operatorname{Ker} Q$ and $y = Qy + (I-Q)y$, we obtain $\operatorname{Im} M \subset (I-Q)Y$. Thus, $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$.

For $u \in \Sigma_\lambda = \{u \in \overline{\Omega} \cap \operatorname{dom} M : Mu = N_\lambda u\}$, we get

$$\begin{aligned}R(u,\lambda) &= \int_0^t (t-s)\varphi_q\left(\int_0^s \lambda f\bigl(r,u(r),u'(r),u''(r)\bigr)\,dr\right) ds \\ &= \int_0^t (t-s)\varphi_q\left(\int_0^s \bigl(\varphi_p\bigl(u''\bigr)\bigr)'\right) ds \\ &= u(t) - u'(0)t = (I-P)u,\end{aligned}$$

i.e. Definition 2.2(c) holds. For $u \in \overline{\Omega}$, we have

$$M\bigl[Pu + R(u,\lambda)\bigr] = \begin{bmatrix} \lambda f(t,u(t),u'(t),u''(t)) \\ T[\lambda f(t,u(t),u'(t),u''(t))] \end{bmatrix} = (I-Q)N_\lambda u.$$

So, Definition 2.2(d) holds. Therefore, N_λ is M-quasi-compact in $\overline{\Omega}$. The proof is completed. □

Theorem 3.1 *Assume that the following conditions hold.*

(H_3) *There exists a nonnegative constant K such that one of* (1) *and* (2) *holds*:
  (1) $Bf(t,A,B,C) > 0$, $t \in [0,1]$, $|B| > K$, $A, C \in \mathbb{R}$,
  (2) $Bf(t,A,B,C) < 0$, $t \in [0,1]$, $|B| > K$, $A, C \in \mathbb{R}$.

(H_4) *There exist nonnegative functions $a(t), b(t), c(t), e(t) \in L^1[0,1]$ such that*

$$\bigl|f(t,x,y,z)\bigr| \le a(t)\varphi_p\bigl(|x|\bigr) + b(t)\varphi_p\bigl(|y|\bigr) + c(t)\varphi_p\bigl(|z|\bigr) + e(t), \quad t \in [0,1], x,y,z \in \mathbb{R},$$

where $\varphi_q(\|a\|_1 + \|b\|_1 + \|c\|_1) < 2^{2-q}$, *if* $1 < p \le 2$; $\varphi_q(2^{p-2}\|a\|_1 + 2^{p-2}\|b\|_1 + \|c\|_1) < 1$, *if* $p \ge 2$.

Then boundary value problem (1.1) *has at least one solution.*

In order to prove Theorem 3.1, we show two lemmas.

Lemma 3.5 *Suppose* (H_3) *and* (H_4) *hold. Then the set*

$$\Omega_1 = \bigl\{u \in \operatorname{dom} M \mid Mu = N_\lambda u, \lambda \in (0,1)\bigr\}$$

is bounded in X.

Proof For $u \in \Omega_1$, we have $QN_\lambda u = 0$, *i.e.* $Tf(t, u(t), u'(t), u''(t)) = 0$. By ($H_3$), there exists a constant $t_0 \in [0,1]$ such that $|u'(t_0)| \leq K$. Since $u(t) = \int_0^t u'(s)\,ds$, $u'(t) = u'(t_0) + \int_{t_0}^t u''(s)\,ds$, we have

$$\bigl|u(t)\bigr| \leq \bigl\|u'\bigr\|_\infty, \qquad \bigl|u'(t)\bigr| \leq K + \bigl\|u''\bigr\|_\infty, \quad t \in [0,1]. \tag{3.2}$$

It follows from $Mu = N_\lambda u$, (H_4), and (3.2) that

$$\begin{aligned}\bigl|u''(t)\bigr| &= \left|\varphi_q\left(\int_0^t \lambda f\bigl(s, u(s), u'(s), u''(s)\bigr)\,ds\right)\right| \\ &\leq \varphi_q\left(\int_0^1 a(t)\varphi_p\bigl(|u|\bigr) + b(t)\varphi_p\bigl(\bigl|u'\bigr|\bigr) + c(t)\varphi_p\bigl(\bigl|u''\bigr|\bigr) + e(t)\,dt\right) \\ &\leq \varphi_q\bigl[\bigl(\|a\|_1 + \|b\|_1\bigr)\varphi_p\bigl(K + \bigl\|u''\bigr\|_\infty\bigr) + \|c\|_1\varphi_p\bigl(\bigl\|u''\bigr\|_\infty\bigr) + \|e\|_1\bigr].\end{aligned}$$

