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Existence and multiplicity of positive solutions for p-Kirchhoff type problem with singularity

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Abstract
In this paper, we consider a class of p-Kirchhoff type problems with a singularity in a bounded domain in R^N. By using the variational method, the existence and multiplicity of positive solutions are obtained.

Keywords: p-Kirchhoff type problem; singularity; Nehari manifold

1 Introduction and main results

The purpose of this paper is to investigate the existence of multiple positive solutions to the following problem:

$$\begin{cases} -M(\int_\Omega |\nabla u|^p\,dx)\Delta_p u = \lambda f(x)u^{-r} + g(x)u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Ω is a smooth bounded domain in R^N, $0 < r < 1 < p < q < p^*$ ($p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = \infty$ if $N \le p$), $M(s) = as^{p-1} + b$ and $a, b, \lambda > 0$. $f, g \in C(\overline{\Omega})$ are nontrivial nonnegative functions.

Problem (1.1) is related to the stationary problem introduced by Kirchhoff in [1]. More precisely, it is the model

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, E, L are constants, which was proposed as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings to describe transversal oscillations of a stretched string. For more details and backgrounds, we refer to [2, 3].

The existence and multiplicity of solutions for the following problem:

$$\begin{cases} -M(\int_\Omega |\nabla u|^p\,dx)\Delta_p u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

on a smooth bounded domain $\Omega \subset R^N$ has been studied in many papers. Liu and Zhao [4] proved (1.2) has at least two nontrivial weak solutions by Morse theory under some restriction on $M(s)$ and $h(x,u)$. In [5], the authors considered the following problem:

$$\begin{cases} -M(\int_\Omega |\nabla u|^p\,dx)\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $M(s) = as + b$, $1 < q < p < r \leq p^*$, they proved the existence of multiplicity nontrivial solutions by using the Nehari manifold when the weight functions $f(x)$ and $g(x)$ change their signs. For more results, we refer to [6–10] and the references therein.

When $p = 2$ and $N = 3$, problem (1.1) reduces to the following singular Kirchhoff type problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda f(x)u^{-r} + \mu g(x)u^q & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $1 < q \leq 5$, the existence of solutions for problem (1.4) has been widely studied (see [11–13]). When $3 < q < 5$ and $\lambda = 1$, Liu and Sun [11] proved that problem (1.4) has at least two positive solutions for $\mu > 0$ small enough. Liao *et al.* showed the multiplicity of positive solutions by the Nehari mainfold in the case of $q = 3$ in [12]. When q is a critical exponents, at least two positive solutions are obtained by variational and perturbation methods in [13].

However, the singular p-Kirchhoff type problems have few been considered, especially $p \neq 2, N \neq 3$. Here we focus on extending the results in [11] and [12]. In fact, the extension is nontrivial and requires a more careful analysis. Our method is based on the Nehari manifold; see [6, 11, 14, 15].

Before starting our main theorems, we make use of the following notations:

• Let $W_0^{1,p}(\Omega)$ be the Sobolev space with norm $\|u\| = (\int_\Omega |\nabla u|^p\,dx)^{\frac{1}{p}}$, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$;

• Let S_z be the best Sobolev constant for the embedding of $W_0^{1,p}(\Omega)$ in $L_z(\Omega)$ with $0 < z < p^*$. Then, for all $u \in W_0^{1,p}(\Omega)\backslash\{0\}$,

$$\|u\|_z \leq S_z^{-\frac{1}{p}}\|u\|.$$

In general, we say that a function $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (1.1) if

$$M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx - \lambda\int_\Omega f|u|^{-r}\varphi\,dx - \int_\Omega g|u|^{q-1}\varphi\,dx = 0$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Thus, the functional corresponding to problem (1.1) is defined by

$$J(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{\lambda}{1-r}\int_\Omega f|u|^{1-r}\,dx - \frac{1}{q}\int_\Omega g|u|^q\,dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $\widehat{M}(s) = \int_0^s M(t)\,dt$.

To obtain the existence results, we introduce the Nehari manifold:

$$N_\lambda = \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} : M(\|u\|^p)\|u\|^p - \lambda\int_\Omega f|u|^{1-r}\,dx - \int_\Omega g|u|^q\,dx = 0\right\},$$

and we define

$$K(u) = (p-1)M\big(\|u\|^p\big)\|u\|^p + pM'\big(\|u\|^p\big)\|u\|^{2p} + \lambda r\int_\Omega f|u|^{1-r}\,dx - (q-1)\int_\Omega g|u|^q\,dx.$$

Now we split N_λ into three disjoint parts as follows:

$$N_\lambda^+ = \big\{u \in N_\lambda : K(u) > 0\big\};$$
$$N_\lambda^0 = \big\{u \in N_\lambda : K(u) = 0\big\};$$
$$N_\lambda^- = \big\{u \in N_\lambda : K(u) < 0\big\}.$$

Let $\lambda^* = \max\{\frac{(1-r)\lambda_1(a)}{p^{\frac{p+1}{p}}}, \frac{(1-r)\lambda_2}{p}\}$, where $\lambda_1(a)$ and λ_2 are given by

$$\lambda_1(a) = \frac{pS_{1-r}^{\frac{1-r}{p}}\sqrt[p]{ab^{p-1}(q-p^2)(\frac{q-p}{p-1})^{p-1}}}{(q+r-1)\|f\|_\infty}\left(\frac{pS_q^{\frac{q}{p}}\sqrt[p]{ab^{p-1}(p^2+r-1)}}{(q+r-1)\|g\|_\infty}\right)^{\frac{2p+r-2}{q-2p+1}}$$

and

$$\lambda_2 = \frac{bS_{1-r}^{\frac{1-r}{p}}(q-p)}{(q+r-1)\|f\|_\infty}\left(\frac{bS_q^{\frac{q}{p}}(p+r-1)}{(q+r-1)\|g\|_\infty}\right)^{\frac{p+r-1}{q-p}},$$

then we state the main theorems.

Theorem 1.1 *Assume that* $p^2 < q < p^*$ *and* $N < 2p$. *Then, for each* $a > 0$ *and* $0 < \lambda < \lambda^*$, *the problem* (1.1) *has at least two positive solutions* $u_\lambda^+ \in N_\lambda^+$ *and* $u_\lambda^- \in N_\lambda^-$.

