Sun and Cui *Boundary Value Problems* (2017) 2017:29
DOI 10.1186/s13661-017-0760-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH
Open Access

Existence results for $(k, n-k)$ conjugate boundary-value problems with integral boundary conditions at resonance with $\dim\ker L = 2$

Qiao Sun[1] and Yujun Cui[2*]

*Correspondence:
cyj720201@163.com
[2]State Key Laboratory of Mining Disaster Prevention and Control Co-founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao, 266590, P.R. China
Full list of author information is available at the end of the article

Abstract

We shall study the existence of solutions for a $(k, n-k)$ conjugate boundary-value problem at resonance with $\dim\ker L = 2$ in this paper. The boundary-value problem is shown as follows:

$$\begin{aligned}
&(-1)^{n-k}\varphi^{(n)}(x) = f\left(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n-1)}(x)\right), \quad x \in [0,1], \\
&\varphi^{(i)}(0) = \varphi^{(j)}(1) = 0, \quad 1 \le i \le k-1, 1 \le j \le n-k-1, \\
&\varphi(0) = \int_0^1 \varphi(x)\,dA(x), \qquad \varphi(1) = \int_0^1 \varphi(x)\,dB(x).
\end{aligned}$$

We can obtain that this boundary-value problem has at least one solution under the conditions we provided through Mawhin's continuation theorem, and an example is also provided for our new results.

Keywords: boundary value problem; resonance; Fredholm operator; Mawhin continuation theorem

1 Introduction

Conjugate boundary-value problems at non-resonance have aroused considerable attention in recent years (see [1–11]), and there is also much research on boundary-value problems at resonance (see [12–21]). However, there are very few papers involving $(k, n-k)$ conjugate boundary-value problems at resonance, especially with $\dim\ker L = 2$. For example, Jiang [13] investigated the following boundary-value problem at resonance with $\dim\ker L = 2$:

$$\begin{aligned}
&(-1)^{n-k}y^{(n)}(t) = f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t)\right) + \varepsilon(t), \quad \text{a.e. } t \in [0,1], \\
&y^{(i)}(0) = y^{(j)}(1) = 0, \quad 0 \le i \le k-1, 0 \le j \le n-k-3, \\
&y^{(n-1)}(1) = \sum_{i=1}^{m} \alpha_i y^{(n-1)}(\xi_i), \qquad y^{(n-2)}(1) = \sum_{j=1}^{l} \beta_j y^{(n-2)}(\eta_j),
\end{aligned}$$

where $1 \le k \le n-3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_l < 1$.

Motivated by [11–13], we shall study the following $(k, n-k)$ conjugate boundary-value problem in the situation of resonance with $\dim\ker L = 2$:

$$(-1)^{n-k}\varphi^{(n)}(x) = f\bigl(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n-1)}(x)\bigr), \quad x \in [0,1], \tag{1}$$

$$\varphi^{(i)}(0) = \varphi^{(j)}(1) = 0, \quad 1 \le i \le k-1, 1 \le j \le n-k-1, \tag{2}$$

$$\varphi(0) = \int_0^1 \varphi(x)\,dA(x), \qquad \varphi(1) = \int_0^1 \varphi(x)\,dB(x), \tag{3}$$

where $1 \le k \le n-1$, $n \ge 2$, $A(x)$, $B(x)$ are left continuous at $x = 1$, right continuous on $[0,1)$; $\int_0^1 u(x)\,dA(x)$ and $\int_0^1 u(x)\,dB(x)$ denote the Riemann-Stieltjes integrals of u with respect to A and B, respectively.

However, there are great differences between this article and the above results, the boundary conditions we study are $\varphi(0) = \int_0^1 \varphi(x)\,dA(x)$ and $\varphi(1) = \int_0^1 \varphi(x)\,dB(x)$. As is well known, it is an original case to study conjugate boundary-value problems with integral boundary conditions in the situation of resonance.

The organization of this paper is as follows. In Section 2, we provide a definition and a theorem which will be used to prove the main results. In Section 3, we will give some lemmas and prove the solvability of problem (1)-(3).

2 Preliminaries

For the convenience of the reader, we recall some definitions and a theorem to be used later.

Definition 2.1 ([22]) Suppose that X and Y are real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero if: (1) $\operatorname{Im} L$ is a closed subspace of Y; (2) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

If X, Y are real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero, and $P : X \to X$, $Q : Y \to Y$ are continuous projectors such that

$$\operatorname{Im} P = \ker L, \qquad \ker Q = \operatorname{Im} L, \qquad X = \ker L \oplus \ker P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q,$$

then we can conclude that

$$L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible. We denote the inverse of the mapping by K_P (generalized inverse operator of L). Let Ω be an open bounded subset of X and $\operatorname{dom} L \cap \Omega \neq \emptyset$, then we say the mapping $N : X \to Y$ is L-compact on $\overline{\Omega}$ if $K_P(I-Q)N : \overline{\Omega} \to X$ is compact and $QN(\overline{\Omega})$ is bounded.

