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Large time behavior for the fractional Ginzburg-Landau equations near the BCS-BEC crossover regime of Fermi gases

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Abstract

In this paper, we consider the fractional Ginzburg-Landau equations near the Bardeen-Cooper-Schrieffer-Bose-Einstein-condensate (BCS-BEC) crossover of atomic Fermi gases. This fractional Ginzburg-Landau equations can be viewed as a generalization of the integral differential equations proposed by Machida and Koyama (Phys. Rev. A 74:033603, 2006). By using the Galerkin method and *a priori* estimates, together with the properties of Sobolev spaces, we first establish the existence and uniqueness of weak solutions to these equations and then we prove the existence of global attractors.

Keywords: fractional Ginzburg-Landau equations; global attractors; weak solution

1 Introduction

In this paper, we consider the fractional Ginzburg-Landau equations for atomic Fermi gases near the BCS-BEC crossover as follows:

$$\begin{aligned}-idu_t &= \left(-\frac{dg^2+1}{U}+a\right)u+g\bigl[a+d(2\nu-2\mu)\bigr]\varphi \\ &\quad -\frac{c}{4m}\Lambda^{2\alpha}u-\frac{g}{4m}(c-d)\Lambda^{2\alpha}\varphi-b|u+g\varphi|^2(u+g\varphi)-idf(x), \qquad (1)\end{aligned}$$

$$i\varphi_t=-i\beta\varphi-\frac{g}{U}u+(2\nu-2\mu)\varphi+\frac{1}{4m}\Lambda^{2\alpha}\varphi+ih(x), \qquad (2)$$

$$u(x,0)=u_0(x), \qquad \varphi(x,0)=\varphi_0(x), \quad x\in\mathbb{R}^n, \qquad (3)$$

$$u(x+2\pi e_i,t)=u(x,t), \qquad \varphi(x+2\pi e_i,t)=\varphi(x,t), \quad x\in\mathbb{R}^n, t\geq 0, \qquad (4)$$

where $u(x,t)$ is the fermion-pair field and $\varphi(x,t)$ is the condensed boson field, μ is the chemical potential, 2ν is the threshold energy of the Feshbach resonance, g is the coupling constant describes the process arising from the Feshbach resonance, in which a boson is created from two fermionic atoms and vice versa. d is generally complex, in the BCS limit d can be considered to be pure imaginary, while in the BEC region, the imaginary part of d vanished. $U>0$, a,b,c,β are real coefficients, the square root of the Laplacian, $\Lambda=(-\Delta)^{\frac{1}{2}}$, is the so-called Zygmund operator. In this paper, we restrict ourselves to the fractional order $\alpha\in(\frac{1}{2},1]$ and the space dimension $n=1$.

The BCS-BEC crossover phenomenon has been experimentally realized by using ultracold gases of ^{6}Li and ^{40}K atoms and it has opened a new era for the study of some longstanding theoretical proposals in many fermion systems. Since the conventional perturbation theory is no longer valid, the equation of state and dynamic properties of the BCS-BEC crossover have become a big challenge for quantum theory. In recent years, a number of groups have studied the atomic Fermi gases from various views which help us to deeply understand the physics of the pseudogap and Berezinskii-Kosterlitz-Thouless transitions in fermionic systems.

In 2006, on the basis of the functional integral formalism, Machida and Koyama [1] constructed a time-dependent Ginzburg-Landau theory for the superfluid atomic Fermi gases near the Feshbach resonance from the fermion-boson model. The time-dependent Ginzburg-Landau theory just can be described as (1)-(2). In the case of $g = 0, \alpha = 1$, (1)-(2) are decoupled and reduced to the conventional time-dependent Ginzburg-Landau (TDGL) equation and the linearized Gross-Pitaevskii (GP) equation, which intensively were studied in the past decades. In the case of $g \neq 0, \alpha = 1$, Chen and Guo [2–4] obtained the global existence and uniqueness of weak solutions to (1)-(2) with periodic boundary conditions, and then by the properties of Besov and Sobolev spaces and matrix theory, together with the energy method, they established the global existence of classical solutions. The existence of global attractors to (1)-(4) was proved by Fang, Jin and Guo [5]. In [6], Guo *et al.* considered the numerical solution of Ginzburg-Landau equations near the BCS-BEC crossover through finite difference method and analyzed its convergence and stability. In [7], the Hopf bifurcation of the above equations was studied and numerical simulations were also given.

In this paper, motivated by [8] and [9], we extend the result to fractional Ginzburg-Landau equations for atomic Fermi gases near the BCS-BEC crossover. When α is not an integer, the fractional dissipation operator $\Lambda^{2\alpha}$ is nonlocal and can be regarded as the infinitesimal generators of Lévy stable diffusion processes. More and more researchers have found that fractional differential equations play an important role in mathematical physics and can be used to describe some physical phenomenon more exactly than integral differential equations. There have been extensive study of fractional differential equations including the fractional Ginzburg-Landau equation [8], the fractional Schrödinger equation [9], the fractional Landau-Lifshitz-Gilbert equation [10], the fractional Landau-Lifshitz equation [11] *etc.* For more details, see [12–15].

The rest of this paper is organized as follows. In the next section, we give some notations. In Section 3, we give *a priori* estimates. In Section 4, we prove the global existence of weak solutions to (1)-(4). Finally, the existence of global attractors is obtained.

2 Notations

Let $\Omega = [0, 2\pi]$, $d = d_r + id_i$, $m = m_r + im_i$, and $|d|^2 = d_r^2 + d_i^2$, $|m|^2 = m_r^2 + m_i^2$. Denote by $L^p(\Omega)$ the usual Sobolev space of the pth-power integrable functions normed by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega \left|f(x)\right|^p \, dx\right)^{\frac{1}{p}}, \qquad \|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} \left|f(x)\right|.$$

When no confusion arises, we set $L^p := L^p(\Omega)$ for $1 \leq p \leq \infty$. Similarly, we define the space $L^p(0,T;X)$ with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p \, dt\right)^{\frac{1}{p}} < \infty \quad (1 \leq p < \infty),$$

$$\|f\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \leq t \leq T} \|f\|_X < \infty.$$

If u is a periodic function, then it can be identified with the Fourier series

$$u = \sum_{j \in \mathbb{Z}} a_j e^{ij\cdot x}.$$

Moreover, $\Lambda^\alpha u$ can be defined by

$$\Lambda^\alpha u = \sum_{j \in \mathbb{Z}} |j|^\alpha a_j e^{ij\cdot x}.$$

Define

$$E = \left\{ u \;\middle|\; u = \sum_{j\in\mathbb{Z}} a_j e^{ij\cdot x}, \sum_{j\in\mathbb{Z}} a_j^2 < \infty, \sum_{j\in\mathbb{Z}} |j|^{2\alpha} a_j^2 < \infty \right\},$$

and H^α denotes a complete space of E with the induced norm

$$\|u\|_{H^\alpha} = \left(\sum_{j\in\mathbb{Z}} a_j^2\right)^{\frac{1}{2}} + \left(\sum_{j\in\mathbb{Z}} |j|^{2\alpha} a_j^2\right)^{\frac{1}{2}}.$$

Then H^α is a Banach space. It is easy to show that H^α is a Hilbert space with the inner product

$$(u,v)_{H^\alpha} = \left(\Lambda^\alpha u, \Lambda^\alpha v\right) = \sum_{j\in\mathbb{Z}} |j|^{2\alpha} a_j b_j,$$

where $\Lambda^\alpha v = \sum_{j\in\mathbb{Z}} |j|^\alpha b_j e^{ij\cdot x}$. For more details, see [9].