Define

$$\Lambda = \inf\left\{\|u\|^{p^2} : u \in W_0^{1,p}(\Omega), \int_\Omega g|u|^{p^2}\,dx = 1\right\}, \tag{1.5}$$

then $\Lambda > 0$ is obtained by some $\phi_\Lambda \in W_0^{1,p}(\Omega)$ with $\int_\Omega g|\phi_\Lambda|^{p^2}\,dx = 1$. In particular,

$$\Lambda\int_\Omega g|u|^{p^2}\,dx \le \|u\|^{p^2} \tag{1.6}$$

and

$$\begin{cases} -\|u\|^p\Delta_p u = \mu g|u|^{p^2-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where μ is an eigenvalue of (1.7), $u \in W_0^{1,p}(\Omega)$ is nonzero and an eigenvector corresponding to μ such that

$$\|u\|^p\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(u\varphi)\,dx = \mu\int_\Omega g|u|^{p^2-1}u\varphi\,dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega);$$

we write

$$I(u) = \|u\|^{p^2}, \quad \text{for } u \in E = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega g|u|^{p^2}\,dx = 1 \right\},$$

and all distinct eigenvalues of (1.7) denoted by $0 < \mu_1 < \mu_2 < \cdots$, we have

$$\mu_1 = \inf_{u \in E} I(u) > 0,$$

where μ_1 is simple, isolated and can be obtained at some $\psi \in E$ and $\psi > 0$ in Ω (see [16]).

Theorem 1.2 *Assume that $p^2 = q < p^*$ and $N < 2p$. Then*

(i) *for each $a \geq \frac{1}{\Lambda}$ and $\lambda > 0$, the problem* (1.1) *has at least one positive solution*;

(ii) *for each $a < \frac{1}{\Lambda}$ and $0 < \lambda < \frac{1-r}{p}\hat{\lambda}$, where*

$$\hat{\lambda} = \frac{bS_{1-r}^{\frac{1-r}{p}}(p^2 - p)}{\|f\|_\infty (p^2 + r - 1)} \left(\frac{b\Lambda(p + r - 1)}{(1 - a\Lambda)(p^2 + r - 1)} \right)^{\frac{p+r-1}{p^2-p}},$$

the problem (1.1) *has at least two positive solutions $u_\lambda^+ \in N_\lambda^+$, $u_\lambda^- \in N_\lambda^-$ and*

$$\lim_{a \to \frac{1}{\Lambda}^-} \inf_{u \in N_\lambda^-} J(u) = \infty.$$

This paper is organized as follows: In Section 2, we present some lemmas which will be used to prove our main results. In Section 3 and Section 4, we will prove Theorems 1.1 and 1.2, respectively.

2 Preliminaries

Lemma 2.1 (i) *If $q \geq p^2$, then the energy functional $J(u)$ is coercive and bounded below in N_λ*;

(ii) *if $q < p^2$, then the energy functional $J(u)$ is coercive and bounded below in $W_0^{1,p}(\Omega)$.*

Proof (i) For $u \in N_\lambda$, we have

$$M\big(\|u\|^p\big)\|u\|^p - \lambda \int_\Omega f|u|^{1-r}\,dx - \int_\Omega g|u|^q\,dx = 0.$$

By the Sobolev inequality,

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\big(\|u\|^p\big) - \frac{\lambda}{1-r}\int_\Omega f|u|^{1-r}\,dx - \frac{1}{q}\int_\Omega g|u|^q\,dx \\ &= \frac{1}{p}\widehat{M}\big(\|u\|^p\big) - \frac{1}{q}M\big(\|u\|^p\big)\|u\|^p - \lambda\frac{q+r-1}{q(1-r)}\int_\Omega f|u|^{1-r}\,dx \\ &\geq \frac{\|u\|^p}{pq}\left(\frac{a(q-p^2)}{p}\|u\|^{p^2-p} + b(q-p)\right) - \lambda\frac{q+r-1}{q(1-r)}\|f\|_\infty S_{1-r}^{\frac{r-1}{p}}\|u\|^{1-r} \\ &\geq \frac{b(q-p)}{pq}\|u\|^p - \lambda\frac{q+r-1}{q(1-r)}\|f\|_\infty S_{1-r}^{\frac{r-1}{p}}\|u\|^{1-r}. \end{aligned}$$

Thus, $J(u)$ is coercive and bounded below in N_λ.

(ii) For $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\big(\|u\|^p\big) - \frac{\lambda}{1-r}\int_\Omega f|u|^{1-r}\,dx - \frac{1}{q}\int_\Omega g|u|^q\,dx \\ &\geq \frac{a}{p^2}\|u\|^{p^2} + \frac{b}{p}\|u\|^p - \frac{\lambda\|f\|_\infty S_{1-r}^{\frac{r-1}{p}}}{1-r}\|u\|^{1-r} - \frac{\|g\|_\infty S_q^{-\frac{q}{p}}}{q}\|u\|^q \\ &= \left(\frac{a}{p^2}\|u\|^{p^2-q} - \frac{\|g\|_\infty S_q^{-\frac{q}{p}}}{q}\right)\|u\|^q + \left(\frac{b}{p}\|u\|^{p+r-1} - \frac{\lambda\|f\|_\infty S_{1-r}^{\frac{r-1}{p}}}{1-r}\right)\|u\|^{1-r}. \end{aligned}$$

Thus, $J(u)$ is coercive and bounded below in $W_0^{1,p}(\Omega)$. □

Lemma 2.2 *If $q > p^2$ and $0 < \lambda < \max\{\lambda_1(a), \lambda_2\}$, then, for all $a > 0$,*
(i) *the submanifold $N_\lambda^0 = \emptyset$;*
(ii) *the submanifold $N_\lambda^\pm \neq \emptyset$.*

Proof (i) Suppose $N_\lambda^0 \neq \emptyset$. Then, for $u \in N_\lambda^0$, we have

$$\begin{aligned} (q+r-1)\|g\|_\infty S_q^{-\frac{q}{p}}\|u\|^q &\geq (q+r-1)\int_\Omega g|u|^q\,dx \\ &= a\big(p^2+r-1\big)\|u\|^{p^2} + b(p+r-1)\|u\|^p \\ &\geq \begin{cases} p\sqrt[p]{ab^{p-1}(p^2+r-1)}\|u\|^{2p-1}, \\ b(p+r-1)\|u\|^p, \end{cases} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \lambda(q+r-1)\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\|u\|^{1-r} &\geq \lambda(q+r-1)\int_\Omega f|u|^{1-r}\,dx \\ &= a\big(q-p^2\big)\|u\|^{p^2} + b(q-p)\|u\|^p \\ &\geq \begin{cases} p\sqrt[p]{ab^{p-1}(q-p^2)(\frac{q-p}{p-1})^{p-1}}\|u\|^{2p-1}, \\ b(q-p)\|u\|^p. \end{cases} \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), for all $u \in N_\lambda^0$, we have

$$\left(\frac{pS_q^{\frac{q}{p}}\sqrt[p]{ab^{p-1}(p^2+r-1)}}{(q+r-1)\|g\|_\infty}\right)^{\frac{1}{q-2p+1}} \leq \|u\| \leq \left(\frac{\lambda(q+r-1)\|f\|_\infty}{pS_{1-r}^{\frac{1-r}{p}}\sqrt[p]{ab^{p-1}(q-p^2)(\frac{q-p}{p-1})^{p-1}}}\right)^{\frac{1}{2p+r-2}}$$

and

$$\left(\frac{bS_q^{\frac{q}{p}}(p+r-1)}{(q+r-1)\|g\|_\infty}\right)^{\frac{1}{q-p}} \leq \|u\| \leq \left(\frac{\lambda(q+r-1)\|f\|_\infty}{bS_{1-r}^{\frac{1-r}{p}}(q-p)}\right)^{\frac{1}{p+r-1}}.$$

Hence, if N_λ^0 is nonempty, then the inequality $\lambda \geq \max\{\lambda_1(a), \lambda_2\}$ must hold.

