Li and Ge *Boundary Value Problems* (2017) 2017:61
DOI 10.1186/s13661-017-0794-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

New positive periodic solutions to singular Rayleigh prescribed mean curvature equations

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Abstract
This paper is concerned with the existence of positive periodic solutions for the prescribed mean curvature Rayleigh equations with a singularity. Our results are based on the Manásevich-Mawhin continuation theorem. The results to be obtained here extend the existing ones in the literature. Moreover, an example is given to illustrate the applicability of our results.

MSC: 34K13; 34C25

Keywords: positive periodic solution; Manásevich-Mawhin continuation theorem; prescribed mean curvature Rayleigh equation; singularity

1 Introduction

The Rayleigh equation arises from many applied fields, such as the physics, mechanics, and engineering technique fields. So, it is meaningful and necessary to study the periodic solutions for the Rayleigh equation. In 1977, Gaines and Mawhin [1] discussed the existence of solutions for the following Rayleigh equation:

$$u''(t) + f\bigl(u'(t)\bigr) + g\bigl(t, u(t)\bigr) = 0.$$

By applying the continuation theorems, Gaines and Mawhin proved that the Rayleigh equation can support periodic solutions.

In recent years, the prescribed mean curvature equation and its modified forms have been studied widely since they arise from some certain problems associated with differential geometry and physics such as combustible gas dynamics; see [2–5] and the references therein. Due to the wide range of application background of the prescribed mean curvature equations, many researchers have worked on the existence of periodic solutions for the prescribed mean curvature equations. For the related papers, we refer the reader to [6–12].

On this basis of work of Gaines and Mawhin [1], some researchers discussed the existence of periodic solutions to some types of prescribed mean curvature Rayleigh equations; see [13, 14] and the references therein. For example, by using Mawhin's continuation theorem in the coincidence degree theory, Li *et al.* [14] considered the periodic solutions for the following prescribed mean curvature Rayleigh equation:

$$\left(\frac{x'(t)}{\sqrt{1+(x'(t))^2}}\right)' + f\bigl(t,x'(t)\bigr) + g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr) = e(t), \tag{1.1}$$

where τ, $e \in C(\mathbb{R},\mathbb{R})$ are T-periodic, and f, $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are T-periodic in the first argument, $T > 0$ is a constant.

Singular equations appear in a great deal of physical models and play an important role in the differential equations. Recently, Lu and Kong in [15] extended the prescribed mean curvature Liénard equations to the singular case and studied the positive periodic solutions for the following prescribed mean curvature Liénard equation with a singularity:

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f\bigl(u(t)\bigr)u'(t) + g\bigl(u(t-\sigma)\bigr) = e(t), \tag{1.2}$$

where $\sigma = kT$, $k = 1,2,\ldots,n$, f and $g : (0,+\infty) \to \mathbb{R}$ are continuous functions, g can be singular at $u = 0$, i.e., $g(u)$ can be unbounded as $u \to 0^+$. $e(t)$ is T-periodic with $\int_0^T e(t)\,dt = 0$. In order to establish the existence result of positive periodic solutions, the authors gave the following conditions:

$[A_1]$ There exist positive constants D_1 and D_2 with $D_1 < D_2$ such that
(1) For each positive continuous T-periodic function $x(t)$ satisfying $\int_0^T g(x(t))\,dt = 0$, there exists a positive point $\tau \in [0,T]$ such that

$$D_1 \le x(\tau) \le D_2;$$

(2) $g(x) < 0$ for all $x \in (0,D_1)$ and $g(x) > 0$ for all $x > D_2$.

$[A_2]$ $g(x(t)) = g_1(x(t)) + g_0(x(t))$, where $g_1 : (0,+\infty) \to \mathbb{R}$ is a continuous function and
(1) There exist positive constants m_0 and m_1 such that $g(x) \le m_0 x + m_1$;
(2) $\int_0^1 g_0(x)\,dx = -\infty$.

$[A_3]$ There exist positive constants γ, c_0, c_1 such that $\gamma < f(x) \le c_0|x| + c_1$.

By applying the Manásevich-Mawhin continuation theorem, the authors proved that equation (1.2) has at least one positive T-periodic solution.

