Cheng and Yao *Boundary Value Problems* (2017) 2017:70
DOI 10.1186/s13661-017-0800-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

New results for Brillouin electron beam focusing system

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Abstract
An experimental conjecture on the existence of positive periodic solutions for the Brillouin electron beam focusing system $x'' + a(1 + \cos 2t)x = \frac{1}{x}$ for $a \in (0, \frac{1}{2})$ is proved by applications of the Manasevich-Mawhin theorem.

MSC: 34K14; 34C25

Keywords: Brillouin electron beam focusing system; positive periodic solution; singular

1 Introduction

In this paper, we consider the 2π-periodic boundary value problem for the equation

$$x'' + a(1 + \cos 2t)x = \frac{1}{x}, \tag{1.1}$$

where $a > 0$ is constant.

The equations arise in the study of electronics and govern the motion of a magnetically focused axially symmetric electron beam under the influence of a Brillouin flow [1]. When the negative pole in a traveling-wave tube is shielded completely by a magnetic field screen, the electron beam focusing system can be described by (1.1). Besides, from a mathematical point of view, equation (1.1) is a singular perturbation of the Mathieu equation.

Motivated by the results of laboratory experiments experts realized in [1], it was conjectured that (1.1) should have a positive periodic solutions if $a \in (0, \frac{1}{4})$ [2]. In the last fifty years, many mathematicians have given birth to extensive literature about this topic (see [3–7]). Although numerical studies back up the experimental conjecture, an analytical proof of the existence of periodic solutions of (1.1) for $a \in (0, \frac{1}{4})$ is still lacking.

The first analytic work on periodic solution of (1.1) was obtained by Ding [3]. Ding proved that (1.1) had at least one positive periodic solution if $a \in (0, \frac{1}{16})$. Afterwards, Ye and Wang [4] obtained that (1.1) had at least one positive periodic solution if $a \in (0, 0.1442)$. In [5], Zhang investigated a kind of singular Liénard equation, and by applications of his theory, they extended the existence result of (1.1) to $a \in (0, 0.1532)$.

However, in the above works, authors were not able to prove or disprove the result which was conjectured in [1]. In this paper, we will show that (1.1) has at least one positive 2π-periodic solution when the parameter $a \in (0, \frac{1}{2})$ other than $(0, \frac{1}{4})$.

2 Brillouin electron beam focusing system

Lemma 2.1 (Manasevich-Mawhin [8]) *Let Ω be an open bounded set in $C_T^1 := \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+T) - x(t) \equiv 0\}$. If*

(i) *The problem*

$$(\phi(x'))' = \lambda\tilde{f}(t,x,x'), \quad x \in C_T^1, \tag{2.1}$$

where $\tilde{f} : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Carathéodory. For each $\lambda \in (0,1)$, problem (2.1) *has no solution on $\partial\Omega$.*

(ii) *The equation*

$$F(a) := \frac{1}{T}\int_0^T \tilde{f}(t,x,x')\,dt = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree of F*

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then problem (2.1) *has at least one periodic solution on $\bar{\Omega}$.*

Lemma 2.2 ([9]) *Suppose that $u \in C_T^1$ and there exists $t_0 \in [0,T]$ such that $|u(t_0)| < d$. Then*

$$\left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{2}} \leq \left(\frac{T}{\pi}\right)\left(\int_0^T |u'(t)|^2 t\right)^{\frac{1}{2}} + dT^{\frac{1}{2}}.$$

Next, we prove that Brillouin electron beam focusing system (1.1) has at least one positive 2π-periodic solution if $a \in (0, \frac{1}{2})$. Firstly, we consider the following singular equation:

$$x''(t) + a(t)x(t) = \frac{1}{x(t)}, \tag{2.2}$$

where $a(t) \in C(\mathbb{R}, [0,+\infty))$ and $a(t+T) = a(t)$, $\forall t \in \mathbb{R}$.