If $1 < p \leq 2$, by Lemma 2.1, we get

$$\bigl|u''(t)\bigr| \leq \varphi_q\bigl(B_1 + A_1\varphi_p\bigl(\bigl\|u''\bigr\|_\infty\bigr)\bigr) \leq 2^{q-2}\bigl[\varphi_q(B_1) + \varphi_q(A_1)\bigl\|u''\bigr\|_\infty\bigr],$$

thus

$$\bigl\|u''\bigr\|_\infty \leq \frac{2^{q-2}\varphi_q(B_1)}{1 - 2^{q-2}\varphi_q(A_1)},$$

where $B_1 = (\|a\|_1 + \|b\|_1)\varphi_p(K) + \|e\|_1$, $A_1 = \|a\|_1 + \|b\|_1 + \|c\|_1$.

If $p > 2$, by Lemma 2.1, we get

$$\bigl|u''(t)\bigr| \leq \varphi_q\bigl(B_2 + A_2\varphi_p\bigl(\bigl\|u''\bigr\|_\infty\bigr)\bigr) \leq \bigl[\varphi_q(B_2) + \varphi_q(A_2)\bigl\|u''\bigr\|_\infty\bigr],$$

thus

$$\bigl\|u''\bigr\|_\infty \leq \frac{\varphi_q(B_2)}{1 - \varphi_q(A_2)},$$

where $B_2 = 2^{p-2}(\|a\|_1 + \|b\|_1)\varphi_p(K) + \|e\|_1$, $A_2 = 2^{p-2}(\|a\|_1 + \|b\|_1) + \|c\|_1$.

These, together with (3.2), mean that Ω_1 is bounded in X. □

Lemma 3.6 *Assume* (H_3) *holds. Then*

$$\Omega_2 = \{u \in \operatorname{Ker} M \mid QNu = 0\}$$

is bounded in X, *where* $N = N_1$.

Proof For $u \in \Omega_2$, we have $u = bt$ and $Tf(t, bt, b, 0) = 0$. By (H_3), we get $|b| \leq K$. So, Ω_2 is bounded. The proof is completed. □

Proof of Theorem 3.1 Let $\Omega = \{u \in X \mid \|u\| < r\}$, where r is large enough such that $K < r < +\infty$ and $\Omega \supset \overline{\Omega_1}$.

By Lemmas 3.5 and 3.6, we know $Mu \neq N_\lambda u$, $u \in \operatorname{dom} M \cap \partial\Omega$ and $QNu \neq 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$.

Let $H(u,\delta) = \rho\delta u + (1-\delta)JQNu$, $\delta \in [0,1]$, $u \in \operatorname{Ker} M \cap \overline{\Omega}$, where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(0,b)^L = bt$, $\rho = \begin{cases} -1, & \text{if } (\mathrm{H}_3)(1) \text{ holds}, \\ 1, & \text{if } (\mathrm{H}_3)(2) \text{ holds}. \end{cases}$

Define a function $\operatorname{Sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$

For $u \in \operatorname{Ker} M \cap \partial\Omega$, we have $u = bt \neq 0$. Thus

$$H(u,\delta) = \rho\delta bt + (1-\delta)\bigl(-Tf(t,bt,b,0)\bigr)t.$$

If $\delta = 1$, $H(u,1) = \rho bt \neq 0$. If $\delta = 0$, by $QNu \neq 0$, we get $H(u,0) = JQN(bt) \neq 0$. For $0 < \delta < 1$, we now prove that $H(u,\delta) \neq 0$. Otherwise, if $H(u,\delta) = 0$, then

$$Tf(t,bt,b,0) = \frac{\rho\delta}{1-\delta} b. \tag{3.3}$$

Since $\|u\| = r > K$, we have $|b| > K$. Thus, $T[bf(t,bt,b,0)] = bTf(t,bt,b,0) = \frac{\rho\delta}{1-\delta} b^2$. So, we have $\operatorname{Sgn}(bf(t,bt,b,0)) = \operatorname{Sgn}\{T[bf(t,bt,b,0)]\} = \operatorname{Sgn}(\frac{\rho\delta}{1-\delta} b^2) = \operatorname{Sgn}(\rho)$. A contradiction with the definition of ρ. So, $H(u,\delta) \neq 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$, $\delta \in [0,1]$.