Based on Lu and Kong in [15], Chen and Kong [16] further study the existence of positive periodic solutions for a prescribed mean curvature p-Laplacian equation with a singularity of repulsive type and a time-varying delay

$$\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+(x'(t))^2}}\right)\right)' + \beta x'(t) + g\bigl(t,x(t),x\bigl(t-\tau(t)\bigr)\bigr) = p(t),$$

where $g : [0,T]\times(0,+\infty)\times(0,+\infty) \to \mathbb{R}$ is a continuous function. g can be singular at $u = 0$, i.e., g can be unbounded as $u \to 0^+$. $\tau, p \in (\mathbb{R},\mathbb{R})$ are T-periodic with $\int_0^T p(t)\,dt = 0$, β is a constant.

Compared with the results in the literature, the prescribed mean curvature Rayleigh equations with singular effects have been scarcely studied.

Inspired by the above facts, in this paper, we further consider the following prescribed mean curvature Rayleigh equations with a singularity:

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f\big(t,u'(t)\big) + g\big(u(t-\tau)\big) = e(t), \tag{1.3}$$

where $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ is a T-periodic function about t and $f(t,0)=0$, $g:(0,+\infty)\to\mathbb{R}$ is a continuous function and has a strong singularity at the origin:

$$\lim_{x\to 0^+}\int_x^1 g(s)\,ds = +\infty. \tag{1.4}$$

$e \in C([0,T],\mathbb{R})$ is a T-periodic function, $0 \le \tau < T$ and τ is a constant. By means of the Manásevich-Mawhin continuation theorem, we prove that (1.3) has at least one positive T-periodic solution.

Remark 1.1 The theorem and methods used to obtain the periodic solutions to (1.1) in [14] and [13] can be applied to the (1.3) if there is no singularity in (1.3). So, we extend the prescribed mean curvature Rayleigh equations to the singular case.

Remark 1.2 If $x \in C^1(\mathbb{R},\mathbb{R})$ with T-periodic, then $f(x)x'$ in equation (1.2) satisfies $\int_0^T f(x(t))x'(t)\,dt = 0$, which is crucial to obtain the priori bounds of T-periodic solutions for equation (1.2). However, the first order derivative term in equation (1.3) is $f(t,x')$. Generally, $\int_0^T f(t,x'(t))\,dt = 0$ does not hold. For example, let us define

$$f\big(t,x'(t)\big) = \big(3-\sin^2 8t\big)x'(t),$$

then it is easy to see that $\int_0^T (3-\sin^2 8t)x'(t)\,dt \neq 0$ for some $x \in C^1(\mathbb{R},\mathbb{R})$. This implies that our method to complete estimate the priori bounds for all T-periodic solutions to equation (1.3) is different from the corresponding ones.

Remark 1.3 From [15] and [16], the conditions composed on $e(t)$ and $p(t)$ are $\int_0^T e(t)\,dt = 0$ and $\int_0^T p(t)\,dt = 0$. But, in this paper, it is unnecessary. For example, let us define

$$e(t) = \frac{e^{\cos^2 8t}}{12},$$

then it is easy to see that $\int_0^T \frac{e^{\cos^2 8t}}{12}\,dt \neq 0$. So, our results can be more general.

2 Preliminary

Throughout this paper, for any T-periodic continuous function $u(t)$, we always use the notations as follows:

$$\|u\|_2 = \left(\int_0^T \big|u(t)\big|^2\,dt\right)^{1/2} \quad \text{and} \quad \|u\|_0 = \max_{t\in[0,T]}\big|u(t)\big|.$$

In order to use Lemma 2.1, let us consider the problem

$$\begin{cases} u'(t) = \phi(v(t)) = \frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -f(t,\phi(v(t))) - g(u(t-\tau)) + e(t). \end{cases} \tag{2.1}$$

Obviously, if $(u(t),v(t))^\top$ is a solution of (2.1), then $u(t)$ is a solution of (1.3).