(ii) Fix $u \in W_0^{1,p}(\Omega)$. Let

$$h_a(t) = at^{p^2-(1-r)}\|u\|^{p^2} + bt^{p-(1-r)}\|u\|^p - t^{q-(1-r)}\int_\Omega g|u|^q\,dx \quad \text{for } a, t \geq 0.$$

We see that $h_a(0) = 0$ and $h_a(t) \to -\infty$ as $t \to \infty$. Since $q > p^2$ and

$$\begin{aligned} h_a'(t) = t^{p+r-2}\Big(& a\big(p^2 + r - 1\big)t^{p^2-p}\|u\|^{p^2} + b(p + r - 1)\|u\|^p \\ & - (q + r - 1)t^{q-p}\int_\Omega g|u|^q\,dx\Big), \end{aligned}$$

there is a unique $t_{a,\max} > 0$ such that $h_a(t)$ reaches its maximum at $t_{a,\max}$, increasing for $t \in [0, t_{a,\max})$ and decreasing for $t \in (t_{a,\max}, \infty)$ with $\lim_{t\to\infty} h_a(t) = -\infty$. Clearly, if $tu \in N_\lambda$, then $tu \in N_\lambda^+$ (or N_λ^-) if and only if $h_a'(t) > 0$ (or < 0). Moreover,

$$t_{0,\max} = \left(\frac{b(p + r - 1)\|u\|^p}{(q + r - 1)\int_\Omega g|u|^q\,dx}\right)^{\frac{1}{q-p}}$$

and

$$\begin{aligned} & h_0(t_{0,\max}) \\ & \quad = b^{\frac{q+r-1}{q-p}}\left[\left(\frac{p + r - 1}{q + r - 1}\right)^{\frac{p+r-1}{q-p}} - \left(\frac{p + r - 1}{q + r - 1}\right)^{\frac{q+r-1}{q-p}}\right]\frac{\|u\|^{\frac{p(q+r-1)}{q-p}}}{(\int_\Omega g|u|^q\,dx)^{\frac{p+r-1}{q-p}}} \\ & \quad \geq \frac{b(q - p)}{q + r - 1}\left(\frac{bS_q^{\frac{q}{p}}(p + r - 1)}{(q + r - 1)\|g\|_\infty}\right)^{\frac{p+r-1}{q-p}}\|u\|^{1-r}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} h_a(0) = 0 < \lambda\int_\Omega f|u|^{1-r}\,dx &\leq \lambda\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\|u\|^{1-r} \\ &< \frac{b(q - p)}{q + r - 1}\left(\frac{bS_q^{\frac{q}{p}}(p + r - 1)}{(q + r - 1)\|g\|_\infty}\right)^{\frac{p+r-1}{q-p}}\|u\|^{1-r} \\ &\leq h_0(t_{0,\max}) < h_a(t_{a,\max}), \end{aligned} \tag{2.3}$$

there exist unique t^+ and t^- such that $0 < t^+ < t_{a,\max} < t^-$,

$$h_a\big(t^+\big) = \lambda\int_\Omega f|u|^{1-r}\,dx = h_a\big(t^-\big)$$

and

$$h_a'\big(t^+\big) > 0 > h_a'\big(t^-\big).$$

That is, $t^+u \in N_\lambda^+$ and $t^-u \in N_\lambda^-$. □

Lemma 2.3 (i) *If* $q = p^2$ *and* $a \geq \frac{1}{\Lambda}$, *then, for all* $\lambda > 0$, $N_\lambda^+ = N_\lambda \neq \emptyset$;
(ii) *if* $q = p^2, a < \frac{1}{\Lambda}$ *and* $0 < \lambda < \hat{\lambda}$, *then* $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ *and* $N_\lambda^\pm \neq \emptyset$.

Proof First, we show that $N_\lambda^+ = N_\lambda$.
Indeed, for all $u \in N_\lambda$, we have

$$\begin{aligned}
&a(p^2+r-1)\|u\|^{p^2} + b(p+r-1)\|u\|^p - (p^2+r-1)\int_\Omega g|u|^{p^2}\,dx \\
&\quad\geq \frac{(a\Lambda-1)(p^2+r-1)}{\Lambda}\|u\|^{p^2} + b(p+r-1)\|u\|^p > 0.
\end{aligned}$$

Therefore, $u \in N_\lambda^+$.
Next, we declare $N_\lambda^+ \neq \emptyset$.
Fix $u \in W_0^{1,p}(\Omega)$. Let

$$\bar{h}(t) = t^{p^2+r-1}\left(a\|u\|^{p^2} - \int_\Omega g|u|^{p^2}\,dx\right) + bt^{p+r-1}\|u\|^p \quad \text{for } a, t \geq 0.$$

Obviously, $\bar{h}(0) = 0$ and $\lim_{t\to\infty}\bar{h}(t) = \infty$. Since

$$\begin{aligned}
\bar{h}'(t) = {}&(p^2+r-1)t^{p^2+r-2}\left(a\|u\|^{p^2} - \int_\Omega g|u|^{p^2}\,dx\right) \\
&+ b(p+r-1)t^{p+r-2}\|u\|^p,
\end{aligned}$$

we can deduce that $\bar{h}(t)$ is increasing for $t \in [0,\infty)$. Thus, there is a unique $t^+ > 0$ such that $\bar{h}(t^+) = \lambda\int_\Omega f|u|^{1-r}\,dx$ and $\bar{h}'(t^+) > 0$. That is, $t^+u \in N_\lambda^+$.
(ii) The proof is similar to Lemma 2.2, we omit it here. □

We write $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and define

$$\alpha^+ = \inf_{u\in N_\lambda^+} J(u); \qquad \alpha^- = \inf_{u\in N_\lambda^-} J(u),$$

then we have the following lemma.

Lemma 2.4 *Suppose that* $q > p^2$ *and* $0 < \lambda < \lambda^*$, *then we have*
(i) $\alpha^+ < 0$;
(ii) $\alpha^- > C_0$, *for some* $C_0 > 0$.
In particular $\alpha^+ = \inf_{u\in N_\lambda} J(u)$.