Theorem 2.1 *Assume that $|a|_\infty := \max_{t\in[0,T]} |a(t)| < \frac{4\pi^2}{T^2}$ holds. Then* (2.2) *has at least one positive T-periodic solution.*

Proof Firstly, we consider the following (homotopy) family of (2.2):

$$x''(t) + \lambda\left(a(t)x(t) - \frac{1}{x(t)}\right) = 0, \quad \lambda \in (0,1]. \tag{2.3}$$

Let $x(t) \in C_T^1$ be an arbitrary solution of (2.3). Integrating (2.3) from 0 to T , we get

$$\int_0^T \left(a(t)x(t) - \frac{1}{x(t)}\right) dt = 0. \tag{2.4}$$

So, we know that there exist positive constants $D_1 < D_2$ and $t_0 \in (0,T)$ such that

$$D_1 \leq x(t_0) \leq D_2. \tag{2.5}$$

Therefore, we have

$$\left|x(t)\right| = \left|x(t_0) + \int_{t_0}^{t} x'(s)\,ds\right| \le D_2 + \int_0^T \left|x'(s)\right|\,ds. \tag{2.6}$$

Let us write $x(t) = \bar{x} + \tilde{x}(t)$, here $\tilde{x}(t) := x(t) - \bar{x}$, and $\bar{x} := \frac{1}{T}\int_0^T x(t)\,dt$. Obviously, $\int_0^T \tilde{x}(t)\,dt = 0$. Now (2.3) for $\tilde{x}(t)$ is

$$\tilde{x}''(t) + \lambda a(t)\big(\bar{x} + \tilde{x}(t)\big) = \lambda \frac{1}{x(t)}, \tag{2.7}$$

since $\bar{x}'' = 0$. Multiplying (2.7) by $\bar{x} - \tilde{x}(t)$, we have

$$\bar{x}\tilde{x}''(t) - \tilde{x}(t)\tilde{x}''(t) + \lambda a(t)\big(\bar{x}^2 - \tilde{x}^2(t)\big) = \lambda \frac{\bar{x} - \tilde{x}(t)}{x(t)}.$$

Integrating this equation over one period and making use of the T-periodicity of $\tilde{x}(t)$, we get

$$-\int_0^T \tilde{x}(t)\tilde{x}''(t)\,dt + \lambda \int_0^T a(t)\big(\bar{x}^2 - \tilde{x}^2(t)\big)\,dt = \lambda \int_0^T \frac{\bar{x} - \tilde{x}(t)}{x(t)}\,dt.$$

So, we have

$$\int_0^T \left|\tilde{x}'(t)\right|^2 dt = \lambda \int_0^T a(t)\tilde{x}^2(t)\,dt - \lambda\bar{x}^2 \int_0^T a(t)\,dt + \lambda \int_0^T \frac{\bar{x} - \tilde{x}(t)}{x(t)}\,dt.$$

Since $a(t) \ge 0$, then $-\bar{x}^2 \int_0^T a(t)\,dt \le 0$. So, we have

$$\begin{aligned}
\int_0^T \left|\tilde{x}'(t)\right|^2 dt &\le \lambda \int_0^T a(t)\tilde{x}^2(t)\,dt + \lambda \int_0^T \frac{\bar{x} - \tilde{x}(t)}{x(t)}\,dt \\
&= \lambda \int_0^T a(t)\tilde{x}^2(t)\,dt + \lambda \int_0^T \frac{2\bar{x} - x(t)}{x(t)}\,dt \\
&= \lambda \int_0^T a(t)\tilde{x}^2(t)\,dt + \lambda \int_0^T \frac{2\bar{x}}{x(t)}\,dt - \lambda T \\
&\le \int_0^T \left|a(t)\right|\left|\tilde{x}(t)\right|^2 dt + 2|\bar{x}| \int_0^T \frac{1}{|x(t)|}\,dt.
\end{aligned}$$

For any $\varepsilon > 0$, there is $g_\varepsilon^+ \in L^2(0,T)$ and $g_\varepsilon^+ > 0$

$$\frac{1}{x(t)} \le \varepsilon x(t) + g_\varepsilon^+(t) \tag{2.8}$$

for all $x(t) > 0$ and a.e. $t \in [0,T]$. So, we have

$$\begin{aligned}
\int_0^T \left|\tilde{x}'(t)\right|^2 &\le \int_0^T \left|a(t)\right|\left|\tilde{x}(t)\right|^2 dt + 2\bar{x} \int_0^T \big(\varepsilon x(t) + g_\varepsilon^+(t)\big)\,dt \\
&\le \int_0^T \left|a(t)\right|\left|\tilde{x}(t)\right|^2 dt + \frac{2}{T}\int_0^T \left|x(t)\right|\,dt \int_0^T \big(\varepsilon x(t) + g_\varepsilon^+(t)\big)\,dt
\end{aligned}$$

$$\begin{aligned}
&\le |a|_\infty \int_0^T \left|\tilde{x}(t)\right|^2 dt + \frac{2\varepsilon}{T}\left(\int_0^T \left|x(t)\right| dt\right)^2 + \frac{2}{T}\int_0^T \left|x(t)\right| dt \int_0^T g_\varepsilon^+(t)\, dt \\
&\le |a|_\infty \int_0^T \left|\tilde{x}(t)\right|^2 dt + 2\varepsilon \int_0^T \left|x(t)\right|^2 dt \\
&\quad + 2\left(\int_0^T \left|x(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \left|g_\varepsilon^+(t)\right|^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

where $|a|_\infty = \max_{t\in[0,T]} |a(t)|$. Since $D_1 \le x(t_0) \le D_2$, by Lemma 2.2, we have

$$\left(\int_0^T \left|x(t)\right|^2 dt\right)^{\frac{1}{2}} \le \left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + D_2\sqrt{T}. \tag{2.9}$$