Lemma 2.1 ([17]) *Assume that there exist positive constants E_1, E_2, E_3 with $E_1 < E_2$ such that the following conditions hold*:

(1) *for each $\lambda \in (0,1]$, each possible positive T-periodic solution $x = (u,v)^\top$ to the system*

$$\begin{cases} u'(t) = \lambda\phi(v(t)) = \lambda\frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -\lambda f(t,\phi(v(t))) - \lambda g(u(t-\tau)) + \lambda e(t), \end{cases}$$

satisfies the inequalities $E_1 < u(t) < E_2$ and $\|u'\|_0 < E_3$ for all $t \in [0,T]$.

(2) *Each possible solution C to the equation $g(C) - \frac{1}{T}\int_0^T e(t)\,dt = 0$ satisfies*

$$E_1 < C < E_2.$$

(3) *We have $(g(E_1) - \frac{1}{T}\int_0^T e(t)\,dt)(g(E_2) - \frac{1}{T}\int_0^T e(t)\,dt) < 0$.*

Then equation (1.3) *has at least one positive T-periodic solution.*

Lemma 2.2 ([18]) *Let $u(t)$ be a continuously differentiable T-periodic function. Then, for any $t_0 \in [0,T]$,*

$$\left(\int_0^T |u(t)|^2\,dt\right)^{1/2} \le \frac{T}{\pi}\left(\int_0^T |u'(t)|^2\,dt\right)^{1/2} + \sqrt{T}|u(t_0)|.$$

For the sake of convenience, we list the following assumptions:

(H1) There exist constants $0 < d_1 < d_2$ such that

$$g(x) - e(t) > 0, \quad \forall x \in (0,d_1), \quad \text{and} \quad g(x) - e(t) < 0, \quad \forall x \in (d_2,+\infty), t \in [0,T].$$

(H2) There exist positive constants m_0 and m_1 such that

$$|g(x)| \le m_0 x + m_1, \quad \forall x \in (0,+\infty).$$

(H3) There exists a positive constant a such that

$$f(t,x)x \ge a|x|^2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

(H4) There exist positive constants β and γ such that

$$f(t,x) \le \beta|x| + \gamma, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

3 Main results

In this section, we will consider the existence of positive periodic solution for (1.3) with a singularity.

First of all, we embed equation (1.3) into the following equation family with a parameter $\lambda \in (0,1]$:

$$\begin{cases} u'(t) = \lambda\phi(v(t)) = \lambda\frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -\lambda f(t,\phi(v(t))) - \lambda g(u(t-\tau)) + \lambda e(t). \end{cases} \tag{3.1}$$

Theorem 3.1 *Suppose the conditions* (H1)-(H4) *hold*, $a\pi > m_0 T$ *and*

$$\beta\sqrt{T}A_0 + \big(m_0 A_1 + m_1 + \|e\|_0 + \gamma\big)T < 1,$$

where $A_0 = \frac{\pi\sqrt{T}(m_0 d_2 + m_1 + \|e\|_0)}{a\pi - m_0 T}$, $A_1 = d_2 + A_0$, *then there exist positive constants* A_1, A_2, A_3 *and* A_4, *which are independent of* λ *such that*

$$A_2 \le u(t) \le A_1, \qquad \|u'\|_0 \le A_3, \qquad \|v\|_0 \le A_4,$$

where $x = (u, v)^\top$ *is any solution to equation* (3.1), $\lambda \in (0, 1]$.

Proof Let $\bar{t}$, $\underline{t}$, respectively, be the global maximum point and global minimum point $u(t)$ on $[0, T]$; then $u'(\bar{t}) = 0$ and $u'(\underline{t}) = 0$. We claim that

$$v'(\underline{t}) \ge 0. \tag{3.2}$$

In fact, if (3.2) does not hold, then there exists $\varepsilon > 0$ such that $v'(t) < 0$ for $t \in (\underline{t} - \varepsilon, \underline{t} + \varepsilon)$. Therefore, $v(t)$ is strictly decreasing for $t \in (\underline{t} - \varepsilon, \underline{t} + \varepsilon)$. Thus, from the first equation of (3.1), we can see that $u'(t)$ is strictly decreasing for $t \in (\underline{t} - \varepsilon, \underline{t} + \varepsilon)$. This contradicts the definition of $\underline{t}$. Therefore, (3.2) is true. From the second equation of (3.1), (3.2) and $f(t, 0) = 0$, we have

$$g\big(u(\underline{t} - \tau)\big) - e(\underline{t}) \le 0. \tag{3.3}$$

In a similar way, we get

$$g\big(u(\bar{t} - \tau)\big) - e(\bar{t}) \ge 0. \tag{3.4}$$

It follows from (H1), (3.3) and (3.4) that

$$u(\underline{t} - \tau) \ge d_1 \quad \text{and} \quad u(\bar{t} - \tau) \le d_2.$$