Proof (i) Let $u \in N_\lambda^+$, it follows that

$$M(\|u\|^p)\|u\|^p - \lambda\int_\Omega f|u|^{1-r}\,dx - \int_\Omega g|u|^q\,dx = 0$$

and

$$\lambda(q+r-1)\int_\Omega f|u|^{1-r}\,dx > a(q-p^2)\|u\|^{p^2} + b(q-p)\|u\|^p.$$

Substituting this into $J(u)$, we have

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\big(\|u\|^p\big) - \frac{\lambda}{1-r}\int_\Omega f|u|^{1-r}\,dx - \frac{1}{q}\int_\Omega g|u|^q\,dx \\ &= \frac{1}{p}\widehat{M}\big(\|u\|^p\big) - \frac{1}{q}M\big(\|u\|^p\big)\|u\|^p - \lambda\frac{q+r-1}{q(1-r)}\int_\Omega f|u|^{1-r}\,dx \\ &< \frac{a(q-p^2)(1-r-p^2)}{p^2q(1-r)}\|u\|^{p^2} + \frac{b(q-p)(1-r-p)}{pq(1-r)}\|u\|^p < 0, \end{aligned}$$

and then $\alpha^+ < 0$.

(ii) Let $u \in N_\lambda^-$. We divide the proof into two cases.

Case (A): $\lambda^* = \frac{(1-r)\lambda_2}{p}$. Since $u \in N_\lambda^-$, and by the Sobolev inequality,

$$\begin{aligned} b(p+r-1)\|u\|^p &\le a\big(p^2+r-1\big)\|u\|^{p^2} + b(p+r-1)\|u\|^p \\ &< (q+r-1)S^{-\frac{q}{p}}\|g\|_\infty\|u\|^q, \end{aligned}$$

which implies

$$\|u\| > \left(\frac{bS_q^{\frac{q}{p}}(p+r-1)}{(q+r-1)\|g\|_\infty}\right)^{\frac{1}{q-p}} \quad \text{for all } u \in N_\lambda^-.$$

Hence,

$$\begin{aligned} J(u) &\ge \frac{a(q-p^2)\|u\|^{p^2}}{p^2q} + \frac{b(q-p)\|u\|^p}{pq} - \lambda\frac{q+r-1}{q(1-r)}\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\|u\|^{1-r} \\ &\ge \|u\|^{1-r}\left(\frac{b(q-p)}{pq}\|u\|^{p+r-1} - \lambda\frac{q+r-1}{q(1-r)}\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\right) \\ &> \left(\frac{bS_q^{\frac{q}{p}}(p+r-1)}{(q+r-1)\|g\|_\infty}\right)^{\frac{1-r}{q-p}}\left[\frac{b(q-p)}{pq}\left(\frac{bS_q^{\frac{q}{p}}(p+r-1)}{(q+r-1)\|g\|_\infty}\right)^{\frac{p+r-1}{q-p}}\right. \\ &\quad \left. - \lambda\frac{q+r-1}{q(1-r)}\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\right] = C_0. \end{aligned}$$

Thus, if $0 < \lambda < \frac{(1-r)\lambda_2}{p}$, then $\alpha^- > C_0 > 0$.

Case (B): $\lambda^* = \frac{(1-r)\lambda_1(a)}{p^{\frac{p+1}{p}}}$. By (2.1), one has

$$p\sqrt[p]{ab^{p-1}\big(p^2+r-1\big)}\|u\|^{2p-1} \le (q+r-1)\|g\|_\infty S_q^{-\frac{q}{p}}\|u\|^q,$$

which implies

$$\|u\| > \left(\frac{pS_q^{\frac{q}{p}}\sqrt[p]{ab^{p-1}(p^2+r-1)}}{(q+r-1)\|g\|_\infty}\right)^{\frac{1}{q-2p+1}} \quad \text{for all } u \in N_\lambda^-.$$

Repeating the argument of case (A), we conclude if $\lambda < \frac{(1-r)\lambda_1(a)}{p^{\frac{p+1}{p}}}$, then $\alpha^- > C_0$ for some $C_0 > 0$. □

Lemma 2.5 *Suppose that* $q = p^2, a < \frac{1}{\Lambda}$ *and* $0 < \lambda < \frac{1-r}{p}\hat{\lambda}$, *then we have*

(i) $\hat{\alpha}^+ < 0$;

(ii) $\hat{\alpha}^- > C_0$, *for some* $C_0 > 0$.

In particular $\hat{\alpha}^+ = \inf_{u\in N_\lambda} J(u)$.

Proof (i) Repeating the same argument of Lemma 2.4(i), we conclude that $\hat{\alpha}^+ < 0$.

(ii) Let $u \in N_\lambda^-$. By (1.6), one has

$$\begin{aligned} b(p+r-1)\|u\|^p &< \left(p^2+r-1\right)\left(\int_\Omega g|u|^{p^2}\,dx - a\|u\|^{p^2}\right) \\ &\le \frac{(1-a\Lambda)(p^2+r-1)}{\Lambda}\|u\|^{p^2}, \end{aligned}$$

which implies that

$$\|u\| > \left(\frac{b\Lambda(p+r-1)}{(1-a\Lambda)(p^2+r-1)}\right)^{\frac{1}{p^2-p}} \quad \text{for all } u \in N_\lambda^-. \tag{2.4}$$

Then we have

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\left(\|u\|^p\right) - \frac{\lambda}{1-r}\int_\Omega f|u|^{1-r}\,dx - \frac{1}{q}\int_\Omega g|u|^q\,dx \\ &\ge \|u\|^{1-r}\left(\frac{(p-1)b}{p^2}\|u\|^{p+r-1} - \lambda\frac{p^2+r-1}{p^2(1-r)}\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\right) \\ &> \left(\frac{b\Lambda(p+r-1)}{(1-a\Lambda)(p^2+r-1)}\right)^{\frac{1-r}{p^2-p}}\left[\frac{(p-1)b}{p^2}\left(\frac{b\Lambda(p+r-1)}{(1-a\Lambda)(p^2+r-1)}\right)^{\frac{p+r-1}{p^2-p}}\right. \\ &\quad \left. - \lambda\frac{p^2+r-1}{p^2(1-r)}\|f\|_\infty S_{1-r}^{-\frac{1-r}{p}}\right]. \end{aligned} \tag{2.5}$$

Thus, if $\lambda < \frac{1-r}{p}\hat{\lambda}$, then $\hat{\alpha}^- > C_0$ for some $C_0 > 0$. □

Lemma 2.6 *For each* $u \in N_\lambda^+$ *(resp.* $u \in N_\lambda^-$*), there exist* $\varepsilon > 0$ *and a continuous function* $f : B(0;\varepsilon) \subset W_0^{1,p}(\Omega) \to R^+$ *such that*

$$f(0) = 1, f(\omega) > 0, f(\omega)(u+\omega) \in N_\lambda^+ \left(\textit{resp. } u \in N_\lambda^-\right), \quad \textit{for all } \omega \in B(0;\varepsilon),$$

where $B(0;\varepsilon) = \{\omega \in W_0^{1,p}(\Omega) : \|\omega\| < \varepsilon\}$.