We denote positive constants by C; they may change from one line to the next line.

3 *A priori* estimates

Lemma 1 *Assume that* $u_0 \in L^2(\Omega), \varphi_0 \in L^2(\Omega), f(x) \in L^2(\Omega), h(x) \in L^2(\Omega)$. *Let* (u,φ) *be the solution to* (1)-(4). *Then, for any* $0 < T < \infty, U > 0, c > 0, m_i d_r + m_r d_i > 0, b > 0$, $m_i > 0$, $d_i > 0$, *and* $\beta > 0$, *we have*

$$\begin{aligned} &\|u + g\varphi\|_{L^2}^2 + \|\varphi\|_{L^2}^2 + \int_0^T \|u + g\varphi\|_{L^4}^4 \, dt \\ &\quad + \int_0^T \left\|\Lambda^\alpha (u + g\varphi)\right\|_{L^2}^2 dt + \int_0^T \left\|\Lambda^\alpha \varphi\right\|_{L^2}^2 dt \leq C, \end{aligned}$$

where the constant C depending on the initial data and T.

Proof From (1)-(2), we have

$$(u+g\varphi)_t = \frac{ia}{d}(u+g\varphi) - \frac{i}{dU}(u+g\varphi) + \frac{ig}{dU}\varphi - \beta g\varphi - \frac{ic}{4md}\Lambda^{2\alpha}(u+g\varphi) - \frac{ib}{d}|u+g\varphi|^2(u+g\varphi) + (f+gh). \tag{5}$$

Multiplying (5) by $\overline{(u+g\varphi)}$, integrating over Ω, and taking the real part, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u+g\varphi\|_{L^2}^2 &= \left(\frac{ad_i}{|d|^2} - \frac{d_i}{|d|^2U}\right)\|u+g\varphi\|_{L^2}^2 + Re\left(\frac{ig}{dU}\int_\Omega \varphi\overline{(u+g\varphi)}\,dx\right)\\ &\quad - Re\left(\beta g\int_\Omega \varphi\overline{(u+g\varphi)}\,dx\right) - \frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2\\ &\quad - \frac{bd_i}{|d|^2}\|u+g\varphi\|_{L^4}^4 + Re\left(\int_\Omega \big(f(x)+gh(x)\big)\overline{(u+g\varphi)}\,dx\right).\end{aligned} \tag{6}$$

Multiplying (2) by $\overline{\varphi}$, integrating over Ω, and taking the imaginary part, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 &= -\beta\|\varphi\|_{L^2}^2 + Re\left(\frac{ig}{U}\int_\Omega (u+g\varphi)\overline{\varphi}\,dx\right)\\ &\quad - \frac{m_i}{4|m|^2}\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2 + Re\left(\int_\Omega h(x)\overline{\varphi}\,dx\right).\end{aligned} \tag{7}$$

Combining the above two equations, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u+g\varphi\|_{L^2}^2 + \|\varphi\|_{L^2}^2\right)\\ &\quad = Re\left(\frac{ig}{dU}\int_\Omega \varphi\overline{(u+g\varphi)}\,dx\right) - Re\left(\beta g\int_\Omega \varphi\overline{(u+g\varphi)}\,dx\right)\\ &\qquad + Re\left(\int_\Omega \big(f(x)+gh(x)\big)\overline{(u+g\varphi)}\,dx\right) + Re\left(\int_\Omega h(x)\overline{\varphi}\,dx\right)\\ &\qquad + Re\left(\frac{ig}{U}\int_\Omega (u+g\varphi)\overline{\varphi}\,dx\right) + \left(\frac{ad_i}{|d|^2} - \frac{d_i}{|d|^2U}\right)\|u+g\varphi\|_{L^2}^2\\ &\qquad - \frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2 - \frac{bd_i}{|d|^2}\|u+g\varphi\|_{L^4}^4\\ &\qquad - \beta\|\varphi\|_{L^2}^2 - \frac{m_i}{4|m|^2}\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2\\ &\quad \le \frac{1}{2}\left|\frac{g}{dU}\right|\|u+g\varphi\|_{L^2}^2 + \frac{1}{2}\left|\frac{g}{dU}\right|\|\varphi\|_{L^2}^2 + \frac{1}{2}|\beta g|\|u+g\varphi\|_{L^2}^2\\ &\qquad + \frac{1}{2}|\beta g|\|\varphi\|_{L^2}^2 + \frac{1}{2}\|u+g\varphi\|_{L^2}^2 + \frac{1}{2}\left(\|f\|_{L^2}^2 + g^2\|h\|_{L^2}^2\right)\\ &\qquad + \frac{1}{2}\|\varphi\|_{L^2}^2 + \frac{1}{2}\|h\|_{L^2}^2 + \frac{1}{2}\frac{|g|}{U}\|u+g\varphi\|_{L^2}^2 + \frac{1}{2}\frac{|g|}{U}\|\varphi\|_{L^2}^2\\ &\qquad + \left(\frac{ad_i}{|d|^2} - \frac{d_i}{|d|^2U}\right)\|u+g\varphi\|_{L^2}^2 - \frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2\\ &\qquad - \frac{bd_i}{|d|^2}\|u+g\varphi\|_{L^4}^4 - \beta\|\varphi\|_{L^2}^2 - \frac{m_i}{4|m|^2}\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}C_1\big(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\big)+\frac{1}{2}\|f\|_{L^2}^2+\frac{1}{2}\big(1+g^2\big)\|h\|_{L^2}^2\\
&\quad-\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\big\|\Lambda^\alpha(u+g\varphi)\big\|_{L^2}^2-\frac{bd_i}{|d|^2}\|u+g\varphi\|_{L^4}^4\\
&\quad-\beta\|\varphi\|_{L^2}^2-\frac{m_i}{4|m|^2}\big\|\Lambda^\alpha\varphi\big\|_{L^2}^2,
\end{aligned}\tag{8}$$

where $C_1=\max\{|\frac{g}{dU}|+\frac{|g|}{U}+|\beta g|+1,|\frac{g}{dU}|+\frac{|g|}{U}+|\beta g|+\frac{2ad_i}{|d|^2}-\frac{2d_i}{|d|^2U}+1\}$.