Thus, we can see that there exists a point $t_0 \in [0, T]$ such that

$$d_1 \le u(t_0) \le d_2. \tag{3.5}$$

Multiplying the second equation of (3.1) by $u'(t)$ and integrating over the interval $[0, T]$, we have

$$\begin{aligned}
0 &= \int_0^T v'(t)u'(t)\,dt = \lambda \int_0^T \frac{v(t)}{\sqrt{1 - v^2(t)}} \cdot v'(t)\,dt \\
&= -\lambda \int_0^T f\left(t, \frac{u'(t)}{\lambda}\right) u'(t)\,dt - \lambda \int_0^T g\big(u(t - \tau)\big)u'(t)\,dt \\
&\quad + \lambda \int_0^T e(t)u'(t)\,dt,
\end{aligned}$$

which together with (H2) and (H3) gives

$$\begin{aligned}
a\int_0^T \left|u'(t)\right|^2 dt
&\le \lambda \int_0^T \left|g\bigl(u(t-\tau)\bigr)\right|\left|u'(t)\right| dt + \lambda \int_0^T \left|e(t)\right|\left|u'(t)\right| dt \\
&\le \lambda \cdot \int_0^T \bigl(m_0\left|u(t-\tau)\right| + m_1\bigr)\left|u'(t)\right| dt + \lambda \cdot \|e\|_0\sqrt{T} \cdot \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} \\
&\le \lambda \cdot m_0\left(\int_0^T \left|u(t)\right|^2 dt\right)^{1/2}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} + \lambda \cdot m_1\sqrt{T}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} \\
&\quad + \lambda \cdot \|e\|_0\sqrt{T} \cdot \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2}.
\end{aligned}$$

By using Lemma 2.2 and (3.5), we have

$$\begin{aligned}
a\int_0^T \left|u'(t)\right|^2 dt
&\le \lambda \cdot m_0\left[\frac{T}{\pi}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} + \sqrt{T}d_2\right]\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} \\
&\quad + \lambda \cdot m_1\sqrt{T}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} + \lambda \cdot \|e\|_0\sqrt{T} \cdot \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} \\
&= \lambda \cdot \frac{m_0 T}{\pi}\int_0^T \left|u'(t)\right|^2 dt + \lambda \cdot m_0 d_2\sqrt{T}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} \\
&\quad + \lambda \cdot m_1\sqrt{T}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} + \lambda \cdot \|e\|_0\sqrt{T} \cdot \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2},
\end{aligned}$$

i.e.,

$$\begin{aligned}
a\left\|u'\right\|_2^2 = a\int_0^T \left|u'(t)\right|^2 dt
&\le \frac{m_0 T}{\pi}\left\|u'\right\|_2^2 + m_0 d_2\sqrt{T}\left\|u'\right\|_2 + m_1\sqrt{T}\left\|u'\right\|_2 + \|e\|_0\sqrt{T}\left\|u'\right\|_2.
\end{aligned}$$

Since $a\pi > m_0 T$, we have

$$\left\|u'\right\|_2 \le \frac{\pi\sqrt{T}(m_0 d_2 + m_1 + \|e\|_0)}{a\pi - m_0 T} := A_0. \tag{3.6}$$

By means of the Hölder inequality, (3.5) and (3.6), we have

$$\begin{aligned}
\|u\|_0 = \max_{t\in[0,T]}\left|u(t)\right| &\le \max_{t\in[0,T]}\left|u(t_0) + \int_{t_0}^t u'(s)\,ds\right| \\
&\le d_2 + \int_0^T \left|u'(s)\right| ds \le d_2 + \sqrt{T}\left\|u'\right\|_2 \\
&\le d_2 + \frac{\pi T(m_0 d_2 + m_1 + \|e\|_0)}{a\pi - m_0 T} := A_1.
\end{aligned}\tag{3.7}$$

Clearly, A_1 is independent of λ.