Theorem 2.1 ([22]; Mawhin continuation theorem) *$L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero, and N is L-compact on $\overline{\Omega}$. The equation $L\varphi = N\varphi$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$ if the following conditions are satisfied*:

(1) *$L\varphi \neq \lambda N\varphi$ for every $(\varphi, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;*
(2) *$N\varphi \notin \operatorname{Im} L$ for every $\varphi \in \ker L \cap \partial\Omega$;*
(3) *$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Let $X = C^{n-1}[0,1]$ with norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \dots, \|u^{(n-1)}\|_\infty\}$, in which $\|u\|_\infty = \max_{x\in[0,1]} |u(x)|$, and $Y = L^1[0,1]$ with norm $\|x\|_1 = \int_0^1 |x(t)|\,dt$. We define an operator L as follows:

$$(L\varphi)(x) = (-1)^{n-k}\varphi^{(n)}(x)$$

with

$$\begin{aligned}\operatorname{dom} L = \Big\{&\varphi \in X : \varphi^{(i)}(0) = \varphi^{(j)}(1) = 0, 1 \le i \le k-1, 1 \le j \le n-k-1,\\ &\varphi(0) = \int_0^1 \varphi(x)\,dA(x), \varphi(1) = \int_0^1 \varphi(x)\,dB(x)\Big\}.\end{aligned}$$

An operator $N : X \to Y$ is defined as

$$(N\varphi)(x) = f\big(x, \varphi(x), \varphi'(x), \dots, \varphi^{(n-1)}(x)\big).$$

So problem (1)-(3) becomes $L\varphi = N\varphi$.

3 Main results

Assume that the following conditions hold in this paper:

$$\text{(H1)}\quad \begin{aligned}&\int_0^1 \Phi_1(x)\,dA(x) = 1, &&\int_0^1 \Phi_2(x)\,dB(x) = 1,\\ &\int_0^1 \Phi_1(x)\,dB(x) = 0, &&\int_0^1 \Phi_2(x)\,dA(x) = 0,\end{aligned}$$

where

$$\begin{aligned}\Phi_1(x) &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\int_x^1 t^{k-1}(1-t)^{n-k-1}\,dt,\\ \Phi_2(x) &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\int_0^x t^{k-1}(1-t)^{n-k-1}\,dt.\end{aligned}$$

$$\text{(H2)}\quad e = \begin{vmatrix} e_1 & e_2 \\ e_3 & e_4 \end{vmatrix} \neq 0,$$

where

$$\begin{aligned}e_1 &= \int_0^1\int_0^1 k(x,y)\Phi_1(x)\,dy\,dA(x), &\qquad e_2 &= \int_0^1\int_0^1 k(x,y)\Phi_1(x)\,dy\,dB(x),\\ e_3 &= \int_0^1\int_0^1 k(x,y)\Phi_2(x)\,dy\,dA(x), &\qquad e_4 &= \int_0^1\int_0^1 k(x,y)\Phi_2(x)\,dy\,dB(x),\end{aligned}$$

$$k(x,y) = \begin{cases} \frac{1}{(k-1)!(n-k-1)!}\int_0^{x(1-y)} t^{k-1}(t+y-x)^{n-k-1}\,dt, & 0 \le x \le y \le 1;\\ \frac{1}{(k-1)!(n-k-1)!}\int_0^{y(1-x)} t^{n-k-1}(t+x-y)^{k-1}\,dt, & 0 \le y \le x \le 1.\end{cases}$$

(H3) $f : [0,1] \times R^n \to R$ satisfies Carathéodory conditions.

(H4) There exist functions $r(x), q_i(x) \in L^1[0,1]$ with $\sum_{i=1}^{n} \|q_i\|_1 < 1$ such that

$$\left|f(x,\varphi_1,\varphi_2,\dots,\varphi_n)\right| \le \sum_{i=1}^{n} q_i(x)|\varphi_i| + r(x),$$

where $x \in [0,1]$, $\varphi_i \in R$.

(H5) There exists a constant $M > 0$ such that if $|\varphi(x)| + |\varphi^{(n-1)}(x)| > M$ for all $x \in [0,1]$, then

$$\int_0^1 \int_0^1 k(x,y) f\big(y,\varphi(y),\varphi'(y),\varphi'(y),\dots,\varphi^{(n-1)}(y)\big)\,dy\,dA(x) \neq 0,$$

or

$$\int_0^1 \int_0^1 k(x,y) f\big(y,\varphi(y),\varphi'(y),\varphi''(y),\dots,\varphi^{(n-1)}(y)\big)\,dy\,dB(x) \neq 0.$$

(H6) There are constants $a, b > 0$ such that one of the following two conditions holds:

$$c_1 \int_0^1 \int_0^1 k(x,y) N\big(c_1\Phi_1(y) + c_2\Phi_2(y)\big)\,dy\,dA(x) < 0, \tag{4}$$

$$c_2 \int_0^1 \int_0^1 k(x,y) N\big(c_1\Phi_1(y) + c_2\Phi_2(y)\big)\,dy\,dB(x) < 0 \tag{5}$$

if $|c_1| > a$ and $|c_2| > b$, or

$$c_1 \int_0^1 \int_0^1 k(x,y) N\big(c_1\Phi_1(y) + c_2\Phi_2(y)\big)\,dy\,dA(x) > 0, \tag{6}$$

$$c_2 \int_0^1 \int_0^1 k(x,y) N\big(c_1\Phi_1(y) + c_2\Phi_2(y)\big)\,dy\,dB(x) > 0 \tag{7}$$

if $|c_1| > a$ and $|c_2| > b$.

Then we can present the following theorem.

Theorem 3.1 *Suppose* (H1)-(H6) *are satisfied, then there must be at least one solution of problem* (1)-(3) *in* X.

To prove the theorem, we need the following lemmas.

Lemma 3.1 *Assume that* (H1) *and* (H2) *hold, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator with index zero. And a linear continuous projector* $Q : Y \to Y$ *can be defined by*

$$(Qu)(x) = (Q_1u)\Phi_1(x) + (Q_2u)\Phi_2(x),$$

where

$$Q_1u = \frac{1}{e}(e_4T_1u - e_3T_2u), \qquad Q_2u = \frac{1}{e}(-e_2T_1u + e_1T_2u),$$

$$T_1u = \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dA(x), \qquad T_2u = \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dB(x).$$

Furthermore, define a linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *as follows*:

$$(K_P u)(x) = \int_0^1 k(x,y)u(y)\,dy + \Phi_1(x)T_1 u + \Phi_2(x)T_2 u$$

such that $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$.