By the homotopy of degree, we get

$$\begin{aligned} \deg(JQN, \Omega \cap \operatorname{Ker} M, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} M, 0\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} M, 0\bigr) = \deg(\rho I, \Omega \cap \operatorname{Ker} M, 0) \neq 0. \end{aligned}$$

By Theorem 2.1, we can see that $Mu = Nu$ has at least one solution in $\overline{\Omega}$. The proof is completed. □

Example Let us consider the following boundary value problem at resonance:

$$\begin{cases} (\varphi_p(u''))'(t) = \frac{1}{8} t \sin x^3 + \frac{1}{16} y^3 + t^3 \sin z^3 + \cos t, \\ u(0) = u''(0) = 0, \qquad u'(1) = 2\int_0^1 tu'(t)\,dt, \end{cases} \tag{3.4}$$

where $p = 4$.

Corresponding to problem (1.1), we have $q = \frac{4}{3}$, $a(t) = \frac{1}{8}t$, $b(t) = \frac{1}{16}$, $c(t) = t^3$, $e(t) = \cos t$, $k(t) = 2t$.

Take $K = 4$. By a simple calculation, we find that the conditions (H_1)-(H_4) hold. By Theorem 3.1, we obtain the result that problem (3.4) has at least one solution.

4 The existence of a solution for problem (1.2)

Let $X = C^2[0,1]$ with norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$, $Y = C[0,1] \times C[0,1] \times C[0,1]$ with norm $\|(y_1,y_2,y_3)\| = \max\{\|y_1\|_\infty, \|y_2\|_\infty, \|y_3\|_\infty\}$, where $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$. We know that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach spaces.

Define operators $M : X \cap \operatorname{dom} M \to Y$, $N_\lambda : X \to Y$ as follows:

$$Mu = \begin{bmatrix} (\varphi_p(u''))'(t) \\ T_1(\varphi_p(u''))'(t) \\ T_2(\varphi_p(u''))'(t) \end{bmatrix}, \qquad N_\lambda u = \begin{bmatrix} \lambda f(t, u(t), u'(t), u''(t)) \\ 0 \\ 0 \end{bmatrix},$$

where $T_1 y = c_1$, $T_2 y = c_2$, $y \in C[0,1]$, c_1, c_2 satisfy

$$\begin{aligned}
&\int_0^1 g(t) \int_0^t \varphi_q\left(\int_0^s y(r) - c_1\,dr\right) ds\,dt = 0, \\
&\int_0^1 h(t) \int_t^1 \varphi_q\left(\int_0^s y(r) - c_2\,dr\right) ds\,dt = 0, \qquad (4.1)\\
&\operatorname{dom} M = \left\{u \in X \mid \varphi_p\left(u''\right) \in C^1[0,1], u''(0) = 0\right\}.
\end{aligned}$$

Lemma 4.1 *For $y \in C[0,1]$, there is only one constant $c_i \in \mathbb{R}$ such that $T_i y = c_i$ with $|c_i| \le \|y\|_\infty$. And $T_i : C[0,1] \to \mathbb{R}$ are continuous, $i = 1, 2$.*

The proof is similar to Lemma 3.1.

It is clear that $u \in \operatorname{dom} M$ is a solution if and only if it satisfies $Mu = Nu$, where $N = N_1$. For convenience, let $(a, b, c)^T := \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Lemma 4.2 *M is a quasi-linear operator.*

Proof It is easy to get $\operatorname{Ker} M = \{a + bt \mid a, b \in \mathbb{R}\} := X_1$.

For $u \in X \cap \operatorname{dom} M$, if $Mu = (y, c_1, c_2)^T$, then c_1, c_2 satisfy (4.1). On the other hand, if $y \in C[0,1]$, $T_1 y = c_1$, $T_2 y = c_2$, take

$$u(t) = \int_0^t (t - s)\varphi_q\left(\int_0^s y(r)\,dr\right) ds.$$

By simple calculation, we get $u \in X \cap \operatorname{dom} M$ and $Mu = (y, c_1, c_2)^T$. Thus

$$\operatorname{Im} M = \left\{(y, c_1, c_2)^T \mid y \in C[0,1], c_1, c_2 \text{ satisfy (4.1)}\right\}.$$

By the continuity of T_i, $i = 1, 2$, we see that $\operatorname{Im} M \subset Y$ is closed. So, M is quasi-linear. The proof is completed. □

Take a projector $P : X \to X_1$ and an operator $Q : Y \to Y_1$ as follows:

$$(Pu)(t) = u(0) + u'(0)t, \qquad Q(y, y_1, y_2)^T = (0, T_1 y_1 - T_1 y, T_2 y_2 - T_2 y)^T,$$

where $Y_1 = \{(0, c_1, c_2)^T \mid c_i \in \mathbb{R}, i = 1, 2\}$. Obviously, $QY = Y_1$, and $\dim Y_1 = \dim X_1$.