From the second equation of (3.1), we have

$$\begin{aligned}\int_0^T \left|v'(t)\right| dt &\leq \lambda \int_0^T \left|f\left(t, \frac{u'(t)}{\lambda}\right)\right| dt + \lambda \int_0^T \left|g\big(u(t-\tau)\big)\right| dt \\ &\quad + \lambda \int_0^T \left|e(t)\right| dt. \end{aligned} \tag{3.8}$$

Furthermore, from (3.7) and (H2), we get

$$\begin{aligned}\int_0^T \left|g\big(u(t-\tau)\big)\right| dt &\leq \int_0^T \big[m_0 u(t-\tau) + m_1\big] dt \\ &\leq m_0 T \cdot \|u\|_0 + m_1 T \\ &\leq m_0 A_1 T + m_1 T. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) and by using (H4), (3.6) and (3.7), we can obtain

$$\begin{aligned}\int_0^T \left|v'(t)\right| dt &\leq \lambda \int_0^T \left|f\left(t, \frac{u'(t)}{\lambda}\right)\right| dt + \lambda \int_0^T \left|g\big(u(t-\tau)\big)\right| dt + \lambda \int_0^T \left|e(t)\right| dt \\ &\leq \lambda \int_0^T \left(\beta \left|\frac{u'(t)}{\lambda}\right| + \gamma\right) dt + m_0 A_1 T + m_1 T + \|e\|_0 T \\ &\leq \beta \int_0^T \left|u'(t)\right| dt + \gamma T + m_0 A_1 T + m_1 T + \|e\|_0 T \\ &\leq \beta \sqrt{T} \left\|u'\right\|_2 + \gamma T + m_0 A_1 T + m_1 T + \|e\|_0 T \\ &\leq \beta \sqrt{T} A_0 + \big(m_0 A_1 + m_1 + \|e\|_0 + \gamma\big) T. \end{aligned} \tag{3.10}$$

Integrating the first equation of (3.1) on the interval $[0, T]$, we have

$$\int_0^T u'(t)\, dt = \int_0^T \frac{v(t)}{\sqrt{1 - v^2(t)}} dt = 0.$$

Then we can see that there exists $\eta \in [0, T]$ such that $v(\eta) = 0$. It implies that

$$\left|v(t)\right| = \left|\int_\eta^t v'(s)\, ds + v(\eta)\right| \leq \int_0^T \left|v'(s)\right| ds,$$

which together with (3.10) yields

$$\begin{aligned}\left|v(t)\right| &\leq \int_0^T \left|v'(s)\right| ds \\ &\leq \beta \sqrt{T} A_0 + \big(m_0 A_1 + m_1 + \|e\|_0 + \gamma\big) T := A_4. \end{aligned} \tag{3.11}$$

Since $\beta \sqrt{T} A_0 + (m_0 A_1 + m_1 + \|e\|_0 + \gamma) T < 1$, we have

$$\|v\|_0 = \max_{t \in [0,T]} \left|v(t)\right| \leq A_4 < 1. \tag{3.12}$$

Clearly, A_4 is independent of λ.

From the first equation of (3.1), we can see that

$$\|u'\|_0 \le \lambda \max_{t\in[0,T]} \frac{|v(t)|}{\sqrt{1-v^2(t)}} \le \lambda \cdot \frac{A_4}{1-A_4^2} \le \frac{A_4}{1-A_4^2} := A_3. \tag{3.13}$$

Clearly, A_3 is independent of λ.