Proof It follows from (H1) that

$$\begin{aligned}
&(-1)^{n-k}\Phi_1^{(n)}(x) = 0, \qquad (-1)^{n-k}\Phi_2^{(n)}(x) = 0, \quad x \in [0,1],\\
&\Phi_1^{(i)}(0) = \Phi_1^{(j)}(1) = 0, \qquad \Phi_2^{(i)}(0) = \Phi_2^{(j)}(1) = 0, \quad 1 \le i \le k-1, 1 \le j \le n-k-1,\\
&\Phi_1(0) = 1, \qquad \Phi_1(1) = 0, \qquad \Phi_2(0) = 0, \quad \Phi_2(1) = 1.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
&\Phi_1(0) = \int_0^1 \Phi_1(x)\,dA(x), \qquad \Phi_2(1) = \int_0^1 \Phi_2(x)\,dB(x),\\
&\Phi_1(1) = \int_0^1 \Phi_1(x)\,dB(x) = 0, \qquad \Phi_2(0) = \int_0^1 \Phi_2(x)\,dA(x) = 0.
\end{aligned}$$

Thus we have

$$\ker L = \{c_1\Phi_1(x) + c_2\Phi_2(x), c_1, c_2 \in R\}.$$

Moreover, we can obtain that

$$\operatorname{Im} L = \left\{u \in Y : \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dA(x) = \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dB(x) = 0\right\}.$$

On the one hand, suppose $u \in \operatorname{Im} L$, then there exists $\varphi \in \operatorname{dom} L$ such that

$$u = L\varphi \in Y.$$

Then we have

$$\varphi(x) = \int_0^1 k(x,y)u(y)\,dy + \varphi(0)\Phi_1(x) + \varphi(1)\Phi_2(x).$$

Furthermore, for $\varphi \in \operatorname{dom} L$, then

$$\begin{aligned}
\varphi(0) &= \int_0^1 \varphi(x)\,dA(x)\\
&= \int_0^1 \left[\int_0^1 k(x,y)u(y)\,dy + \varphi(0)\Phi_1(x) + \varphi(1)\Phi_2(x)\right] dA(x)\\
&= \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dA(x) + \varphi(0)\int_0^1 \Phi_1(x)\,dA(x) + \varphi(1)\int_0^1 \Phi_2(x)\,dA(x).
\end{aligned}$$

Using this together with (H1), we can get

$$\varphi(0) = \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dA(x) + \varphi(0),$$

it means $\int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dA(x) = 0$. And

$$\begin{aligned}
\varphi(1) &= \int_0^1 \varphi(x)\,dB(x) \\
&= \int_0^1 \left[\int_0^1 k(x,y)u(y)\,dy + \varphi(0)\Phi_1(x) + \varphi(1)\Phi_2(x) \right] dB(x) \\
&= \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dB(x) + \varphi(0) \int_0^1 \Phi_1(x)\,dB(x) + \varphi(1) \int_0^1 \Phi_2(x)\,dB(x).
\end{aligned}$$

So we obtain that

$$\int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dB(x) = 0.$$

Thus

$$\operatorname{Im} L \subset \left\{ u : \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dA(x) = \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dB(x) = 0 \right\}.$$

On the other hand, if $u \in Y$ satisfies

$$\int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dA(x) = \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dB(x) = 0,$$

we let

$$\varphi(x) = \int_0^1 k(x,y)u(y)\,dy + \Phi_1(x) + \Phi_2(x),$$

then we conclude that

$$(L\varphi)(x) = (-1)^{n-k}\varphi^{(n)}(x) = u(x),$$

$$\varphi^{(i)}(0) = \varphi^{(j)}(1) = 0, \quad 1 \le i \le k-1, 1 \le j \le n-k-1,$$

and

$$\begin{aligned}
\varphi(0) &= \int_0^1 k(0,y)u(y)\,dy + \Phi_1(0) + \Phi_2(0) = 1, \\
\varphi(1) &= \int_0^1 k(1,y)u(y)\,dy + \Phi_1(1) + \Phi_2(1) = 1.
\end{aligned}$$

Besides,

$$\int_0^1 \varphi(x)\,dA(x) = \int_0^1 \int_0^1 k(x,y)u(y)\,dy\,dA(x) + \int_0^1 \Phi_1(x)\,dA(x) + \int_0^1 \Phi_2(x)\,dA(x) = 1,$$

and

$$\int_0^1 \varphi(x)\,dB(x) = \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dB(x) + \int_0^1 \Phi_1(x)\,dB(x) + \int_0^1 \Phi_2(x)\,dB(x) = 1.$$

Therefore

$$\varphi(0) = \int_0^1 \varphi(x)\,dA(x), \qquad \varphi(1) = \int_0^1 \varphi(x)\,dB(x).$$

That is, $\varphi \in \operatorname{dom} L$, hence, $u \in \operatorname{Im} L$. In conclusion,

$$\operatorname{Im} L = \left\{ u \in Y : \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dA(x) = \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dB(x) = 0 \right\}.$$