Proof For $u \in N_\lambda^+$, define $F : W_0^{1,p}(\Omega) \times R \to R$ as follows:

$$\begin{aligned} F(\omega,t) &= at^{p^2+r-1}\left(\int_\Omega \left|\nabla(u+\omega)\right|^p dx\right)^p + bt^{p+r-1}\int_\Omega \left|\nabla(u+\omega)\right|^p dx \\ &\quad - t^{q+r-1}\int_\Omega g|u|^q\,dx - \lambda\int_\Omega f|u|^{1-r}\,dx. \end{aligned}$$

Since $u \in N_\lambda^+$, it is easily seen that $F(0,1) = 0$ and $F_t(0,1) > 0$. Then by the implicit function theorem at the point $(0,1)$, we can see that there exist $\varepsilon > 0$ and a continuous function

$f : B(0;\varepsilon) \subset W_0^{1,p}(\Omega) \to R^+$ such that

$$f(0) = 1, f(\omega) > 0, f(\omega)(u + \omega) \in N_\lambda^+, \quad \text{for all } \omega \in B(0;\varepsilon).$$

In the same way, we can prove the case $u \in N_\lambda^-$. □

Remark 2.1 The proof of Lemma 2.6 is inspired by [11].

3 Proof of Theorem 1.1

By Lemma 2.1 and the Ekeland variational principle [17], there exists a minimizing sequence $\{u_n\} \subset N_\lambda^+$ such that

$$\text{(i)} \quad J(u_n) < \alpha^+ + \frac{1}{n};$$

$$\text{(ii)} \quad J(u) > J(u_n) - \frac{1}{n}\|u - u_n\|, \quad \forall u \in N_\lambda^+.$$

Note that $J(|u_n|) = J(u_n)$. We may assume that $u_n \geq 0$ in Ω. Using Lemma 2.1 again, we can see that there is a constant $C_1 > 0$ such that, for all $n \in N^+$, $\|u_n\| \leq C_1$. Thus, there exist a subsequence (still denoted by $\{u_n\}$) and u_λ^+ in $W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u_\lambda^+ \text{ weakly in } W_0^{1,p}(\Omega),$$
$$u_n \to u_\lambda^+ \text{ strongly in } L^{1-r}(\Omega),$$
$$u_n \to u_\lambda^+ \text{ strongly in } L^{q}(\Omega),$$
$$u_n \to u_\lambda^+ \text{ a.e. in } \Omega.$$

Now we conclude that $u_\lambda^+ \in N_\lambda^+$ is a positive solution of (1.1). The proof is inspired by Liu and Sun [11]. In order to prove the claim, we divide the arguments into six steps.

Step 1: u_λ^+ is not identically zero.

Indeed, it is an immediate conclusion of the following inequalities:

$$J(u_\lambda^+) \leq \underline{\lim}_{n\to\infty} J(u_n) = \alpha^+ < 0.$$

Step 2: There exists C_2 such that up to a subsequence we have

$$a(p^2 + r - 1)\|u_n\|^{p^2} + b(p + r - 1)\|u_n\|^p - (q + r - 1)\int_\Omega g|u_\lambda^+|^q\,dx > C_2. \tag{3.1}$$

In order to prove (3.1), it suffices to verify

$$a(p^2 + r - 1)\overline{\lim}_{n\to\infty}\|u_n\|^{p^2} + b(p + r - 1)\overline{\lim}_{n\to\infty}\|u_n\|^p > (q + r - 1)\int_\Omega g|u_\lambda^+|^q\,dx. \tag{3.2}$$

Since $u_n \in N_\lambda^+$,

$$a(p^2 + r - 1)\|u_n\|^{p^2} + b(p + r - 1)\|u_n\|^p \geq (q + r - 1)\int_\Omega g|u_\lambda^+|^q\,dx. \tag{3.3}$$

It follows that

$$a(p^2+r-1)\varlimsup_{n\to\infty}\|u_n\|^{p^2}+b(p+r-1)\varlimsup_{n\to\infty}\|u_n\|^p\geq(q+r-1)\int_\Omega g\left|u_\lambda^+\right|^q dx.$$

Suppose by contradiction that

$$a(p^2+r-1)\varlimsup_{n\to\infty}\|u_n\|^{p^2}+b(p+r-1)\varlimsup_{n\to\infty}\|u_n\|^p=(q+r-1)\int_\Omega g\left|u_\lambda^+\right|^q dx. \tag{3.4}$$

Then, from (3.3) and (3.4), one has

$$a(p^2+r-1)\lim_{n\to\infty}\|u_n\|^{p^2}+b(p+r-1)\lim_{n\to\infty}\|u_n\|^p=(q+r-1)\int_\Omega g\left|u_\lambda^+\right|^q dx.$$

Thus $\|u_n\|^p$ converges to a positive number A that satisfies

$$a(p^2+r-1)A^p+b(p+r-1)A=(q+r-1)\int_\Omega g\left|u_\lambda^+\right|^q dx$$

and

$$a(q-p^2)A^p+b(q-p)A=\lambda(q+r-1)\int_\Omega f\left|u_\lambda^+\right|^{1-r} dx.$$

On the other hand, by (2.3), we have

$$\begin{aligned}
0&\leq(\lambda^*-\lambda)\int_\Omega f|u_n|^{1-r}dx\\
&< b^{\frac{q+r-1}{q-p}}\left(\frac{p+r-1}{q+r-1}\right)^{\frac{p+r-1}{q-p}}\left(\frac{q-p}{q+r-1}\right)\frac{\|u_n\|^{\frac{p(q+r-1)}{q-p}}}{(\int_\Omega g|u_n|^q dx)^{\frac{p+r-1}{q-p}}}-\lambda\int_\Omega f|u_n|^{1-r}dx\\
&\to b^{\frac{q+r-1}{q-p}}\left(\frac{p+r-1}{q+r-1}\right)^{\frac{p+r-1}{q-p}}\left(\frac{q-p}{q+r-1}\right)\frac{A^{\frac{q+r-1}{q-p}}}{(\frac{a(p^2+r-1)A^p+b(p+r-1)A}{q+r-1})^{\frac{p+r-1}{q-p}}}\\
&\quad-\frac{a(q-p^2)A^p+b(q-p)A}{q+r-1}\\
&< b^{\frac{q+r-1}{q-p}}\left(\frac{p+r-1}{q+r-1}\right)^{\frac{p+r-1}{q-p}}\left(\frac{q-p}{q+r-1}\right)\frac{A^{\frac{q+r-1}{q-p}}}{(\frac{b(p+r-1)A}{q+r-1})^{\frac{p+r-1}{q-p}}}\\
&\quad-\frac{a(q-p^2)A^p+b(q-p)A}{q+r-1}\\
&=-\frac{a(q-p^2)}{q+r-1}A^p<0,
\end{aligned}$$

which is impossible. Hence, (3.1) and (3.2) must hold.