Noticing that $c>0, m_id_r+m_rd_i>0, b>0$, $m_i>0$, $d_i>0$, and $\beta>0$, we have

$$\frac{d}{dt}\big(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\big)\le C_1\big(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\big)+\|f\|_{L^2}^2+\big(1+g^2\big)\|h\|_{L^2}^2.$$

By Gronwall's inequality, we have

$$\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\le C.$$

From (8), we can also deduce that

$$\int_0^T\|u+g\varphi\|_{L^4}^4\,dt+\int_0^T\big\|\Lambda^\alpha(u+g\varphi)\big\|_{L^2}^2\,dt++\int_0^T\big\|\Lambda^\alpha\varphi\big\|_{L^2}^2\,dt\le C.$$

Thus we complete the proof. □

Lemma 2 *Let $u_0\in L^2(\Omega),\varphi_0\in L^2(\Omega),f\in L^2(\Omega),h\in L^2(\Omega)$. Assume that $U>0$, $c>0$, $m_id_r+m_rd_i>0$, $b>0$, $m_i>0$, $d_i>0$ and $\beta>0$, we have*

$$(u+g\varphi)_t\in L^{\frac{4}{3}}\big(0,T;H^{-\alpha}(\Omega)\big),\qquad \varphi_t\in L^2\big(0,T;H^{-\alpha}(\Omega)\big).$$

Proof It follows from Lemma 1 that

$$u+g\varphi\in L^2\big(0,T;H^\alpha(\Omega)\big),\qquad \varphi\in L^2\big(0,T;H^\alpha(\Omega)\big).$$

For any $\psi\in H^\alpha(\Omega)$, we have

$$\begin{aligned}
\int_0^t\big((u+g\varphi)_t,\psi\big)\,dt&=\frac{ia}{d}\int_0^t\big((u+g\varphi),\psi\big)\,dt-\frac{i}{dU}\int_0^t\big((u+g\varphi),\psi\big)\,dt\\
&\quad+\left(\frac{ig}{dU}-\beta g\right)\int_0^t(\varphi,\psi)\,dt-\frac{ic}{4md}\int_0^t\big(\Lambda^{2\alpha}(u+g\varphi),\psi\big)\,dt\\
&\quad-\frac{ib}{d}\int_0^t\big(|u+g\varphi|^2(u+g\varphi),\psi\big)\,dt+\int_0^t(f+gh,\psi)\,dt,\\
\int_0^t(\varphi_t,\psi)\,dt&=-\beta\int_0^t(\varphi,\psi)\,dt+\frac{ig}{U}\int_0^t(u,\psi)\,dt-i(2\nu-2\mu)\int_0^t(\varphi,\psi)\,dt\\
&\quad-\frac{i}{4m}\int_0^t\big(\Lambda^{2\alpha}\varphi,\psi\big)\,dt+\int_0^t(h,\psi)\,dt.
\end{aligned}$$

By Hölder's inequality, we have

$$\left|\int_0^t(u+g\varphi,\psi)\,dt\right|\le C\|u+g\varphi\|_{L^2(0,t;L^2(\Omega))}\|\psi\|_{L^2(0,t;L^2(\Omega))},$$

$$\left|\int_0^t (\varphi,\psi)\,dt\right| \le C\|\varphi\|_{L^2(0,t;L^2(\Omega))}\|\psi\|_{L^2(0,t;L^2(\Omega))},$$
$$\left|\int_0^t \bigl(\Lambda^{2\alpha}(u+g\varphi),\psi\bigr)\,dt\right| \le C\|u+g\varphi\|_{L^2(0,t;H^\alpha(\Omega))}\|\psi\|_{L^2(0,t;H^\alpha(\Omega))},$$
$$\left|\int_0^t \bigl(|u+g\varphi|^2(u+g\varphi),\psi\bigr)\,dt\right| \le C\|u+g\varphi\|^3_{L^4(0,t;L^4(\Omega))}\|\psi\|_{L^4(0,t;L^4(\Omega))},$$
$$\left|\int_0^t \bigl(\Lambda^{2\alpha}\varphi,\psi\bigr)\,dt\right| \le C\|\varphi\|_{L^2(0,t;H^\alpha(\Omega))}\|\psi\|_{L^2(0,t;H^\alpha(\Omega))},$$
$$\left|\int_0^t (f+gh,\psi)\,dt\right| \le C\bigl(\|f\|_{L^2(0,t;L^2(\Omega))}+\|h\|_{L^2(0,t;L^2(\Omega))}\bigr)\|\psi\|_{L^2(0,t;L^2(\Omega))},$$
$$\left|\int_0^t (h,\psi)\,dt\right| \le C\|h\|_{L^2(0,t;L^2(\Omega))}\|\psi\|_{L^2(0,t;L^2(\Omega))}.$$

Applying the Sobolev embedding

$$\|\psi\|_{L^4(\Omega)} \le C\|\psi\|_{H^\alpha(\Omega)},$$

we have

$$\left|\int_0^t \bigl((u+g\varphi)_t,\psi\bigr)\,dt\right| \le C\|\psi\|_{L^4(0,t;H^\alpha(\Omega))}, \qquad \left|\int_0^t (\varphi_t,\psi)\,dt\right| \le C\|\psi\|_{L^2(0,t;H^\alpha(\Omega))}.$$

Therefore

$$\bigl\|(u+g\varphi)_t\bigr\|_{L^{\frac{4}{3}}(0,t;H^{-\alpha}(\Omega))} \le C, \qquad \bigl\|\varphi_t\bigr\|_{L^2(0,t;H^{-\alpha}(\Omega))} \le C.$$

Thus the proof of Lemma 2 is completed. □

Let $I_\phi(t) = ((u+g\varphi,\phi_1),(\varphi,\phi_2))$, $\phi = (\phi_1,\phi_2)$.

Lemma 3 *Under the conditions as in Lemma* 1, *for any* $\phi_1 \in L^2(\Omega)$, $\phi_2 \in L^2(\Omega)$, $I_\phi(t)$ *is a continuous function with respect to* t.

Proof First, let $\phi = (\phi_1,\phi_2) \in C^\infty(\Omega)\times C^\infty(\Omega)$. We have

$$\begin{aligned}
I_{\phi_1}(t) &= (u+g\varphi,\phi_1) = (u_0+g\varphi_0,\phi_1) + \frac{ia}{d}\int_0^t \bigl((u+g\varphi),\phi_1\bigr)\,dt \\
&\quad - \frac{i}{dU}\int_0^t \bigl((u+g\varphi),\phi_1\bigr)\,dt + \left(\frac{ig}{dU}-\beta g\right)\int_0^t (\varphi,\phi_1)\,dt \\
&\quad - \frac{ic}{4md}\int_0^t \bigl(\Lambda^{2\alpha}(u+g\varphi),\phi_1\bigr)\,dt - \frac{ib}{d}\int_0^t \bigl(|u+g\varphi|^2(u+g\varphi),\phi_1\bigr)\,dt \\
&\quad + \int_0^t (f+gh,\phi_1)\,dt,
\end{aligned}$$
$$\begin{aligned}
I_{\phi_2}(t) &= (\varphi,\phi_2) = (\varphi_0,\phi_2) - \beta\int_0^t (\varphi,\phi_2)\,dt + \frac{ig}{U}\int_0^t (u,\phi_2)\,dt \\
&\quad - i(2\nu-2\mu)\int_0^t (\varphi,\phi_2)\,dt - \frac{i}{4m}\int_0^t \bigl(\Lambda^{2\alpha}\varphi,\phi_2\bigr)\,dt + \int_0^t (h,\phi_2)\,dt.
\end{aligned}$$