We define a linear operator $P : X \to X$ as

$$(P\varphi)(x) = \Phi_1(x)\varphi(0) + \Phi_2(x)\varphi(1),$$

then

$$\begin{aligned}
\bigl(P^2\varphi\bigr)(x) &= \bigl(P(P\varphi)\bigr)(x) \\
&= \Phi_1(x)\bigl[(P\varphi)(0)\bigr] + \Phi_2(x)\bigl[(P\varphi)(1)\bigr] \\
&= \Phi_1(x)\bigl[\Phi_1(0)\varphi(0) + \Phi_2(0)\varphi(1)\bigr] + \Phi_2(x)\bigl[\Phi_1(1)\varphi(0) + \Phi_2(1)\varphi(1)\bigr] \\
&= \Phi_1(x)\varphi(0) + \Phi_2(x)\varphi(1).
\end{aligned}$$

It is obvious that $P^2\varphi = P\varphi$ and $\operatorname{Im} P = \ker L$. For any $\varphi \in X$, together with $\varphi = (\varphi - P\varphi) + P\varphi$, we have $X = \ker P + \ker L$. It is easy to obtain that $\ker L \cap \ker P = \{0\}$, which implies

$$X = \ker P \oplus \ker L.$$

Next, an operator $Q : Y \to Y$ is defined as follows:

$$(Qu)(x) = (Q_1 u)\Phi_1(x) + (Q_2 u)\Phi_2(x),$$

where

$$\begin{aligned}
&Q_1 u = \frac{1}{e}(e_4 T_1 u - e_3 T_2 u), \qquad Q_2 u = \frac{1}{e}(-e_2 T_1 u + e_1 T_2 u), \\
&T_1 u = \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dA(x), \qquad T_2 u = \int_0^1\int_0^1 k(x,y)u(y)\,dy\,dB(x).
\end{aligned}$$

Obviously, $e_1 = T_1(\Phi_1(x))$, $e_2 = T_2(\Phi_1(x))$, $e_3 = T_1(\Phi_2(x))$, $e_4 = T_2(\Phi_2(x))$. Noting that

$$\begin{aligned}
\bigl(Q^2 u\bigr)(x) &= \bigl(Q_1(Qu)\bigr)(x)\Phi_1(x) + \bigl(Q_2(Qu)\bigr)(x)\Phi_2(x) \\
&= \bigl[Q_1\bigl((Q_1 u)\Phi_1(x) + (Q_2 u)\Phi_2(x)\bigr)\bigr]\Phi_1(x) \\
&\quad + \bigl[Q_2\bigl((Q_1 u)\Phi_1(x) + (Q_2 u)\Phi_2(x)\bigr)\bigr]\Phi_2(x),
\end{aligned}$$

since

$$\begin{aligned}
Q_1\big((Q_1u)\Phi_1(x)\big) &= \frac{1}{e}\big(e_4T_1\big(\Phi_1(x)\big) - e_3T_2\big(\Phi_1(x)\big)\big)Q_1u \\
&= \frac{1}{e}(e_4e_1 - e_3e_2)Q_1u = Q_1u, \\
Q_1\big((Q_2u)\Phi_2(x)\big) &= \frac{1}{e}\big(e_4T_1\big(\Phi_2(x)\big) - e_3T_2\big(\Phi_2(x)\big)\big)Q_2u \\
&= \frac{1}{e}(e_4e_3 - e_3e_4)Q_2u = 0, \\
Q_2\big((Q_1u)\Phi_1(x)\big) &= \frac{1}{e}\big(-e_2T_1\big(\Phi_1(x)\big) + e_1T_2\big(\Phi_1(x)\big)\big)Q_1u \\
&= \frac{1}{e}(-e_2e_1 + e_1e_2)Q_1u = 0, \\
Q_2\big((Q_2u)\Phi_2(x)\big) &= \frac{1}{e}\big(-e_2T_1\big(\Phi_2(x)\big) + e_1T_2\big(\Phi_2(x)\big)\big)Q_2u \\
&= \frac{1}{e}(-e_2e_3 + e_1e_4)Q_2u = Q_2u,
\end{aligned}$$

so

$$\big(Q^2u\big)(x) = (Q_1u)\Phi_1(x) + (Q_2u)\Phi_2(x) = (Qu)(x).$$

And since $u \in \ker Q$, we have $e_4T_1u - e_3T_2u = 0$, $-e_2T_1u + e_1T_2u = 0$, it follows from (H2) that $T_1u = T_2u = 0$, so $u \in \operatorname{Im} L$, that is, $\ker Q \subset \operatorname{Im} L$, and obviously, $\operatorname{Im} L \subset \ker Q$. So $\ker Q = \operatorname{Im} L$. For any $u \in Y$, because $u = (u - Qu) + Qu$, we have $Y = \operatorname{Im} L + \operatorname{Im} Q$. Moreover, together with $Q^2u = Qu$, we can get $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Above all, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$.

To sum up, we can get that $\operatorname{Im} L$ is a closed subspace of Y; $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$; that is, L is a Fredholm operator of index zero.

We now define an operator $K_P : Y \to X$ as follows:

$$(K_Pu)(x) = \int_0^1 k(x,y)u(y)\,dy + \Phi_1(x)T_1u + \Phi_2(x)T_2u.$$

For any $u \in \operatorname{Im} L$, we have $T_1u = 0$, $T_2u = 0$. Consequently,

$$(K_Pu)(x) = \int_0^1 k(x,y)u(y)\,dy, \qquad (K_Pu)(0) = 0, \qquad (K_Pu)(1) = 0.$$

So

$$\begin{aligned}
&(K_Pu)(x) \in \ker P, \qquad (K_Pu)(0) = \int_0^1 (K_Pu)(x)\,dA(x), \\
&(K_Pu)(1) = \int_0^1 (K_Pu)(x)\,dB(x).
\end{aligned}$$