By the continuity and boundedness of T_i, $i = 1, 2$, we can easily see that Q is continuous and bounded in Y. It follows from Lemma 3.3 that $Q(I - Q)(y, y_1, y_2)^T = (0, 0, 0)^T$, $y, y_1, y_2 \in C[0,1]$.

Define an operator $R : X \times [0,1] \to X_2$ as

$$R(u, \lambda)(t) = \int_0^t (t - s)\varphi_q\left(\int_0^s \lambda f\left(r, u(r), u'(r), u''(r)\right) dr\right) ds,$$

where $\operatorname{Ker} M \oplus X_2 = X$. By (H_2) and the Arzela-Asscoli theorem, we can easily see that $R : \overline{\Omega} \times [0,1] \to X_2 \cap \operatorname{dom} M$ is continuous and compact, where $\Omega \subset X$ is an open bounded set.

Lemma 4.3 *Assume that* $\Omega \subset X$ *is an open bounded set. Then* N_λ *is* M*-quasi-compact in* $\overline{\Omega}$.

Proof It is clear that $\operatorname{Im} P = \operatorname{Ker} M$, $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$ and $R(\cdot,0) = 0$. For $u \in \overline{\Omega}$,

$$(I-Q)N_\lambda u = \begin{bmatrix} \lambda f(t,u(t),u'(t),u''(t)) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -T_1\lambda f(t,u(t),u'(t),u''(t)) \\ -T_2\lambda f(t,u(t),u'(t),u''(t)) \end{bmatrix} = \begin{bmatrix} \lambda f(t,u(t),u'(t),u''(t)) \\ T_1\lambda f(t,u(t),u'(t),u''(t)) \\ T_2\lambda f(t,u(t),u'(t),u''(t)) \end{bmatrix} \in \operatorname{Im} M.$$

Since $\operatorname{Im} M \subset \operatorname{Ker} Q$ and $y = Qy + (I-Q)y$, we obtain $\operatorname{Im} M \subset (I-Q)Y$. Thus, $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$.

For $u \in \Sigma_\lambda = \{u \in \overline{\Omega} \cap \operatorname{dom} M : Mu = N_\lambda u\}$, we get

$$\begin{aligned} R(u,\lambda) &= \int_0^t (t-s)\varphi_q\left(\int_0^s \lambda f\big(r,u(r),u'(r),u''(r)\big)\,dr\right) ds \\ &= \int_0^t (t-s)\varphi_q\left(\int_0^s \big(\varphi_p(u'')\big)'\right) ds \\ &= u(t) - u(0) - u'(0)t = (I-P)u, \end{aligned}$$

i.e. Definition 2.2(c) holds. For $u \in \overline{\Omega}$, we have

$$M\big[Pu + R(u,\lambda)\big] = \begin{bmatrix} \lambda f(t,u(t),u'(t),u''(t)) \\ T_1\lambda f(t,u(t),u'(t),u''(t)) \\ T_2\lambda f(t,u(t),u'(t),u''(t)) \end{bmatrix} = (I-Q)N_\lambda u.$$

Thus, Definition 2.2(d) holds. Therefore, N_λ is M-quasi-compact in $\overline{\Omega}$. The proof is completed. □

Theorem 4.1 *Assume that the following conditions hold*:

(H_5) *There exists a nonnegative constant* L *such that if* $|u(t)| > L$, $t \in [0,1]$ *then either*

$$T_1 f\big(t,u(t),u'(t),u''(t)\big) \neq 0$$

or

$$T_2 f\big(t,u(t),u'(t),u''(t)\big) \neq 0.$$

(H_6) *There exist nonnegative constants* K_1, K_2 *such that one of* (1) *and* (2) *holds*:

(1)

$$Bf(t,A,B,C) > 0, \quad t \in [0,1], |B| > K_1, A, C \in \mathbb{R},$$

and

$$Af(t,A,B,C) > 0, \quad t \in [0,1], |B| \leq K_1, |A| > K_2, C \in \mathbb{R}.$$