For $0 \le t_1, t_2 \le T$ and $|t_2 - t_1| < 1$, we have

$$\begin{aligned}
\left|I_{\phi_1}(t_1) - I_{\phi_1}(t_2)\right| &\le C\left(\|\phi_1\|_{L^\infty} + \left\|\Lambda^{2\alpha}\phi_1\right\|_{L^\infty}\right)\|u + g\varphi\|_{L^2(t_1,t_2;L^2)}|t_1 - t_2|^{\frac{1}{2}} \\
&\quad + C\|\phi_1\|_{L^\infty}\|\varphi\|_{L^2(t_1,t_2;L^2)}|t_1 - t_2|^{\frac{1}{2}} \\
&\quad + C\|\phi_1\|_{L^\infty}\|u + g\varphi\|^3_{L^4(t_1,t_2;L^4)}|t_1 - t_2|^{\frac{1}{4}} \\
&\quad + C\|\phi_1\|_{L^\infty}\left(\|f\|_{L^2(t_1,t_2;L^2)} + \|h\|_{L^2(t_1,t_2;L^2)}\right)|t_1 - t_2|^{\frac{1}{2}} \\
&\le C|t_1 - t_2|^{\frac{1}{4}}, \\
\left|I_{\phi_2}(t_1) - I_{\phi_2}(t_2)\right| &\le C\left(\|\phi_2\|_{L^\infty} + \left\|\Lambda^{2\alpha}\phi_2\right\|_{L^\infty}\right)\|\varphi\|_{L^2(t_1,t_2;L^2)}|t_1 - t_2|^{\frac{1}{2}} \\
&\quad + C\|\phi_2\|_{L^\infty}\|u + g\varphi\|_{L^2(t_1,t_2;L^2)}|t_1 - t_2|^{\frac{1}{2}} \\
&\quad + C\|\phi_2\|_{L^\infty}\|h\|_{L^2(t_1,t_2;L^2)}|t_1 - t_2|^{\frac{1}{2}} \\
&\le C|t_1 - t_2|^{\frac{1}{2}} \le C|t_1 - t_2|^{\frac{1}{4}}.
\end{aligned}$$

Then

$$\left|I_\phi(t_1) - I_\phi(t_2)\right| \le \left|I_{\phi_1}(t_1) - I_{\phi_1}(t_2)\right| + \left|I_{\phi_2}(t_1) - I_{\phi_2}(t_2)\right| \le C|t_1 - t_2|^{\frac{1}{4}}.$$

Therefore, the continuity of $I_\phi(t)$ follows.

Next we use a density argument to extend the result for $\phi \in L^2(\Omega) \times L^2(\Omega)$. Let $\epsilon > 0$ be an arbitrary positive number, for $\phi \in L^2(\Omega) \times L^2(\Omega)$, we may select some $\phi^\epsilon \in C^\infty(\Omega) \times C^\infty(\Omega)$ such that $\|\phi^\epsilon - \phi\|_{L^2(\Omega)} \le \epsilon$. By the triangle inequality and Hölder's inequality, we have

$$\begin{aligned}
\left|I_\phi(t_1) - I_\phi(t_2)\right| &\le \left|I_{\phi_1}(t_1) - I_{\phi_1^\epsilon}(t_1)\right| + \left|I_{\phi_1^\epsilon}(t_1) - I_{\phi_1^\epsilon}(t_2)\right| \\
&\quad + \left|I_{\phi_1}(t_2) - I_{\phi_1^\epsilon}(t_2)\right| + \left|I_{\phi_2}(t_1) - I_{\phi_2^\epsilon}(t_1)\right| \\
&\quad + \left|I_{\phi_2^\epsilon}(t_1) - I_{\phi_2^\epsilon}(t_2)\right| + \left|I_{\phi_2}(t_2) - I_{\phi_2^\epsilon}(t_2)\right| \\
&\le \epsilon C + \left|I_{\phi^\epsilon}(t_1) - I_{\phi^\epsilon}(t_2)\right|.
\end{aligned}$$

Since I_{ϕ^ϵ} is continuous in t and ϵ is arbitrary, the continuity of $I_\phi(t)$ follows for $\phi \in L^2 \times L^2$. □

4 Global existence of weak solutions

In this section, we will show the global existence of the weak solutions. We first give the definition of weak solutions.

Definition Let $u_0 \in L^2(\Omega), \varphi_0 \in L^2(\Omega), f(x) \in L^2(\Omega), h(x) \in L^2(\Omega)$, we say that (u, φ) is a weak solution of (1)-(4) if

(i) for all $T > 0$, $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^\alpha(\Omega)), \varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^\alpha(\Omega))$;

(ii) for all $\psi \in C^\infty(Q_T)$, we have

$$\begin{aligned}
&-id(u,\psi) + id(u_0,\psi) \\
&\quad = \left(-\frac{dg^2+1}{U} + a\right)\int_0^t (u,\psi)\,dt + g\big[a + d(2\nu - 2\mu)\big]\int_0^t (\varphi,\psi)\,dt \\
&\qquad - \frac{c}{4m}\int_0^t \big(\Lambda^\alpha u, \Lambda^\alpha \psi\big)\,dt - \frac{g}{4m}(c-d)\int_0^t \big(\Lambda^\alpha \varphi, \Lambda^\alpha \psi\big)\,dt \\
&\qquad - b\int_0^t \big(|u+g\varphi|^2(u+g\varphi),\psi\big)\,dt + id\int_0^t \big(f(x),\psi\big)\,dt, \\
&i(\varphi,\psi) - i(\varphi_0,\psi) = -i\beta\int_0^t (\varphi,\psi)\,dt - \frac{g}{U}\int_0^t (u,\psi)\,dt + (2\nu - 2\mu)\int_0^t (\varphi,\psi)\,dt \\
&\qquad\qquad + \frac{1}{4m}\int_0^t \big(\Lambda^\alpha \varphi, \Lambda^\alpha \psi\big)\,dt + i\int_0^t \big(h(x),\psi\big)\,dt,
\end{aligned}$$

with initial conditions

$$\big(u(x,0),\psi\big) = \big(u_0(x),\psi\big), \qquad \big(\varphi(x,0),\psi\big) = \big(\varphi_0(x),\psi\big),$$

where $Q_T = (0,T)\times\Omega$.

Next, we recall the following lemmas which will be used later.

Lemma 4 *Let B_0, B, B_1 be three Banach spaces such that*

$$B_0 \subset B \subset B_1,$$

where the injections are continuous and B_0, B_1 are reflexive, and the injection $B_0 \to B$ is compact. Denote

$$W = \left\{ v \,\middle|\, v \in L^{p_0}(0,T;B_0), v' = \frac{dv}{dt} \in L^{p_1}(0,T;B_1) \right\},$$

where T is finite and $1 < p_0, p_1 < \infty$. Then W equipped with the norm

$$\|v\|_{L^{p_0}(0,T;B_0)} + \|v'\|_{L^{p_1}(0,T;B_1)}$$

is a Banach space and the embedding $W \hookrightarrow L^{p_0}(0,T;B)$ is compact.

Lemma 5 *Assume that D is a bounded domain in $\mathbb{R}^n_x \times \mathbb{R}_t$, functions $f_l, f \in L^q(D)$ $(1 < q < \infty)$ and*

$$\|f_l\|_{L^q(D)} \le C, \quad f_l \to f \text{ in } D \text{ a.e.}$$

Then $f_l \to f$ weakly in $L^q(D)$.

Lemma 6 *X is a Banach space, suppose that $g \in L^p(0,T;X)$, $\frac{\partial g}{\partial t} \in L^p(0,T;X)$ $(1 \le p \le \infty)$. Then $g \in C([0,T];X)$ (after possibly being redefined on a set of measure zero).*

Now, we state our main result as follows.