In addition, it is easy to know that

$$(K_Pu)^{(i)}(0) = 0, \quad 1 \le i \le k-1; \qquad (K_Pu)^{(j)}(1) = 0, \quad 1 \le j \le n-k-1,$$

then $(K_P u)(x) \in \operatorname{dom} L$. Therefore

$$K_P u \in \operatorname{dom} L \cap \ker P, \quad u \in \operatorname{Im} L.$$

Next we will prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \ker P}$. It is clear that

$$(LK_P u)(x) = u(x), \quad u \in \operatorname{Im} L.$$

For each $v \in \operatorname{dom} L \cap \ker P$, we have

$$\begin{aligned}
(K_P L v)(x) &= \int_0^1 k(x,y)(-1)^{n-k} v^{(n)}(y)\,dy + \Phi_1(x) \int_0^1 \int_0^1 k(x,y)(-1)^{n-k} v^{(n)}(y)\,dy\,dA(x) \\
&\quad + \Phi_2(x) \int_0^1 \int_0^1 k(x,y)(-1)^{n-k} v^{(n)}(y)\,dy\,dB(x) \\
&= v(x) - v(0)\Phi_1(x) - v(1)\Phi_2(x) + \Phi_1(x) \int_0^1 \big(v(x) - v(0)\Phi_1(x) \\
&\quad - v(1)\Phi_2(x)\big)\,dA(x) + \Phi_2(x) \int_0^1 \big(v(x) - v(0)\Phi_1(x) - v(1)\Phi_2(x)\big)\,dB(x) \\
&= v(x) + \Phi_1(x) \int_0^1 v(x)\,dA(x) + \Phi_2(x) \int_0^1 v(x)\,dB(x) \\
&= v(x) + v(0)\Phi_1(x) + v(1)\Phi_2(x) \\
&= v(x).
\end{aligned}$$

It implies that $K_P L v = v$. So $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. Thus the lemma holds. □

Lemma 3.2 *N is L-compact on $\overline{\Omega}$ if* $\operatorname{dom} L \cap \overline{\Omega} \neq 0$, *where Ω is a bounded open subset of X.*

Proof We can get easily that QN is bounded. From (H3) we know that there exists $M_0(x) \in L^1$ such that $|(I-Q)N\varphi| \leq M_0(x)$, a.e. $x \in [0,1]$, $\varphi \in \overline{\Omega}$. Hence $K_P(I-Q)N(\overline{\Omega})$ is bounded. By the Lebesgue dominated convergence theorem and condition (H3), we can obtain that $K_P(I-Q)N(\overline{\Omega})$ is continuous. In addition, for $\{\int_0^1 k(x,y)(I-Q)N\varphi(y)\,dy + \Phi_1(x)\int_0^1\int_0^1 k(x,y)(I-Q)N\varphi(y)\,dy\,dA(x) + \Phi_2(x)\int_0^1\int_0^1 k(x,y)(I-Q)N\varphi(y)\,dy\,dB(x)\}$ is equicontinuous, by the Ascoli-Arzela theorem, we get $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Thus, N is L-compact. The proof is completed. □

Lemma 3.3 *The set* $\Omega_1 = \{\varphi \in \operatorname{dom} L \setminus \ker L : L\varphi = \lambda N\varphi, \lambda \in [0,1]\}$ *is bounded if* (H1)-(H5) *are satisfied.*

Proof Take $\varphi \in \Omega_1$, then $N\varphi \in \operatorname{Im} L$, thus we have

$$\int_0^1 \int_0^1 k(x,y) f\big(y, \varphi(y), \varphi'(y), \ldots, \varphi^{(n-1)}(y)\big)\,dy\,dA(x) = 0 \tag{8}$$

and

$$\int_0^1 \int_0^1 k(x,y) f\big(y, \varphi(y), \varphi'(y), \ldots, \varphi^{(n-1)}(y)\big)\,dy\,dB(x) = 0. \tag{9}$$

By this together with (H5) we know that there exists $x_0 \in [0,1]$ such that

$$\left|\varphi(x_0)\right| + \left|\varphi^{(n-1)}(x_0)\right| \le M.$$

And $\varphi^{(i)}(0) = \varphi^{(j)}(1) = 0$, $1 \le i \le k-1$, $1 \le j \le n-k-1$, hence there exists at least a point $\theta_i \in [0,1]$ such that $\varphi^{(i)}(\theta_i) = 0$, $i = 1,2,\ldots,n-2$. Thus, we get $\varphi^{(i)}(x) = \int_{\theta_i}^{x} \varphi^{(i+1)}(t)\,dt$, $i = 1,2,\ldots,n-2$. So,

$$\left\|\varphi^{(i)}\right\|_\infty \le \left\|\varphi^{(i+1)}\right\|_1 \le \left\|\varphi^{(i+1)}\right\|_\infty, \quad i = 1,2,\ldots,n-2. \tag{10}$$

From

$$\left\|\varphi^{(n-1)}(x)\right\|_\infty = \max_{x\in[0,1]}\left|\varphi^{(n-1)}(x)\right|$$

and

$$\begin{aligned}\varphi^{(n-1)}(x) &= \varphi^{(n-1)}(x_0) + \int_{x_0}^{x} \varphi^{(n)}(t)\,dt \\ &= \varphi^{(n-1)}(x_0) + \int_{x_0}^{x} (-1)^{n-k} f\left(t,\varphi(t),\varphi'(t),\ldots,\varphi^{(n-1)}(t)\right)dt,\end{aligned}$$

it follows from (H4) and (10) that

$$\begin{aligned}\left|\varphi^{(n-1)}(x)\right| &\le \left|\varphi^{(n-1)}(x_0)\right| + \left|\int_{x_0}^{x} \left|\varphi^{(n)}(t)\right| dt\right| \\ &\le M + \sum_{i=1}^{n} \|q_i\|_1 \left\|\varphi^{(i-1)}\right\|_\infty + \|r\|_1 \\ &\le M_1 + c'\|\varphi\|_\infty + c''\left\|\varphi^{(n-1)}\right\|_\infty,\end{aligned} \tag{11}$$

where $c' = \|q_1\|_1$, $c'' = \sum_{i=2}^{n} \|q_i\|_1$, $M_1 = M + \|r\|_1$.