(2)

$$Bf(t,A,B,C) < 0, \quad t \in [0,1], |B| > K_1, A, C \in \mathbb{R},$$

and

$$Af(t,A,B,C) < 0, \quad t \in [0,1], |A| > K_2, |B| \leq K_1, C \in \mathbb{R}.$$

(H_7) *There exist nonnegative functions* $a(t), b(t), c(t), e(t) \in L^1[0,1]$ *such that*

$$|f(t,x,y,z)| \leq a(t)\varphi_p(|x|) + b(t)\varphi_p(|y|) + c(t)\varphi_p(|z|) + e(t), \quad t \in [0,1], x, y, z \in \mathbb{R},$$

where $\varphi_q(\|a\|_1 + \|b\|_1 + \|c\|_1) < 2^{2-q}$, *if* $1 < p \leq 2$; $\varphi_q(2^{p-2}\|a\|_1 + 2^{p-2}\|b\|_1 + \|c\|_1) < 1$, *if* $p \geq 2$.

Then boundary value problem (1.2) *has at least one solution.*

In order to prove Theorem 4.1, we show two lemmas.

Lemma 4.4 *Suppose* (H_5)-(H_7) *hold. Then the set*

$$\Omega_1 = \{u \in \operatorname{dom} M \mid Mu = N_\lambda u, \lambda \in (0,1)\}$$

is bounded in X.

Proof For $u \in \Omega_1$, we have $QN_\lambda u = 0$, *i.e.* $T_i f(t, u(t), u'(t), u''(t)) = 0$, $i = 1, 2$. By (H_5) and (H_6), there exist constants $t_0, t_1 \in [0,1]$ such that $|u(t_0)| \leq L$, $|u'(t_1)| \leq K_1$. Since $u(t) = u(t_0) + \int_{t_0}^t u'(s)\,ds$, $u'(t) = u'(t_1) + \int_{t_1}^t u''(s)\,ds$, then

$$|u(t)| \leq L + \|u'\|_\infty, \qquad |u'(t)| \leq K_1 + \|u''\|_\infty, \quad t \in [0,1]. \tag{4.2}$$

It follows from $Mu = N_\lambda u$, (H_7), and (4.2) that

$$\begin{aligned} |u''(t)| &= \left|\varphi_q\left(\int_0^t \lambda f(s, u(s), u'(s)' u''(s))\,ds\right)\right| \\ &\leq \varphi_q\left(\int_0^1 a(t)\varphi_p(|u|) + b(t)\varphi_p(|u'|) + c(t)\varphi_p(|u''|) + e(t)\,dt\right) \\ &\leq \varphi_q\big(\|a\|_1\varphi_p(K_1 + L + \|u''\|_\infty) + \|b\|_1\varphi_p(K_1 + \|u''\|_\infty) \\ &\quad + \|c\|_1\varphi_p(\|u''\|_\infty) + \|e\|_1\big). \end{aligned}$$

If $1 < p \leq 2$, by Lemma 2.1, we get

$$|u''(t)| \leq \varphi_q(B_1 + A_1\varphi_p(\|u''\|_\infty)) \leq 2^{q-2}[\varphi_q(B_1) + \varphi_q(A_1)\|u''\|_\infty],$$

thus

$$\|u''\|_\infty \le \frac{2^{q-2}\varphi_q(B_1)}{1-2^{q-2}\varphi_q(A_1)},$$

where $B_1 = \|a\|_1\varphi_p(K_1+L) + \|b\|_1\varphi_p(K_1) + \|e\|_1$, $A_1 = \|a\|_1 + \|b\|_1 + \|c\|_1$.

If $p > 2$, by Lemma 2.1, we get

$$|u''(t)| \le \varphi_q\bigl(B_2 + A_2\varphi_p(\|u''\|_\infty)\bigr) \le \bigl[\varphi_q(B_2) + \varphi_q(A_2)\|u''\|_\infty\bigr],$$

thus

$$\|u''\|_\infty \le \frac{\varphi_q(B_2)}{1-\varphi_q(A_2)},$$

where $B_2 = 2^{p-2}\|a\|_1\varphi_p(K_1+L) + 2^{p-2}\|b\|_1\varphi_p(K_1) + \|e\|_1$, $A_2 = 2^{p-2}\|a\|_1 + 2^{p-2}\|b\|_1 + \|c\|_1$.

These, together with (4.2), mean that Ω_1 is bounded in X. □

Lemma 4.5 *Assume* (H_6) *holds. Then*

$$\Omega_2 = \{u \in \operatorname{Ker} M \mid QNu = 0\}$$

is bounded in X, *where* $N = N_1$.