Theorem 1 *Let* $\frac{1}{2} < \alpha \le 1$, $u_0 \in L^2(\Omega), \varphi_0 \in L^2(\Omega), f(x) \in L^2(\Omega), h(x) \in L^2(\Omega)$, $U > 0, c > 0, m_i d_r + m_r d_i > 0, b > 0, m_i > 0, d_i > 0$, *and* $\beta > 0$, *then there exists a unique weak solution* (u, φ) *to* (1)-(4) *such that*

$$u + g\varphi \in C\big([0,T]; w - L^2(\Omega)\big) \cap L^2\big(0,T; H^\alpha(\Omega)\big),$$
$$\varphi \in C\big([0,T]; w - L^2(\Omega)\big) \cap L^2\big(0,T; H^\alpha(\Omega)\big),$$

and

$$u + g\varphi \in L^4\big(0,T; L^4(\Omega)\big),$$

where we say that $u + g\varphi \in C([0,T]; w - L^2(\Omega))$ *and* $\varphi \in C([0,T]; w - L^2(\Omega))$, *if* $(u + g\varphi, \phi_1) \in C([0,T])$ *and* $(\varphi, \phi_2) \in C([0,T])$ *for any* $\phi = (\phi_1, \phi_2) \in L^2 \times L^2$.

Proof By Galerkin's method, we look for approximate solutions $(u_N(x,t), \varphi_N(x,t))$ for equations (1)-(4) in the form

$$u_N(x,t) = \sum_{j=1}^{N} \alpha_{jN}(t)\omega_j(x), \qquad \varphi_N(x,t) = \sum_{j=1}^{N} \beta_{jN}(t)\omega_j(x),$$

where $\omega_j(x) = e^{ij\cdot x}$ and $\alpha_{jN}(t), \beta_{jN}(t)$ satisfy the following system of ordinary differential equations:

$$\begin{aligned}
-id(u_{Nt}, \omega_j) = {}& \left(-\frac{dg^2+1}{U} + a\right)(u_N, \omega_j) + g\big[a + d(2\nu - 2\mu)\big](\varphi_N, \omega_j) \\
& - \frac{c}{4m}\big(\Lambda^{2\alpha} u_N, \omega_j\big) - \frac{g}{4m}(c-d)\big(\Lambda^{2\alpha}\varphi_N, \omega_j\big) \\
& - b\big(|u_N + g\varphi_N|^2(u_N + g\varphi_N), \omega_j\big) + id\big(f(x), \omega_j\big), \qquad (9)
\end{aligned}$$

$$\begin{aligned}
i(\varphi_{Nt}, \omega_j) = {}& -i\beta(\varphi_N, \omega_j) - \frac{g}{U}(u_N, \omega_j) + (2\nu - 2\mu)(\varphi_N, \omega_j) \\
& + \frac{1}{4m}\big(\Lambda^{2\alpha}\varphi_N, \omega_j\big) + i\big(h(x), \omega_j\big), \quad 1 \le j \le N, \qquad (10)
\end{aligned}$$

$$u_N(x,0) = u_{N0}(x) \to u_0(x), \qquad \varphi_N(x,0) = \varphi_{N0}(x) \to \varphi_0(x) \quad \text{in } L^2(\Omega). \qquad (11)$$

The local existence theory for nonlinear ordinary differential equations ensures that the initial value problem (9)-(11) has at least one solution on $[0, t_m]$. By a priori estimates, we know that there exists a global solution for the initial value problem of the nonlinear ordinary differential system (9)-(11) on $[0, T]$.

Similar to the proof of Lemma 1 and Lemma 2, we have

$$u_N + g\varphi_N \in L^2\big(0,T; H^\alpha(\Omega)\big), \qquad u_{Nt} + g\varphi_{Nt} \in L^{\frac{4}{3}}\big(0,T; H^{-\alpha}(\Omega)\big),$$
$$u_N + g\varphi_N \in L^4\big(0,T; L^4(\Omega)\big),$$
$$\varphi_N \in L^2\big(0,T; H^\alpha(\Omega)\big), \qquad \varphi_{Nt} \in L^2\big(0,T; H^{-\alpha}(\Omega)\big).$$

Applying the compactness lemma (Lemma 4), there exist two subsequences, still denoted by $\{u_N + g\varphi_N\}, \{\varphi_N\}$, such that

$$u_N + g\varphi_N \to u + g\varphi \quad \text{strongly in } L^2\big(0, T; L^2(\Omega)\big) \text{ and a.e.}$$
$$\varphi_N \to \varphi \quad \text{strongly in } L^2\big(0, T; L^2(\Omega)\big) \text{ and a.e.}$$

Since $u_N + g\varphi_N \in L^4(0, T; L^4(\Omega))$, we can deduce that

$$\big\| |u_N + g\varphi_N|^2 (u_N + g\varphi_N) \big\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))} \le C.$$

From Lemma 5, we have

$$|u_N + g\varphi_N|^2 (u_N + g\varphi_N) \rightharpoonup |u + g\varphi|^2 (u + g\varphi) \quad \text{weakly in } L^{\frac{4}{3}}\big(0, T; L^{\frac{4}{3}}(\Omega)\big).$$

For any $\phi \in L^2 \times L^2$, we see that $\{(u + g\varphi, \phi_1)\}_N$ and $\{(\varphi, \phi_2)\}_N$ are equicontinuous in $C([0, T])$ from Lemma 3. On the other hand, it follows from Lemma 1 that $\{(u + g\varphi, \phi_1)\}_N$ and $\{(\varphi, \phi_2)\}_N$ are uniformly bounded in $C([0, T])$. Therefore, by Arzela-Ascoli theorem, we have $\{(u + g\varphi, \phi_1)\}_N$ and $\{(\varphi, \phi_2)\}_N$ are compact in $C([0, T])$.

Letting $N \to \infty$, we immediately obtain

$$\begin{aligned}
&\int_0^t (u_N + g\varphi_N, \psi)\, dt \to \int_0^t (u + g\varphi, \psi)\, dt, \\
&\int_0^t \big(\Lambda^\alpha (u_N + g\varphi_N), \Lambda^\alpha \psi\big)\, dt \to \int_0^t \big(\Lambda^\alpha (u + g\varphi), \Lambda^\alpha \psi\big)\, dt, \\
&\int_0^t \big(|u_N + g\varphi_N|^2 (u_N + g\varphi_N), \psi\big)\, dt \to \int_0^t \big(|u + g\varphi|^2 (u + g\varphi), \psi\big)\, dt, \\
&\int_0^t (\varphi_N, \psi)\, dt \to \int_0^t (\varphi, \psi)\, dt, \\
&\int_0^t \big(\Lambda^\alpha \varphi_N, \Lambda^\alpha \psi\big)\, dt \to \int_0^t \big(\Lambda^\alpha \varphi, \Lambda^\alpha \psi\big)\, dt.
\end{aligned}$$

Therefore, we have for any $\psi \in C^\infty(Q_T)$

$$\begin{aligned}
-id(u, \psi) + id(u_0, \psi) = {} & \left(-\frac{dg^2 + 1}{U} + a\right) \int_0^t (u, \psi)\, dt + g\big[a + d(2\nu - 2\mu)\big] \int_0^t (\varphi, \psi)\, dt \\
& - \frac{c}{4m} \int_0^t \big(\Lambda^\alpha u, \Lambda^\alpha \psi\big)\, dt - \frac{g}{4m}(c - d) \int_0^t \big(\Lambda^\alpha \varphi, \Lambda^\alpha \psi\big)\, dt \\
& - b \int_0^t \big(|u + g\varphi|^2 (u + g\varphi), \psi\big)\, dt - id \int_0^t (f, \psi)\, dt, \\
i(\varphi, \psi) - i(\varphi_0, \psi) = {} & -i\beta \int_0^t (\varphi, \psi)\, dt - \frac{g}{U} \int_0^t (u, \psi)\, dt + (2\nu - 2\mu) \int_0^t (\varphi, \psi)\, dt \\
& + \frac{1}{4m} \int_0^t \big(\Lambda^\alpha \varphi, \Lambda^\alpha \psi\big)\, dt + i \int_0^t (h, \psi)\, dt.
\end{aligned}$$

Next we show the initial conditions hold.