Step 3: For nonnegative $\varphi\in W_0^{1,p}(\Omega)$ and $t>0$ small, we can find $f_n(t):=f_n(t\varphi)$ such that $f_n(0)=1$ and $f_n(t)(u_n+t\varphi)\in N_\lambda^+$ for each $u_n\in N_\lambda^+$ by Lemma 2.6. $f'_{n+}(0)\in[-\infty,\infty]$ is denoted by the right derivative of $f_n(t)$ at zero. We claim that there exists $C_3>0$ such that

$f'_{n+}(0) > -C_3$ for all $n \in N^+$. Since $u_n, f_n(t)(u_n + t\varphi) \in N_\lambda$, we deduce that

$$0 = a\|u_n\|^{p^2} + b\|u_n\|^p - \lambda \int_\Omega f|u_n|^{1-r}\,dx - \int_\Omega g|u_n|^q\,dx$$

and

$$\begin{aligned}0 &= af_n^{p^2}(t)\|u_n + t\varphi\|^{p^2} + bf_n^p(t)\|u_n + t\varphi\|^p - \lambda f_n^{1-r}(t)\int_\Omega f|u_n + t\varphi|^{1-r}\,dx \\ &\quad - f_n^q(t)\int_\Omega g|u_n + t\varphi|^q\,dx.\end{aligned}$$

Thus

$$\begin{aligned}0 &= a\big(f_n^{p^2}(t) - 1\big)\|u_n + t\varphi\|^{p^2} + a\big(\|u_n + t\varphi\|^{p^2} - \|u_n\|^{p^2}\big) \\ &\quad + b\big(f_n^p(t) - 1\big)\|u_n + t\varphi\|^p + b\big(\|u_n + t\varphi\|^p - \|u_n\|^p\big) \\ &\quad - \lambda\big(f_n^{1-r}(t) - 1\big)\int_\Omega f|u_n + t\varphi|^{1-r}\,dx - \lambda\int_\Omega f\big(|u_n + t\varphi|^{1-r} - |u_n|^{1-r}\big)\,dx \\ &\quad - \big(f_n^q(t) - 1\big)\int_\Omega g|u_n + t\varphi|^q\,dx - \int_\Omega g\big(|u_n + t\varphi|^q - |u_n|^q\big)\,dx \\ &\le a\big(f_n^{p^2}(t) - 1\big)\|u_n + t\varphi\|^{p^2} + a\big(\|u_n + t\varphi\|^{p^2} - \|u_n\|^{p^2}\big) \\ &\quad + b\big(f_n^p(t) - 1\big)\|u_n + t\varphi\|^p + b\big(\|u_n + t\varphi\|^p - \|u_n\|^p\big) \\ &\quad - \lambda\big(f_n^{1-r}(t) - 1\big)\int_\Omega f|u_n + t\varphi|^{1-r}\,dx - \big(f_n^q(t) - 1\big)\int_\Omega g|u_n + t\varphi|^q\,dx.\end{aligned}$$

Then, dividing by $t > 0$ and letting $t \to 0$, we have

$$\begin{aligned}0 &\le ap^2\|u_n\|^{p^2} f'_{n+}(0) + ap^2\|u_n\|^{p^2-p}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx \\ &\quad + bp\|u_n\|^p f'_{n+}(0) + bp\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx \\ &\quad - \lambda(1-r)f'_{n+}(0)\int_\Omega f|u_n|^{1-r}\,dx - qf'_{n+}(0)\int_\Omega g|u_n|^q\,dx \\ &= f'_{n+}(0)\bigg(ap^2\|u_n\|^{p^2} + bp\|u_n\|^p - \lambda(1-r)\int_\Omega f|u_n|^{1-r}\,dx - q\int_\Omega g|u_n|^q\,dx\bigg) \\ &\quad + ap^2\|u_n\|^{p^2-p}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx + bp\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx \\ &= f'_{n+}(0)\bigg(a\big(p^2 + r - 1\big)\|u_n\|^{p^2} + b(p + r - 1)\|u_n\|^p - (q + r - 1)\int_\Omega g|u_n|^q\,dx\bigg) \\ &\quad + ap^2\|u_n\|^{p^2-p}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx + bp\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx.\end{aligned}$$

One deduces from (3.1)

$$f'_{n+}(0) \ge -\frac{ap^2\|u_n\|^{p^2-p}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx + bp\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx}{a(p^2 + r - 1)\|u_n\|^{p^2} + b(p + r - 1)\|u_n\|^p - (q + r - 1)\int_\Omega g|u_n|^q\,dx}.$$

Therefore, by the boundedness of $\{u_n\}$, we conclude that $\{f'_{n+}(0)\}$ is bounded from below.

Step 4: Choose n^* large enough such that $\frac{(1-r)C_1}{n} < \frac{C_2}{2}$ for all $n > n^*$. Then we claim that there exists C_4 such that $f'_{n+}(0) < C_4$ for each $n > n^*$. Without loss of generality, we may suppose $f'_{n+}(0) \geq 0$. Then from condition (ii), we have

$$\begin{aligned}
&\bigl|f_n(t) - 1\bigr| \frac{\|u_n\|}{n} + \bigl|tf_n(t)\bigr| \frac{\|\varphi\|}{n} \\
&\geq \frac{1}{n} \bigl\|f_n(t)(u_n + t\varphi) - u_n\bigr\| \\
&\geq J(u_n) - J\bigl(f_n(t)(u_n + t\varphi)\bigr) \\
&= \frac{a(p^2 + r - 1)}{p^2(1-r)} \bigl(f_n^{p^2}(t) - 1\bigr) \|u_n + t\varphi\|^{p^2} + \frac{a(p^2 + r - 1)}{p^2(1-r)} \bigl(\|u_n + t\varphi\|^{p^2} - \|u_n\|^{p^2}\bigr) \\
&\quad + \frac{b(p + r - 1)}{p(1-r)} \bigl(f_n^{p}(t) - 1\bigr) \|u_n + t\varphi\|^{p} + \frac{b(p + r - 1)}{p(1-r)} \bigl(\|u_n + t\varphi\|^{p} - \|u_n\|^{p}\bigr) \\
&\quad - \frac{q + r - 1}{q(1-r)} \bigl(f_n^{q}(t) - 1\bigr) \int_\Omega g|u_n|^q \, dx - \frac{q + r - 1}{q(1-r)} f_n^q(t) \int_\Omega g\bigl(|u_n + t\varphi|^q - |u_n|^q\bigr) \, dx.
\end{aligned}$$

Then, dividing by $t > 0$ and letting $t \to 0$, we deduce

$$\begin{aligned}
&f'_{n+}(0) \frac{\|u_n\|}{n} + \frac{\|\varphi\|}{n} \\
&\geq \frac{a(p^2 + r - 1)}{1-r} f'_{n+}(0) \|u_n\|^{p^2} + \frac{a(p^2 + r - 1)}{1-r} \|u_n\|^{p^2 - p} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi \, dx \\
&\quad + \frac{b(p + r - 1)}{1-r} f'_{n+}(0) \|u_n\|^{p} + \frac{b(p + r - 1)}{1-r} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi \, dx \\
&\quad - \frac{q + r - 1}{1-r} f'_{n+}(0) \int_\Omega g|u_n|^q \, dx - \frac{q + r - 1}{1-r} \int_\Omega g|u_n|^{q-1} \varphi \, dx.
\end{aligned} \tag{3.5}$$