In the following, we will prove that there exists a positive constant A_2 which is dependent of λ such that

$$u(t) \ge A_2. \tag{3.14}$$

Indeed, it follows from the second equation of (3.1) that

$$v'(t+\tau) = -\lambda f\left(t+\tau, \frac{u'(t+\tau)}{\lambda}\right) - \lambda g\bigl(u(t)\bigr) + \lambda e(t+\tau). \tag{3.15}$$

Multiplying both sides of (3.15) by $u'(t)$ and integrating on $[\xi, t]$, here $\xi \in [0,T]$, we get

$$\begin{aligned}
\lambda \int_{u(\xi)}^{u(t)} g_0(u)\,du &= \lambda \int_\xi^t g_0\bigl(u(s)\bigr)u'(s)\,ds \\
&= -\int_\xi^t v'(t+\tau)u'(t)\,dt - \lambda \int_\xi^t f\left(t+\tau, \frac{u'(t+\tau)}{\lambda}\right)u'(t)\,dt \\
&\quad + \lambda \int_\xi^t e(t+\tau)u'(t)\,dt;
\end{aligned}$$

then

$$\begin{aligned}
\lambda \left| \int_{u(\xi)}^{u(t)} g_0(u)\,du \right| &= \int_0^T \bigl|v'(t+\tau)\bigr|\bigl|u'(t)\bigr|\,dt \\
&\quad + \lambda \int_0^T \left| f\left(t+\tau, \frac{u'(t+\tau)}{\lambda}\right)\right| \cdot \bigl|u'(t)\bigr|\,dt \\
&\quad + \lambda \int_0^T \bigl|e(t+\tau)\bigr|\bigl|u'(t)\bigr|\,dt.
\end{aligned} \tag{3.16}$$

Furthermore, by (3.10) and (3.13) we obtain

$$\begin{aligned}
\int_0^T \bigl|v'(t+\tau)\bigr|\bigl|u'(t)\bigr|\,dt &\le \|u'\|_0 \cdot \int_0^T \bigl|v'(t+\tau)\bigr|\,dt \\
&\le \lambda \cdot \frac{A_4}{1-A_4^2} \cdot \bigl[\beta\sqrt{T}A_0 + \bigl(m_0A_1 + m_1 + \|e\|_0 + \gamma\bigr)T\bigr].
\end{aligned} \tag{3.17}$$

By using (H4) and (3.13), we have

$$\begin{aligned}
&\int_0^T \left| f\left(t+\tau, \frac{u'(t+\tau)}{\lambda}\right)\right| \cdot \bigl|u'(t)\bigr|\,dt \\
&\quad \le \int_0^T \left(\beta\left|\frac{u'(t+\tau)}{\lambda}\right| + \gamma\right)\bigl|u'(t)\bigr|\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le \beta \int_0^T \left|\frac{u'(t+\tau)}{\lambda}\right| \cdot \left|u'(t)\right| dt + \gamma \int_0^T \left|u'(t)\right| dt \\
&\le \frac{\beta T}{\lambda} \cdot \left\|u'\right\|_0^2 + \gamma T \cdot \left\|u'\right\|_0 \\
&\le \frac{\beta T}{\lambda} \cdot \left(\lambda \cdot \frac{A_4}{1-A_4^2}\right)^2 + \gamma T \cdot \lambda \cdot \frac{A_4}{1-A_4^2} \\
&\le \beta T \cdot \lambda \cdot \left(\frac{A_4}{1-A_4^2}\right)^2 + \gamma T \cdot \lambda \cdot \frac{A_4}{1-A_4^2}.
\end{aligned}
\tag{3.18}
$$

Substituting (3.17) and (3.18) into (3.16), we obtain

$$
\begin{aligned}
\lambda \left|\int_{u(\xi)}^{u(t)} g_0(u)\, du\right| &\le \int_0^T \left|v'(t+\tau)\right|\left|u'(t)\right| dt + \lambda \int_0^T \left|f\left(t+\tau, \frac{u'(t+\tau)}{\lambda}\right)\right|\left|u'(t)\right| dt \\
&\quad + \lambda \int_0^T \left|e(t+\tau)\right|\left|u'(t)\right| dt \\
&\le \lambda \cdot \frac{A_4}{1-A_4^2} \cdot \left[\beta \sqrt{T} A_0 + \left(m_0 A_1 + m_1 + \|e\|_0 + \gamma\right) T\right] \\
&\quad + \beta T \cdot \lambda^2 \cdot \left(\frac{A_4}{1-A_4^2}\right)^2 + \gamma T \cdot \lambda^2 \cdot \frac{A_4}{1-A_4^2} + \lambda^2 \cdot \|e\|_0 \cdot \left(\frac{A_4}{1-A_4^2}\right)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\left|\int_{u(\xi)}^{u(t)} g_0(u)\, du\right| &\le \frac{A_4}{1-A_4^2} \cdot \left[\beta \sqrt{T} A_0 + \left(m_0 A_1 + m_1 + \|e\|_0 + \gamma\right) T\right] \\
&\quad + \beta T \cdot \left(\frac{A_4}{1-A_4^2}\right)^2 + \gamma T \cdot \frac{A_4}{1-A_4^2} + \|e\|_0 \cdot \frac{A_4}{1-A_4^2}.
\end{aligned}
$$