By $u_N + g\varphi_N \in L^2(0,T;H^\alpha(\Omega)), u_{Nt} + g\varphi_{Nt} \in L^2(0,T;H^{-\alpha}(\Omega))$, and Lemma 6, we have

$$u_N + g\varphi_N \in C\big([0,T],H^{-\alpha}(\Omega)\big).$$

Similarly, $\varphi_N \in C([0,T],H^{-\alpha}(\Omega))$. Then

$$\begin{aligned}&u_N(0) + g\varphi_N(0) \rightharpoonup u(0) + g\varphi(0) \quad \text{weakly in } H^{-\alpha}(\Omega),\\&\varphi_N(0) \rightharpoonup \varphi(0) \quad \text{weakly in } H^{-\alpha}(\Omega).\end{aligned}$$

But from (12), we have

$$\begin{aligned}&u_N(0) + g\varphi_N(0) \to u_0 + g\varphi_0 \quad \text{in } L^2(\Omega),\\&\varphi_N(0) \to \varphi_0 \quad \text{in } L^2(\Omega).\end{aligned}$$

Therefore

$$u(0) + g\varphi(0) = u_0 + g\varphi_0, \qquad \varphi(0) = \varphi_0.$$

Assume that (u,φ) and $(u^\star,\varphi^\star)$ are the two different solutions of (1)-(4). For $w = u - u^\star$, $v = \varphi - \varphi^\star$, we have

$$\begin{aligned}(w+gv)_t ={}& -\frac{i}{dU}(w+gv) + \frac{ig}{dU}v + \frac{ia}{d}(w+gv) - \frac{ic}{4md}\Lambda^{2\alpha}(w+gv)\\&- \frac{ib}{d}\big(|u+g\varphi|^2(u+g\varphi) - \left|u^\star + g\varphi^\star\right|^2\big(u^\star + g\varphi^\star\big)\big), \qquad (12)\\v_t ={}& -\beta v + \frac{ig}{U}w - i(2v - 2\mu)v - \frac{i}{4m}\Lambda^{2\alpha}v, \qquad (13)\\w(x,0) ={}& 0, \qquad v(x,0) = 0. \qquad (14)\end{aligned}$$

Multiplying (12)-(13) by $\overline{(w+gv)}$ and $\overline{v}$, respectively, then integrating over Ω and taking the real part, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w+gv\|_{L^2}^2\\&\quad= \left(\frac{ad_i}{|d|^2} - \frac{d_i}{|d|^2U}\right)\|w+gv\|_{L^2}^2 + Re\left(\frac{ig}{dU}\int_\Omega v\overline{(w+gv)}\,dx\right)\\&\qquad- \frac{c(m_id_r + m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2\\&\qquad- Re\left(\frac{ib}{d}\int_\Omega\big(|u+g\varphi|^2(u+g\varphi) - \left|u^\star + g\varphi^\star\right|^2\big(u^\star+g\varphi^\star\big)\big)\overline{(w+gv)}\,dx\right),\\&\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 = -\beta\|v\|_{L^2}^2 + Re\left(\frac{ig}{U}\int_\Omega w\overline{v}\,dx\right) - \frac{m_i}{4|m|^2}\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2.\end{aligned}$$

Combining the above two equations yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|w+gv\|_{L^2}^2+\|v\|_{L^2}^2\right)\\
&\quad\le\left(\frac{ad_i}{|d|^2}-\frac{d_i}{|d|^2U}+\frac{g}{2|d|U}+\frac{g}{2U}\right)\|w+gv\|_{L^2}^2-\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^2\\
&\qquad+\frac{3b}{|d|}\int_\Omega\max\left(|u+g\varphi|^2,\left|u^\star+g\varphi^\star\right|^2\right)|w+gv|^2\,dx+\left(\frac{g}{2|d|U}+\frac{g}{2U}-\beta\right)\|v\|_{L^2}^2\\
&\quad\le\left(\frac{ad_i}{|d|^2}-\frac{d_i}{|d|^2U}+\frac{g}{2|d|U}+\frac{g}{2U}\right)\|w+gv\|_{L^2}^2-\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^2\\
&\qquad+\frac{3b}{|d|}\int_\Omega\max\left(|u+g\varphi|^2,\left|u^\star+g\varphi^\star\right|^2\right)|w+gv|^2\,dx+\left(\frac{g}{2|d|U}+\frac{g}{2U}-\beta\right)\|v\|_{L^2}^2\\
&\quad\le\left(\frac{ad_i}{|d|^2}-\frac{d_i}{|d|^2U}+\frac{g}{2|d|U}+\frac{g}{2U}\right)\|w+gv\|_{L^2}^2-\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^2\\
&\qquad+\frac{3b}{|d|}\left(\int_\Omega|u+g\varphi|^4\,dx\right)^{\frac{1}{2}}\left(\int_\Omega|w+gv|^4\,dx\right)^{\frac{1}{2}}\\
&\qquad+\frac{3b}{|d|}\left(\int_\Omega\left|u^\star+g\varphi^\star\right|^4\,dx\right)^{\frac{1}{2}}\left(\int_\Omega|w+gv|^4\,dx\right)^{\frac{1}{2}}+\left(\frac{g}{2|d|U}+\frac{g}{2U}-\beta\right)\|v\|_{L^2}^2\\
&\quad\le\left(\frac{ad_i}{|d|^2}-\frac{d_i}{|d|^2U}+\frac{g}{2|d|U}+\frac{g}{2U}\right)\|w+gv\|_{L^2}^2-\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^2\\
&\qquad+\frac{3b}{|d|}\left(\|u+g\varphi\|_{L^4}^2+\left\|u^\star+g\varphi^\star\right\|_{L^4}^2\right)\|w+gv\|_{L^2}^{2(1-\frac{1}{4\alpha})}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^{\frac{1}{2\alpha}}\\
&\qquad+\left(\frac{g}{2|d|U}+\frac{g}{2U}-\beta\right)\|v\|_{L^2}^2\\
&\quad\le\left(\frac{ad_i}{|d|^2}-\frac{d_i}{|d|^2U}+\frac{g}{2|d|U}+\frac{g}{2U}\right)\|w+gv\|_{L^2}^2-\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^2\\
&\qquad+\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha(w+gv)\right\|_{L^2}^2+C\left(\|u+g\varphi\|_{L^4}^{\frac{8\alpha}{4\alpha-1}}+\left\|u^\star+g\varphi^\star\right\|_{L^4}^{\frac{8\alpha}{4\alpha-1}}\right)\|w+gv\|_{L^2}^2\\
&\qquad+\left(\frac{g}{2|d|U}+\frac{g}{2U}-\beta\right)\|v\|_{L^2}^2\\
&\quad\le C\left(\|u+g\varphi\|_{L^4}^{\frac{8\alpha}{4\alpha-1}}+\left\|u^\star+g\varphi^\star\right\|_{L^4}^{\frac{8\alpha}{4\alpha-1}}+1\right)\left(\|w+gv\|_{L^2}^2+\|v\|_{L^2}^2\right),
\end{aligned}
$$

where we have used the Gagliardo-Nirenberg inequality, the Young inequality, and the elementary inequality

$$|a|^pa-|b|^pb\le(p+1)\max\left(|a|^p,|b|^p\right)|a-b|.$$

Note that $w(x,0)=0$ and $v(x,0)=0$. By Gronwall's inequality, we have

$$w+gv=0,\qquad v=0.$$

Thus we complete the proof. □

5 Global attractors

Finally, we consider the large time behavior of the solution to (1)-(4) in $L^2\times L^2$. Before doing so, we give the following lemma which will be used later.