From (3.5) and the choice of n^*, we have

$$\begin{aligned}
\frac{\|\varphi\|}{n} \geq{}& \frac{C_2}{2(1-r)} f'_{n+}(0) + \frac{a(p^2 + r - 1)}{1-r} \|u_n\|^{p^2 - p} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi \, dx \\
&+ \frac{b(p + r - 1)}{1-r} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi \, dx - \frac{q + r - 1}{1-r} \int_\Omega g|u_n|^{q-1} \varphi \, dx.
\end{aligned}$$

Namely,

$$\begin{aligned}
\frac{C_2}{2(1-r)} f'_{n+}(0) \leq{}& \frac{\|\varphi\|}{n} - \frac{a(p^2 + r - 1)}{1-r} \|u_n\|^{p^2 - p} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi \, dx \\
&- \frac{b(p + r - 1)}{1-r} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi \, dx + \frac{q + r - 1}{1-r} \int_\Omega g|u_n|^{q-1} \varphi \, dx.
\end{aligned}$$

Therefore, by the boundedness of $\{u_n\}$, we conclude $\{f'_{n+}(0)\}_{n>n^*}$ is bounded from above.

Step 5: $u_\lambda^+ > 0$ a.e. in Ω and for nonnegative $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\bigl(a\bigl\|u_\lambda^+\bigr\|^{p^2 - p} + b\bigr) \int_\Omega \bigl|\nabla u_\lambda^+\bigr|^{p-2} \nabla u_\lambda^+ \nabla\varphi \, dx - \lambda \int_\Omega f\bigl|u_\lambda^+\bigr|^{-r} \varphi \, dx - \int_\Omega g\bigl|u_\lambda^+\bigr|^{q-1} \varphi \, dx \\
&\geq 0.
\end{aligned} \tag{3.6}$$

Similar to the argument in Step 4, one can obtain

$$\begin{aligned}
&f'_{n+}(0)\frac{\|u_n\|}{n}+\frac{\|\varphi\|}{n}\\
&\quad\geq -f'_{n+}(0)\left(a\|u_n\|^{p^2}+b\|u_n\|^{p}-\int_\Omega g|u_n|^q\,dx-\lambda\int_\Omega f|u_n|^{1-r}\varphi\,dx\right)\\
&\qquad -a\|u_n\|^{p^2-p}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx-b\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx\\
&\qquad +\int_\Omega g|u_n|^{q-1}\varphi\,dx+\varliminf_{t\to0^+}\frac{\lambda}{1-r}\int_\Omega\frac{f(|u_n+t\varphi|^{1-r}-|u_n|^{1-r})}{t}\,dx\\
&\quad= -a\|u_n\|^{p^2-p}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx-b\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx\\
&\qquad +\int_\Omega g|u_n|^{q-1}\varphi\,dx+\varliminf_{t\to0^+}\frac{\lambda}{1-r}\int_\Omega\frac{f(|u_n+t\varphi|^{1-r}-|u_n|^{1-r})}{t}\,dx.
\end{aligned}\tag{3.7}$$

Since $f(|u_n+t\varphi|^{1-r}-|u_n|^{1-r})\geq 0, \forall t>0$, by Fatou's lemma, we obtain

$$\int_\Omega f|u_n|^{-r}\varphi\,dx\leq\varliminf_{t\to0^+}\frac{1}{1-r}\int_\Omega\frac{f(|u_n+t\varphi|^{1-r}-|u_n|^{1-r})}{t}\,dx.\tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}
&\lambda\int_\Omega f|u_n|^{-r}\varphi\,dx\\
&\quad\leq\frac{1}{n}\big(f'_{n+}(0)\|u_n\|+\|\varphi\|\big)+a\|u_n\|^{p^2-p}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx\\
&\qquad +b\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx-\int_\Omega g|u_n|^{q-1}\varphi\,dx\\
&\quad\leq\frac{C_1\cdot\max\{C_3,C_4\}+\|\varphi\|}{n}+a\|u_n\|^{p^2-p}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx\\
&\qquad +b\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx-\int_\Omega g|u_n|^{q-1}\varphi\,dx,
\end{aligned}$$

for all $n>n^*$.

Passing to the limit as $n\to\infty$, one has

$$\begin{aligned}
\varliminf_{n\to\infty}\lambda\int_\Omega f|u_n|^{-r}\varphi\,dx\leq a\varliminf_{n\to\infty}\|u_n\|^{p^2-p}\int_\Omega\big|\nabla u_\lambda^+\big|^{p-2}\nabla u_\lambda^+\nabla\varphi\,dx\\
+b\int_\Omega\big|\nabla u_\lambda^+\big|^{p-2}\nabla u_\lambda^+\nabla\varphi\,dx-\int_\Omega g\big|u_\lambda^+\big|^{q-1}\varphi\,dx.
\end{aligned}$$

Then using Fatou's lemma again, we infer that

$$\begin{aligned}
&\lambda\int_\Omega f\big|u_\lambda^+\big|^{-r}\varphi\,dx\\
&\quad\leq a\varliminf_{n\to\infty}\|u_n\|^{p^2-p}\int_\Omega\big|\nabla u_\lambda^+\big|^{p-2}\nabla u_\lambda^+\nabla\varphi\,dx\\
&\qquad +b\int_\Omega\big|\nabla u_\lambda^+\big|^{p-2}\nabla u_\lambda^+\nabla\varphi\,dx-\int_\Omega g\big|u_\lambda^+\big|^{q-1}\varphi\,dx.
\end{aligned}\tag{3.9}$$

Since $u_n \to u_\lambda^+$ a.e. in Ω, we get $u_\lambda^+ \geq 0$ a.e. in Ω. Thus, one infers from (3.9) that

$$\lambda \int_\Omega f \left|u_\lambda^+\right|^{1-r} dx \leq a \varliminf_{n\to\infty} \|u_n\|^{p^2-p} \left\|u_\lambda^+\right\|^p + b\left\|u_\lambda^+\right\|^p - \int_\Omega g\left|u_\lambda^+\right|^q dx. \tag{3.10}$$

On the other hand

$$\begin{aligned} a \varliminf_{n\to\infty} \|u_n\|^{p^2-p} \left\|u_\lambda^+\right\|^p + b\left\|u_\lambda^+\right\|^p &\leq a \varlimsup_{n\to\infty} \|u_n\|^{p^2} + b \varlimsup_{n\to\infty} \|u_n\|^p \\ &= \lambda \int_\Omega f\left|u_\lambda^+\right|^{1-r} dx + \int_\Omega g\left|u_\lambda^+\right|^q dx. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\varliminf_{n\to\infty} \|u_n\|^p = \varlimsup_{n\to\infty} \|u_n\|^p = \left\|u_\lambda^+\right\|^p. \tag{3.12}$$

Thus, (3.6) can be obtained by inserting (3.12) into (3.9). Moreover, from (3.6), one has

$$\int_\Omega \left|\nabla u_\lambda^+\right|^{p-2} \nabla u_\lambda^+ \nabla\varphi \, dx \geq 0, \quad \forall \varphi \in W_0^{1,p}(\Omega), \varphi \geq 0.$$

Therefore, using the strong maximum principle for weak solutions (see [18]), we obtain $u_\lambda^+ > 0$ a.e. in Ω.