Proof For $u \in \Omega_2$, we have $u = a + bt$ and $Q(Nu) = 0$. By (H_6), we see that there exists a constant $t_0 \in [0,1]$ such that $|u(t_0)| = |a + bt_0| \le K_2$, $|u'(t)| = |b| \le K_1$. So, Ω_2 is bounded. The proof is completed. □

Proof of Theorem 4.1 Let $\Omega = \{u \in X \mid \|u\| < r\}$, where r is large enough such that $K_1 + K_2 < r < +\infty$ and $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2}$.

By Lemmas 4.4 and 4.5, we know $Mu \ne N_\lambda u$, $u \in \operatorname{dom} M \cap \partial\Omega$ and $QNu \ne 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$.

Let $H(u,\delta) = \rho\delta u + (1-\delta)JQNu$, $\delta \in [0,1]$, $u \in \operatorname{Ker} M \cap \overline{\Omega}$, where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(0,a,b)^T = a + bt$, $\rho = \begin{cases} -1, & \text{if } (\mathrm{H}_6)(1) \text{ holds,} \\ 1, & \text{if } (\mathrm{H}_6)(2) \text{ holds.}\end{cases}$

Take the function $\operatorname{Sgn}(x)$ is the same as the one in Proof of Theorem 3.1.

For $u \in \operatorname{Ker} M \cap \partial\Omega$, we have $u = a + bt \ne 0$. Thus

$$H(u,\delta) = \rho\delta(a+bt) + (1-\delta)\bigl(-T_1 f(t,a+bt,b,0) - T_2 f(t,a+bt,b,0)t\bigr).$$

If $\delta = 1$, $H(u,1) = \rho(a+bt) \ne 0$. If $\delta = 0$, by $QNu \ne 0$, we get $H(u,0) = JQN(a+bt) \ne 0$. For $0 < \delta < 1$, we now prove that $H(u,\delta) \ne 0$. Otherwise, if $H(u,\delta) = 0$, then

$$T_1 f(t,a+bt,b,0) = \frac{\rho\delta}{1-\delta}a, \qquad T_2 f(t,a+bt,b,0) = \frac{\rho\delta}{1-\delta}b. \tag{4.3}$$

Since $\|u\| = \max\{\|a+bt\|_\infty, |b|\} = r > K_1 + K_2$, we have either $|b| > K_1$ or $\|a+bt\|_\infty > K_1 + K_2$. If $|b| > K_1$, then $T_2 bf(t,a+bt,b,0) = bT_2 f(t,a+bt,b,0) = \frac{\rho\delta}{1-\delta}b^2$. So, we have $\operatorname{Sgn}(bf(t,a+bt,b,0)) = \operatorname{Sgn}(T_2 bf(t,a+bt,b,0)) = \operatorname{Sgn}(\frac{\rho\delta}{1-\delta}b^2) = \operatorname{Sgn}(\rho)$. This is a contradiction with the definition of ρ. If $|b| \le K_1$, then $\|a+bt\|_\infty > K_1 + K_2$. Thus $\min_{t\in[0,1]}|a+bt| >$

K_2 and $\mathrm{Sgn}(a) = \mathrm{Sgn}(a + bt)$. By $T_1 af(t, a + bt, b, 0) = aT_1 f(t, a + bt, b, 0) = \frac{\rho\delta}{1-\delta}a^2$, we get $\mathrm{Sgn}(T_1(a + bt)f(t, a + bt, b, 0)) = \mathrm{Sgn}(T_1 af(t, a + bt, b, 0)) = \mathrm{Sgn}(\frac{\rho\delta}{1-\delta}a^2) = \mathrm{Sgn}(\rho)$. This is a contradiction with the definition of ρ, too. So, $H(u, \delta) \neq 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$, $\delta \in [0, 1]$.

By the homotopy of degree, we get

$$\begin{aligned}\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) &= \deg\bigl(H(\cdot, 0), \Omega \cap \operatorname{Ker} M, 0\bigr) \\ &= \deg\bigl(H(\cdot, 1), \Omega \cap \operatorname{Ker} M, 0\bigr) = \deg(\rho I, \Omega \cap \operatorname{Ker} M, 0) \neq 0.\end{aligned}$$

By Theorem 2.1, we find that (1.2) has at least one solution in $\overline{\Omega}$. The proof is completed. □

Competing interests
The author declares that she has no competing interests.

Author's contributions
All results belong to WJ.

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