Lemma 7 ([16]) *Let g, h and y be three nonnegative locally integrable functions on (t_0, ∞) such that y' is locally integrable on (t_0, ∞), and which satisfy*

$$\frac{dy}{dt} \le gy + h$$

and

$$\int_t^{t+r} g(s)\,ds \le a_1, \qquad \int_t^{t+r} h(s)\,ds \le a_2, \qquad \int_t^{t+r} y(s)\,ds \le a_3, \quad t \ge t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \le \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \quad t \ge t_0.$$

Theorem 2 *Assume that $\alpha \in (\frac{1}{2}, 1]$, $U > 0, c > 0, m_i d_r + m_r d_i > 0, b > 0$, $m_i > 0$, $d_i > 0$, $0 < g < 2$ and $\beta > \frac{\frac{g}{|d|U} + \frac{g}{U} + 1}{2-g}$, then the solution operator $S(t) : S(t)(u_0 + g\varphi_0, \varphi_0) = (u + g\varphi, \varphi)$, for all $t > 0$, well defines a semigroup in the space $L^2 \times L^2$ and the following statements hold*:

(i) *For any $t > 0$, $S(t)$ is continuous in $L^2 \times L^2$.*
(ii) *For any $(u_0, \varphi_0) \in L^2 \times L^2$, $S(t)$ is continuous from $[0, T]$ to $L^2 \times L^2$.*
(iii) *For any $t > 0$, $S(t)$ is compact in $L^2 \times L^2$.*
(iv) *The semigroup $\{S(t)\}_{t \ge 0}$ possesses a global attractor $\mathcal{A}$ in $L^2 \times L^2$.*

Proof It is easy to show that the solution (u, φ) to (1)-(4) well defines a semigroup $S(t)$ on $L^2 \times L^2$.

First, we consider the absorbing set in $L^2 \times L^2$. Similarly to the proof of Lemma 1, we have

$$\begin{aligned} &\frac{d}{dt}\left(\|u + g\varphi\|_{L^2}^2 + \|\varphi\|_{L^2}^2\right) - \left(\frac{g}{|d|U} + \frac{g}{U} + \frac{2ad_i}{|d|^2} - \frac{2d_i}{|d|^2 U} + \beta g + 1\right)\|u + g\varphi\|_{L^2}^2 \\ &\quad + \left(2\beta - \frac{g}{|d|U} - \frac{g}{U} - \beta g - 1\right)\|\varphi\|_{L^2}^2 + \frac{2bd_i}{|d|^2}\|u + g\varphi\|_{L^4}^4 + \frac{m_i}{4|m|^2}\left\|\Lambda^\alpha \varphi\right\|_{L^2}^2 \\ &\quad + \frac{c(m_i d_r + m_r d_i)}{4|m|^2|d|^2}\left\|\Lambda^\alpha (u + g\varphi)\right\|_{L^2}^2\,ds \\ &\le \|f\|_{L^2}^2 + \left(1 + g^2\right)\|h\|_{L^2}^2. \end{aligned} \tag{15}$$

Since

$$\frac{2bd_i}{|d|^2\epsilon}\|u + g\varphi\|_{L^2}^2 \le \frac{bd_i}{|d|^2}\|u + g\varphi\|_{L^4}^4 + \frac{bd_i}{|d|^2\epsilon^2}|\Omega|.$$

Thus we can choose ϵ_0 small enough such that

$$\frac{4bd_i}{|d|^2\epsilon_0} - \frac{g}{|d|U} - \frac{g}{U} - \frac{2ad_i}{|d|^2} + \frac{2d_i}{|d|^2 U} - \beta g - 1 > 0.$$

Define

$$R_1 = \min\left\{\frac{4bd_i}{|d|^2\epsilon_0} - \frac{g}{|d|U} - \frac{g}{U} - \frac{2ad_i}{|d|^2} + \frac{2d_i}{|d|^2 U} - \beta g - 1, 2\beta - \frac{g}{|d|U} - \frac{g}{U} - \beta g - 1\right\} > 0,$$

we have

$$\frac{d}{dt}\left(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\right)+R_1\left(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\right)\le\frac{2bd_i}{|d|^2\epsilon_0^2}|\Omega|+\|f\|_{L^2}^2+\left(1+g^2\right)\|h\|_{L^2}^2.$$

Applying Gronwall's inequality yields

$$\begin{aligned}\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2&\le e^{-R_1t}\left(\|u_0+g\varphi_0\|_{L^2}^2+\|\varphi_0\|_{L^2}^2\right)\\&\quad+\left(\frac{2bd_i}{|d|^2\epsilon_0^2}|\Omega|+\|f\|_{L^2}^2+\left(1+g^2\right)\|h\|_{L^2}^2\right)\left(1-e^{-R_1t}\right),\quad t\ge0.\end{aligned}$$

Therefore

$$\lim_{t\to\infty}\left(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\right)\le\rho_0^2,$$

where $\rho_0^2=\frac{2bd_i}{|d|^2\epsilon_0^2}|\Omega|+\|f\|_{L^2}^2+(1+g^2)\|h\|_{L^2}^2$.

It follows that the existence of an absorbing ball in $L^2\times L^2$. Indeed, for any $\rho>\rho_0$, denote by $\mathcal{B}_0$ the ball $B(0,\rho)$, for any bounded set $\mathcal{B}$, there exists a positive $t_0=\frac{1}{R_1}\log\frac{\rho^2}{\rho^2-\rho_0^2}$ such that $S(t)\mathcal{B}\subset\mathcal{B}_0$ for all $t\ge t_0$.

Next we consider the absorbing set in $H^\alpha\times H^\alpha$.

Integrating (15) from t to $t+1$, and using the definition of R_1, we have

$$\begin{aligned}&\left\|u(t+1)+g\varphi(t+1)\right\|_{L^2}^2+\left\|\varphi(t+1)\right\|_{L^2}^2+R_1\int_t^{t+1}\left(\|u+g\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2\right)ds\\&\qquad+\frac{c(m_id_r+m_rd_i)}{4|m|^2|d|^2}\int_t^{t+1}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2ds+\frac{m_i}{4|m|^2}\int_t^{t+1}\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2ds\\&\quad\le\left\|u(t)+g\varphi(t)\right\|_{L^2}^2+\left\|\varphi(t)\right\|_{L^2}^2+\frac{2bd_i}{|d|^2\epsilon_0^2}|\Omega|+\|f\|_{L^2}^2+\left(1+g^2\right)\|h\|_{L^2}^2.\end{aligned}$$

So for all $t\ge t_0$, we see that

$$\int_t^{t+1}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2+\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2ds\le a_1$$

are uniformly bounded.