From the strong force condition (1.4), we know that (3.14) holds. Therefore, from (3.7), (3.12), (3.13) and (3.14), we can see that the proof of Theorem 3.1 is now completed. □

Theorem 3.2 *Assume that the conditions in Theorem* 3.1 *hold, then equation* (1.3) *has at least one positive T-periodic solution.*

Proof Define

$$
0 < E_1 < \min\{d_1, A_2\}, \qquad E_2 > \max\{d_2, A_1\}, \qquad E_3 > A_3.
$$

It follows from (3.5), (3.7), (3.13) and (3.14) that

$$
E_1 < u(t) < E_2, \qquad \left\|u'\right\|_0 < E_3. \tag{3.19}
$$

Then we can see that the condition (1) of Lemma 2.1 is satisfied.

For a possible solution C to the equation

$$
g(C) - \frac{1}{T}\int_0^T e(t)\, dt = 0,
$$

it is easy to see that $E_1 < C < E_2$ is satisfied. Thus, the condition (2) of Lemma 2.1 is satisfied.

Finally, we prove that the condition (3) of Lemma 2.1 is also satisfied. In fact, from (H1), we have

$$g(E_1) - \frac{1}{T}\int_0^T e(t)\,dt > 0,$$

and

$$g(E_2) - \frac{1}{T}\int_0^T e(t)\,dt < 0,$$

which implies that the condition (3) of Lemma 2.1 is also satisfied. Therefore, by application of Lemma 2.1, we conclude that (1.3) has at least one positive T-periodic solution. □

4 Example

Consider the following prescribed mean curvature Rayleigh equations with a singularity:

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + \big(3 - \sin^2 8t\big)u'(t) - \frac{1}{6u(t-1)} + \frac{u(t-1)}{4} = \frac{e^{\cos^2 8t}}{12}. \tag{4.1}$$

Conclusion *Problem* (4.1) *has at least one positive* $\pi/8$*-periodic solution.*

Proof Corresponding to equation (1.3), we have

$$f\big(t,u'(t)\big) = \big(3 - \sin^2 8t\big)u'(t), \qquad e(t) = \frac{e^{\cos^2 8t}}{12},$$
$$g\big(u(t-1)\big) = -\frac{1}{6u(t-1)} + \frac{u(t-1)}{4}, \qquad T = \frac{\pi}{8}.$$

It is easy to see that (H1)-(H4) hold if we choose

$$d_2 = 1, \qquad m_0 = \frac{1}{4}, \qquad m_1 = \frac{1}{6}, \qquad a = 2, \qquad \beta = 3, \qquad \gamma = \frac{1}{8}.$$

Moreover, $a\pi > m_0 T$ and

$$A_0 = \frac{\pi\sqrt{T}(m_0 d_2 + m_1 + \|e\|_0)}{a\pi - m_0 T} \approx 0.15915, \qquad A_1 = d_2 + A_0 \approx 1.15915,$$

then we have

$$\beta\sqrt{T}A_0 + \big(m_0 A_1 + m_1 + \|e\|_0 + \gamma\big)T \approx 0.56026 < 1.$$

Hence, the conditions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.2, we can see that equation (4.1) has at least one positive $\pi/8$-periodic solution. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed to obtaining new results in this article and also read and approved the final manuscript.

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Acknowledgements
The authors are grateful for the referee's helpful suggestions and comments. This work is supported by the Fundamental Research Funds for the Central Universities (Grant No. 2016B07514, No. 2015B27914).

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 31 December 2016 Accepted: 12 April 2017 Published online: 24 April 2017

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