Step 6: u_λ^+ is a weak solution of (1.1), and $u_\lambda^+ \in N_\lambda^+$. By (3.12), we have $u_n \to u_\lambda^+$ strongly in $W_0^{1,p}(\Omega)$, and so $u_\lambda^+ \in N_\lambda^+$. Assume $\phi \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$, define $\Psi \in W_0^{1,p}(\Omega)$ by $\Psi := (u_\lambda^+ + \varepsilon\phi)^+$. Then from Step 5 it follows

$$\begin{aligned} 0 &\leq \int_\Omega \left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\Psi - \lambda f\left|u_\lambda^+\right|^{-r}\Psi - g\left|u_\lambda^+\right|^{q-1}\Psi\right] dx \\ &= \int_{[u_\lambda^+ + \varepsilon\phi > 0]} \left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\left(u_\lambda^+ + \varepsilon\phi\right)\right. \\ &\quad \left. - \lambda f\left|u_\lambda^+\right|^{-r}\left(u_\lambda^+ + \varepsilon\phi\right) - g\left|u_\lambda^+\right|^{q-1}\left(u_\lambda^+ + \varepsilon\phi\right)\right] dx \\ &= \left(\int_\Omega - \int_{[u_\lambda^+ + \varepsilon\phi \leq 0]}\right)\left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\left(u_\lambda^+ + \varepsilon\phi\right)\right. \\ &\quad \left. - \lambda f\left|u_\lambda^+\right|^{-r}\left(u_\lambda^+ + \varepsilon\phi\right) - g\left|u_\lambda^+\right|^{q-1}\left(u_\lambda^+ + \varepsilon\phi\right)\right] dx \\ &= a\left\|u_\lambda^+\right\|^{p^2} + b\left\|u_\lambda^+\right\|^p - \lambda\int_\Omega f\left|u_\lambda^+\right|^{1-r} dx - \int_\Omega g\left|u_\lambda^+\right|^q dx \\ &\quad + \varepsilon\int_\Omega \left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\phi - \lambda f\left|u_\lambda^+\right|^{-r}\phi - g\left|u_\lambda^+\right|^{q-1}\phi\right] dx \\ &\quad - \int_{[u_\lambda^+ + \varepsilon\phi \leq 0]} \left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\left(u_\lambda^+ + \varepsilon\phi\right)\right. \\ &\quad \left. - \lambda f\left|u_\lambda^+\right|^{-r}\left(u_\lambda^+ + \varepsilon\phi\right) - g\left|u_\lambda^+\right|^{q-1}\left(u_\lambda^+ + \varepsilon\phi\right)\right] dx \\ &= \varepsilon\int_\Omega \left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\phi - \lambda f\left|u_\lambda^+\right|^{-r}\phi - g\left|u_\lambda^+\right|^{q-1}\phi\right] dx \\ &\quad - \int_{[u_\lambda^+ + \varepsilon\phi \leq 0]} \left[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+ \nabla\left(u_\lambda^+ + \varepsilon\phi\right)\right. \end{aligned}$$

$$
\begin{aligned}
&\quad - \lambda f\left|u_\lambda^+\right|^{-r}\left(u_\lambda^+ + \varepsilon\phi\right) - g\left|u_\lambda^+\right|^{q-1}\left(u_\lambda^+ + \varepsilon\phi\right)\big]\,dx \\
&\le \varepsilon\int_\Omega\big[\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+\nabla\phi - \lambda f\left|u_\lambda^+\right|^{-r}\phi - g\left|u_\lambda^+\right|^{q-1}\phi\big]\,dx \\
&\quad - \varepsilon\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\int_{[u_\lambda^+ + \varepsilon\phi\le 0]}\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+\nabla\phi\,dx.
\end{aligned}
$$

Since the measure of the domain of integration $[u_\lambda^+ + \varepsilon\phi \le 0]$ tends to zero as $\varepsilon \to 0$, it follows $\int_{[u_\lambda^+ + \varepsilon\phi\le 0]} |\nabla u_\lambda^+|^{p-2}\nabla u_\lambda^+\nabla\phi\,dx \to 0$. Dividing by ε and letting $\varepsilon \to 0$, we have

$$
\left(a\left\|u_\lambda^+\right\|^{p^2-p} + b\right)\int_\Omega\left|\nabla u_\lambda^+\right|^{p-2}\nabla u_\lambda^+\nabla\phi\,dx - \lambda\int_\Omega f\left|u_\lambda^+\right|^{-r}\phi\,dx - \int_\Omega g\left|u_\lambda^+\right|^{q-1}\phi\,dx \ge 0.
$$

Notice that ϕ is arbitrary, the inequality also holds for $-\phi$, so it follows that u_λ^+ is a weak solution of (1.1). Moreover, from (3.2) and (3.12), we deduce that $u_\lambda^+ \in N_\lambda^+$.

A similar argument shows that there exists another solution $u_\lambda^- \in N_\lambda^-$.

4 Proof of Theorem 1.2

(i) By Lemma 2.3(i), we write $N_\lambda = N_\lambda^+$ and define

$$
\theta^+ = \inf_{u\in N_\lambda^+} J(u).
$$

Similar to Lemma 2.5(i), we have $\theta^+ < 0$. Applying Lemma 2.2(i) and the Ekeland variational principle, we see that there exists a minimizing sequence $\{u_n\}$ for $J(u)$ in N_λ^+ such that

$$
\begin{aligned}
&\text{(i)}\quad J(u_n) < \theta^+ + \frac{1}{n}; \\
&\text{(ii)}\quad J(u) > J(u_n) - \frac{1}{n}\|u - u_n\|, \quad \forall u \in N_\lambda^+.
\end{aligned}
$$

Repeating the same argument as Theorem 1.1, we can see that $u_\lambda \in N_\lambda^+$ is a positive solution of the problem (1.1).

(ii) Similar to the proof of Theorem 1.1, we know that the problem (1.1) has at least two positive solutions $u_\lambda^+ \in N_\lambda^+$ and $u_\lambda^- \in N_\lambda^-$. Moreover, combining (2.4) with (2.5), we have

$$
\lim_{a\to\frac{1}{\Lambda}^-}\left\|u_\lambda^-\right\| = \infty
$$

and

$$
\lim_{a\to\frac{1}{\Lambda}^-}\inf_{u\in N_\lambda^-} J(u) = \infty.
$$

This completes the proof of Theorem 1.2.

Remark 4.1 The results of Theorems 1.1 and 1.2 extend the results of [11, 12]. The results from the cited work correspond to our results for the case $p = 2$ and $N = 3$. From these two references, we obtained the motivation for this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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