Multiplying (5) by $\overline{\Lambda^{2\alpha}(u+g\varphi)}$, then integrating over Ω and taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2-\left(\frac{g}{|d|U}+\frac{2ad_i}{|d|^2}-\frac{2d_i}{|d|^2U}+\beta g\right)\left\|\Lambda^\alpha(u+g\varphi)\right\|_{L^2}^2\\&\qquad-\left(\frac{g}{|d|U}+\beta g\right)\left\|\Lambda^\alpha\varphi\right\|_{L^2}^2\\&\quad\le-\frac{c(m_id_r+m_rd_i)}{2|m|^2|d|^2}\left\|\Lambda^{2\alpha}(u+g\varphi)\right\|_{L^2}^2\\&\qquad+Re\left(\frac{2ib}{d}\int_\Omega|u+g\varphi|^2(u+g\varphi)\overline{\Lambda^{2\alpha}(u+g\varphi)}\,dx\right)\\&\qquad+Re\left(\frac{2ib}{d}\int_\Omega(f+gh)\overline{\Lambda^{2\alpha}(u+g\varphi)}\,dx\right)\end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{c(m_i d_r + m_r d_i)}{2|m|^2|d|^2}\left\|\Lambda^{2\alpha}(u+g\varphi)\right\|_{L^2}^2 + \frac{c(m_i d_r + m_r d_i)}{8|m|^2|d|^2}\left\|\Lambda^{2\alpha}(u+g\varphi)\right\|_{L^2}^2 \\
&\quad + C_2\|u+g\varphi\|_{L^6}^6 + \frac{c(m_i d_r + m_r d_i)}{4|m|^2|d|^2}\left\|\Lambda^{2\alpha}(u+g\varphi)\right\|_{L^2}^2 + C_3\|f+gh\|_{L^2}^2.
\end{aligned}
$$

Since

$$
\|u+g\varphi\|_{L^6}^6 \leq \delta\left\|\Lambda^{2\alpha}(u+g\varphi)\right\|_{L^2}^2 + C(\delta)\|u+g\varphi\|_{L^2}^{\frac{6\alpha-1}{2(3\alpha-1)}},
$$

we can choose δ small enough such that

$$
C_2\|u+g\varphi\|_{L^6}^6 \leq \frac{c(m_i d_r + m_r d_i)}{8|m|^2|d|^2}\left\|\Lambda^{2\alpha}(u+g\varphi)\right\|_{L^2}^2 + C_4.
$$

Therefore

$$
\begin{aligned}
&\frac{d}{dt}\left\|\Lambda^{\alpha}(u+g\varphi)\right\|_{L^2}^2 - \left(\frac{g}{|d|U} + \frac{2ad_i}{|d|^2} - \frac{2d_i}{|d|^2U} + \beta g\right)\left\|\Lambda^{\alpha}(u+g\varphi)\right\|_{L^2}^2 \\
&\quad - \left(\frac{g}{|d|U} + \beta g\right)\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 \\
&\leq C_3\|f+gh\|_{L^2}^2 + C_4. \qquad (16)
\end{aligned}
$$

Multiplying (2) by $\overline{\Lambda^{2\alpha}\varphi}$, integrating over Ω, and taking the imaginary part, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 &= -\beta\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 + Re\left(\frac{ig}{U}\int_{\Omega} u\overline{\Lambda^{2\alpha}\varphi}\,dx\right) + Re\left(\int_{\Omega} h(x)\overline{\Lambda^{2\alpha}\varphi}\,dx\right) \\
&\quad - \frac{m_i}{4|m|^2}\left\|\Lambda^{2\alpha}\varphi\right\|_{L^2}^2 \\
&\leq -\beta\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 + Re\left(\frac{ig}{U}\int_{\Omega}(u+g\varphi)\overline{\Lambda^{2\alpha}\varphi}\,dx\right) + \frac{m_i}{4|m|^2}\left\|\Lambda^{2\alpha}\varphi\right\|_{L^2}^2 \\
&\quad + C\|h\|_{L^2}^2 - \frac{m_i}{4|m|^2}\left\|\Lambda^{2\alpha}\varphi\right\|_{L^2}^2 \\
&\leq -\beta\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 + \frac{|g|^2}{2\beta U^2}\left\|\Lambda^{\alpha}(u+g\varphi)\right\|_{L^2}^2 + \frac{\beta}{2}\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 + C\|h\|_{L^2}^2 \\
&\leq -\frac{\beta}{2}\left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2 + \frac{|g|^2}{2\beta U^2}\left\|\Lambda^{\alpha}(u+g\varphi)\right\|_{L^2}^2 + C\|h\|_{L^2}^2. \qquad (17)
\end{aligned}
$$

Combining (16) and (17) yields

$$
\frac{d}{dt}\left(\left\|\Lambda^{\alpha}(u+g\varphi)\right\|_{L^2}^2 + \left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2\right) \leq R_2\left(\left\|\Lambda^{\alpha}(u+g\varphi)\right\|_{L^2}^2 + \left\|\Lambda^{\alpha}\varphi\right\|_{L^2}^2\right) + C_5,
$$

where $R_2 = \max\{\frac{g}{|d|U} + \frac{2ad_i}{|d|^2} - \frac{2d_i}{|d|^2U} + \beta g + \frac{g^2}{\beta U^2}, \frac{g}{|d|U} + \beta g - \beta\}$.

Since

$$
\int_t^{t+1} R_2\,ds \leq R_2, \qquad \int_t^{t+1} C_5\,ds \leq a_2,
$$

applying the uniform Gronwall's inequality (Lemma 7), we have

$$\left\| \Lambda^{\alpha}(u + g\varphi) \right\|_{L^2}^2 + \left\| \Lambda^{\alpha}\varphi \right\|_{L^2}^2 \le (a_1 + a_2)e^{R_2}, \quad t \ge t_0 + 1. \tag{18}$$

The above inequality can ensure us the existence of an absorbing ball in $H^{\alpha} \times H^{\alpha}$. In fact, let $\mathcal{B}$ be a bounded set in $H^{\alpha} \times H^{\alpha}$. Obviously, it is also a bounded set in $L^2 \times L^2$ and $S(t)\mathcal{B} \subset \mathcal{B}_0$ for $t \ge t_0$. From (18), it follows that $S(t)\mathcal{B} \subset \mathcal{B}_1$, where $\mathcal{B}_1 = B(0, \rho_1)$ is a ball with radius $\rho_1^2 = \rho_0^2 + (a_1 + a_2)e^{R_2}$ in $H^{\alpha} \times H^{\alpha}$. Since the embedding $H^{\alpha} \times H^{\alpha} \hookrightarrow L^2 \times L^2$ is compact, we obtain

$$\bigcup_{t \ge t_0 + 1} S(t)\mathcal{B} \text{ is relatively compact in } L^2 \times L^2.$$

Thus the assertion of item (iii) has been proved.

Notice that items (i) and (ii) can be verified similar to the proof of Lemmas 1-4 and item (iv) is a direct corollary of items (i)-(iii). Thus the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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