In addition, for

$$\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} \varphi'(t)\,dt,$$

from (10) we have

$$\|\varphi\|_\infty \le M + \left\|\varphi^{(n-1)}\right\|_\infty. \tag{12}$$

Besides, $\|\varphi\| = \max\{\|\varphi\|_\infty, \|\varphi^{(n-1)}\|_\infty\}$. If $\|\varphi\|_\infty \ge \|\varphi^{(n-1)}\|_\infty$, by (11) and (12) we have

$$\left\|\varphi^{(n-1)}\right\|_\infty \le \frac{M_1 + c'\|\varphi\|_\infty}{1-c''}$$

and

$$\|\varphi\|_\infty \le M + \frac{M_1 + c'\|\varphi\|_\infty}{1-c''},$$

so $\|\varphi\|_\infty \le \frac{1}{1-c'-c''}[(1-c'')M + M_1]$.

If $\|\varphi^{(n-1)}\|_\infty > \|\varphi\|_\infty$, then by (11) and (12) we have

$$\begin{aligned}\left\|\varphi^{(n-1)}\right\|_\infty &\le M_1 + c'\left(M + \left\|\varphi^{(n-1)}\right\|_\infty\right) + c''\left\|\varphi^{(n-1)}\right\|_\infty \\ &\le M_1 + c'M + \left(c' + c''\right)\left\|\varphi^{(n-1)}\right\|_\infty,\end{aligned}$$

so $\|\varphi^{(n-1)}\|_\infty \le \frac{1}{1-c'-c''}(M_1 + c'M)$. Above all, $\|\varphi\| \le M_X$, where

$$M_X = \max\left\{\frac{1}{1-c'-c''}\left[\left(1-c''\right)M + M_1\right], \frac{1}{1-c'-c''}\left(M_1 + c'M\right)\right\}.$$

Above all, we know Ω_1 is bounded. The proof of the lemma is completed. □

Lemma 3.4 *The set* $\Omega_2 = \{\varphi : \varphi \in \ker L, N\varphi \in \operatorname{Im} L\}$ *is bounded if* (H1)-(H3), (H6) *hold*.

Proof Let $\varphi \in \Omega_2$, then $\varphi(x) \equiv c_1\Phi_1(x) + c_2\Phi_2(x)$, and $N\varphi \in \operatorname{Im} L$, so we can get

$$c_1 \int_0^1 \int_0^1 k(x,y) f\left(y, c_1\Phi_1(y) + c_2\Phi_2(y), \dots, c_1\Phi_1^{(n-1)}(y) + c_2\Phi_2^{(n-1)}(y)\right) dy\, dA(x) = 0$$

and

$$c_2 \int_0^1 \int_0^1 k(x,y) f\left(y, c_1\Phi_1(y) + c_2\Phi_2(y), \dots, c_1\Phi_1^{(n-1)}(y) + c_2\Phi_2^{(n-1)}(y)\right) dy\, dB(x) = 0.$$

According to (H6), we have $|c_1| \le a$, $|c_2| \le b$, that is to say, Ω_2 is bounded. We complete the proof. □

Lemma 3.5 *The set* $\Omega_3 = \{\varphi \in \ker L : \lambda J\varphi + \alpha(1-\lambda)QN\varphi = 0, \lambda \in [0,1]\}$ *is bounded if conditions* (H1)-(H3), (H6) *are satisfied, where* $J : \ker L \to \operatorname{Im} L$ *is a linear isomorphism given by* $J(c_1\Phi_1(x) + c_2\Phi_2(x)) = \frac{1}{e}(e_4c_1 - e_3c_2)\Phi_1(x) + \frac{1}{e}(-e_2c_1 + e_1c_2)\Phi_2(x)$, *and*

$$\alpha = \begin{cases} -1, & \textit{if } (4)\text{-}(5) \textit{ hold}; \\ 1, & \textit{if } (6)\text{-}(7) \textit{ hold}. \end{cases}$$

Proof Suppose that $\varphi \in \Omega_3$, we have $\varphi(x) = c_1\Phi_1(x) + c_2\Phi_2(x)$, and

$$\lambda c_1 = -\alpha(1-\lambda)T_1N\varphi, \qquad \lambda c_2 = -\alpha(1-\lambda)T_2N\varphi.$$

If $\lambda = 0$, by condition (H6) we have $|c_1| \le a$, $|c_2| \le b$. If $\lambda = 1$, then $c_1 = c_2 = 0$. If $\lambda \in (0,1)$, we suppose $|c_1| \ge a$ or $|c_2| \ge b$, then

$$\lambda c_1^2 = -\alpha(1-\lambda)c_1T_1N\varphi < 0$$

or

$$\lambda c_2^2 = -\alpha(1-\lambda)c_2T_2N\varphi < 0,$$

which contradicts with $\lambda c_1^2 > 0$, $\lambda c_2^2 > 0$. So the lemma holds. □

Then Theorem 3.1 can be proved now.

Proof of Theorem 3.1 Suppose that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i} \cup \{0\}$ is a bounded open subset of X. From Lemma 3.2 we know that N is L-compact on $\overline{\Omega}$. In view of Lemmas 3.3 and 3.4, we can get

(1) $L\varphi \neq \lambda N\varphi$, for every $(\varphi, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;

(2) $N\varphi \notin \operatorname{Im} L$, for every $\varphi \in \ker L \cap \partial\Omega$.

Set $H(\varphi, \lambda) = \lambda J\varphi + \alpha(1-\lambda)QN\varphi$. It follows from Lemma 3.5 that $H(\varphi, \lambda) \neq 0$ for any $\varphi \in \partial\Omega \cap \ker L$. So, by the homotopy of degree, we have

$$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) = \deg(\alpha J, \Omega \cap \ker L, 0) \neq 0.$$

All the conditions of Theorem 2.1 are satisfied. So there must be at least one solution of problem (1)-(3) in X. The proof of Theorem 3.1 is completed. □

4 Example

We now present an example to illustrate our main theorem. Consider the following boundary-value problem:

$$\begin{aligned}
&\varphi^{(4)}(x) = \frac{\pi}{24}\left|\varphi(x)\right| + \frac{1}{12}\sin\varphi'(x) + \frac{1}{4}\sin\varphi''(x) + \frac{1}{6}\varphi'''(x)\arctan\left(\frac{1}{5}\varphi'''(x)\right) + x, \\
&\quad x \in [0,1], \\
&\varphi'(0) = \varphi'(1) = 0, \qquad \varphi(0) = -\frac{5}{11}\varphi\left(\frac{1}{2}\right) + \frac{16}{11}\varphi\left(\frac{1}{4}\right), \\
&\varphi(1) = \frac{40}{13}\varphi\left(\frac{1}{2}\right) - \frac{27}{13}\varphi\left(\frac{1}{3}\right).
\end{aligned}$$

Obviously, $n = 4$, $k = 2$, and

$$A(x) = \begin{cases} 0, & x \le \frac{1}{4}; \\ \frac{16}{11}, & \frac{1}{4} < x \le \frac{1}{2}; \\ 1, & \frac{1}{2} < x \le 1; \end{cases} \qquad B(x) = \begin{cases} 0, & x \le \frac{1}{3}; \\ -\frac{27}{13}, & \frac{1}{3} < x \le \frac{1}{2}; \\ 1, & \frac{1}{2} < x \le 1. \end{cases}$$

Let $\Phi_1(x) = 2x^3 - 3x^2 + 1$, $\Phi_2(x) = -2x^3 + 3x^2$, then

$$\begin{aligned}
&\int_0^1 \Phi_1(x)\,dA(x) = -\frac{5}{11}\Phi_1\left(\frac{1}{2}\right) + \frac{16}{11}\Phi_1\left(\frac{1}{4}\right) = 1, \\
&\int_0^1 \Phi_2(x)\,dB(x) = \frac{40}{13}\Phi_2\left(\frac{1}{2}\right) - \frac{27}{13}\Phi_2\left(\frac{1}{3}\right) = 1,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \Phi_1(x)\,dB(x) = \frac{40}{13}\Phi_1\left(\frac{1}{2}\right) - \frac{27}{13}\Phi_1\left(\frac{1}{3}\right) = 0, \\
&\int_0^1 \Phi_2(x)\,dA(x) = -\frac{5}{11}\Phi_2\left(\frac{1}{2}\right) + \frac{16}{11}\Phi_2\left(\frac{1}{4}\right) = 0,
\end{aligned}$$

thus (H1) is satisfied. By calculation, we can obtain that $e = \begin{vmatrix} e_1 & e_2 \\ e_3 & e_4 \end{vmatrix} \neq 0$, so (H2) holds. Let

$$f(x, \varphi, \varphi', \varphi'', \varphi''') = \frac{\pi}{24}|\varphi| + \frac{1}{12}\sin\varphi' + \frac{1}{4}\sin\varphi'' + \frac{1}{6}\varphi''' \arctan\left(\frac{1}{5}\varphi'''\right) + x,$$

then

$$\left|f(x, \varphi, \varphi', \varphi'', \varphi''')\right| \leq \frac{\pi}{24}|\varphi| + \frac{1}{12}\left|\varphi'\right| + \frac{1}{4}\left|\varphi''\right| + \frac{\pi}{12}\left|\varphi'''\right| + 1,$$

where

$$q_1 = \frac{\pi}{24}, \qquad q_2 = \frac{1}{12}, \qquad q_3 = \frac{1}{4}, \qquad q_4 = \frac{\pi}{12}, \qquad r(x) = 1.$$

Taking $M = 11$, we have $|\varphi'''(x)| + |\varphi(x)| > 11$,

$$\begin{cases} f(x, \varphi, \varphi', \varphi'', \varphi''') \geq \frac{\pi}{24} \cdot 5 - \frac{1}{12} - \frac{1}{4} > 0, & \text{if } |\varphi(x)| \geq 5; \\ f(x, \varphi, \varphi', \varphi'', \varphi''') \geq -\frac{1}{12} - \frac{1}{4} + \frac{1}{6} \cdot 5 \cdot \frac{\pi}{4} > 0, & \text{if } |\varphi'''(x)| \geq 5, \end{cases}$$

for $k(x,y) > 0$,

$$\int_0^1 \int_0^1 k(x,y) f\big(y, \varphi(y), \varphi'(y), \varphi''(y), \varphi'''(y)\big)\, dy\, dA(x) \neq 0$$

and

$$\int_0^1 \int_0^1 k(x,y) f\big(y, \varphi(y), \varphi'(y), \varphi''(y), \varphi'''(y)\big)\, dy\, dB(x) \neq 0.$$

Hence (H5) holds. Finally, taking $a = \frac{8}{\pi}$, $b = \frac{8}{\pi}$, when $|c_1| > a$, $|c_2| > b$,

$$\begin{cases} f(x, \varphi, \varphi', \varphi'', \varphi''') > \frac{\pi}{24} \cdot (\frac{8}{\pi}\Phi_1(x) + \frac{8}{\pi}\Phi_2(x)) - \frac{1}{12} - \frac{1}{4} = 0, & \text{if } c_1 \cdot c_2 > 0; \\ f(x, \varphi, \varphi', \varphi'', \varphi''') > -\frac{1}{12} - \frac{1}{4} + \frac{1}{6} \cdot 12(\frac{16}{\pi}) \cdot \arctan(\frac{1}{5} \cdot 12 \cdot \frac{16}{\pi}) > 0, & \text{if } c_1 \cdot c_2 < 0, \end{cases}$$

then we obtain

$$c_1 \int_0^1 \int_0^1 k(x,y) f\big(y, c_1\Phi_1(y) + c_2\Phi_2(y), \ldots, c_1\Phi_1'''(y) + c_2\Phi_2'''(y)\big)\, dy\, dA(x) > 0$$

and

$$c_2 \int_0^1 \int_0^1 k(x,y) f\big(y, c_1\Phi_1(y) + c_2\Phi_2(y), \ldots, c_1\Phi_1'''(y) + c_2\Phi_2'''(y)\big)\, dy\, dB(x) > 0,$$

then condition (H6) is satisfied. It follows from Theorem 3.1 that there must be at least one solution in $C^3[0,1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this article. All the authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shandong University of Science and Technology, Qingdao, 266590, P.R. China. [2]State Key Laboratory of Mining Disaster Prevention and Control Co-founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao, 266590, P.R. China.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (11571207) and the Tai'shan Scholar Engineering Construction Fund of Shandong Province of China and SDUST graduate innovation project SDKDY-170342.

Received: 16 November 2016 Accepted: 21 February 2017 Published online: 28 February 2017

References
1. Agarwal, RP, O'Regan, D: Positive solutions for $(p, n-p)$ conjugate boundary-value problems. J. Differ. Equ. **150**, 462-473 (1998)
2. Agarwal, RP, O'Regan, D: Multiplicity results for singular conjugate, focal, and (N, P) problems. J. Differ. Equ. **170**, 142-156 (2001)
3. Cui, Y, Zou, Y: Monotone iterative technique for $(k, n-k)$ conjugate boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2015**, 69 (2015)
4. Eloe, PW, Henderson, J: Singular nonlinear $(k, n-k)$ conjugate boundary value problems. J. Differ. Equ. **133**, 136-151 (1997)
5. He, X, Ge, W: Positive solutions for semipositone $(p, n-p)$ right focal boundary value problems. Appl. Anal. **81**, 227-240 (2002)
6. Jiang, D: Positive solutions to $(k, n-k)$ conjugate boundary-value problems. Acta Math. Sin. **44**, 541-548 (2001) (in Chinese)
7. Jiang, W, Zhang, J: Positive solutions for $(k, n-k)$ conjugate eigenvalue problems in Banach spaces. Nonlinear Anal. **71**, 723-729 (2009)
8. Kong, L, Wang, J: The Green's function for $(k, n-k)$ boundary value problems and its application. J. Math. Anal. Appl. **255**, 404-422 (2001)
9. Lin, X, Jiang, D, Li, X: Existence and uniqueness of solutions for singular $(k, n-k)$ conjugate boundary value problems. Comput. Math. Appl. **52**, 375-382 (2006)
10. Ma, R: Positive solutions for semipositone $(k, n-k)$ conjugate boundary-value problems. J. Math. Anal. Appl. **252**, 220-229 (2000)
11. Zhang, G, Sun, J: Positive solutions of singular $(k, n-k)$ multi-point boundary value problems. Acta Math. Sin. **49**, 391-398 (2006) (in Chinese)
12. Jiang, W, Qiu, J: Solvability of $(k, n-k)$ conjugate boundary-value problems at resonance. Electron. J. Differ. Equ. **114**, 1 (2012)
13. Jiang, W: Existence of solutions for $(k, n-k-2)$ conjugate boundary-value problems at resonance with $\dim\ker L = 2$. Electron. J. Differ. Equ. **226**, 1 (2013)
14. Liu, Y, Ge, W: Solvability of nonlocal boundary-value problems for ordinary differential equations of higher order. Nonlinear Anal. **57**, 435-458 (2004)
15. Ma, R: Existence results of a m-point boundary-value problem at resonance. J. Math. Anal. Appl. **294**, 147-157 (2004)
16. Sun, Q, Cui, Y: Solvability of $(k, n-k)$ conjugate boundary value problems with integral boundary conditions at resonance. J. Funct. Spaces **2016**, Article ID 3454879 (2016)
17. Xue, C, Ge, W: The existence of solutions for multi-point boundary value problem at resonance. Acta Math. Sin. **48**, 281-290 (2005)
18. Zhang, X, Feng, M, Ge, W: Existence result of second-order differential equations with integral boundary conditions at resonance. J. Math. Anal. Appl. **353**, 311-319 (2009)
19. Iyase, SA: Existence results for a fourth order multipoint boundary value problem at resonance. J. Niger. Math. Soc. **34**, 259-266 (2015)
20. Svatoslav, S: Existence results for functional boundary value problems at resonance. Math. Slovaca **48**, 43-55 (1998)
21. Feng, W, Webb, JRL: Solvability of three point boundary value problems at resonance. Nonlinear Anal. **30**, 3227-3238 (1997)
22. Mawhin, J: Topological Degree Methods in Nonlinear Boundary-Value Problems. NSF-CBMS Regional Conference Series in Mathematics. Am. Math. Soc., Providence (1979)