By applications of Wirtinger's inequality (in [10] Lemma 2.4) and (2.9), we have

$$\begin{aligned}
\int_0^T \left|\tilde{x}'(t)\right|^2 &\le |a|_\infty \left(\frac{T}{2\pi}\right)^2 \int_0^T \left|\tilde{x}'(t)\right|^2 dt + 2\varepsilon\left(\left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + D_2\sqrt{T}\right)^2 \\
&\quad + 2\left(\left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + D_2\sqrt{T}\right)\left\|g_\varepsilon^+\right\|_2 \\
&= |a|_\infty \left(\frac{T}{2\pi}\right)^2 \int_0^T \left|\tilde{x}'(t)\right|^2 dt + 2\varepsilon\left(\frac{T}{\pi}\right)^2 \int_0^T \left|x'(t)\right|^2 dt \\
&\quad + 2\left(2D_2\sqrt{T}\varepsilon + \left\|g_\varepsilon^+\right\|_2\right)\left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad + 2D_2^2 T\varepsilon + 2D_2\sqrt{T}\left\|g_\varepsilon^+\right\|_2,
\end{aligned}$$

where $\|g_\varepsilon^+\|_2 = (\int_0^T |g_\varepsilon^+(t)|^2\, dt)^{\frac{1}{2}}$. Since $\tilde{x}'(t) = x'(t)$, then we have

$$\begin{aligned}
\int_0^T \left|x'(t)\right|^2 &\le \left(|a|_\infty\left(\frac{T}{2\pi}\right)^2 + 2\varepsilon\left(\frac{T}{\pi}\right)^2\right)\int_0^T \left|x'(t)\right|^2 dt \\
&\quad + 2\left(2D_2\sqrt{T}\varepsilon + \left\|g_\varepsilon^+\right\|_2\right)\left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad + 2D_2^2 T\varepsilon + 2D_2\sqrt{T}\left\|g_\varepsilon^+\right\|_2.
\end{aligned}$$

From $|a|_\infty < \frac{4\pi^2}{T^2}$ for $\varepsilon > 0$ sufficiently small, there exists a positive constant M_1' such that

$$\int_0^T \left|x'(t)\right|^2 dt \le M_1'.$$

From (2.6) and by applying Hölder's inequality, we have

$$|x|_\infty \le D_2 + \int_0^T \left|x'(s)\right| ds \le D_2 + \sqrt{T}\left(\int_0^T \left|x'(s)\right|^2 ds\right)^{\frac{1}{2}} \le D_2 + \sqrt{T}M_1'^{\frac{1}{2}} = M_1. \tag{2.10}$$

On the other hand, from $x(0) = x(T)$, we know that there is a point $t_1 \in [0, T]$ such that $x'(t_1) = 0$, and then $|x'(t)| = |x'(t_1) + \int_{t_1}^t x''(s)\, ds| \le \int_0^T |x''(s)|\, ds$. From (2.3) and (2.8), we have

$$
\begin{aligned}
|x'|_\infty &\le \int_0^T |x''(t)|\,dt \\
&\le \lambda \int_0^T |a(t)||x(t)|\,dt + \lambda \int_0^T \frac{1}{x(t)}\,dt \\
&\le \lambda |a|_\infty M_1 T + \lambda \int_0^T \big(\varepsilon x(t) + g_\varepsilon^+(t)\big)\,dt \\
&\le \lambda |a|_\infty M_1 T + \lambda \varepsilon M_1 T + \lambda \sqrt{T}\left(\int_0^T |g_\varepsilon^+(t)|^2\,dt\right)^{\frac{1}{2}} \\
&\le \lambda |a|_\infty M_1 T + \lambda \varepsilon M_1 T + \lambda \sqrt{T}\|g_\varepsilon^+\|_2 := \lambda M_2,
\end{aligned}
$$

i.e.,

$$|x'|_\infty \le \lambda M_2. \tag{2.11}$$

Multiplying (2.3) by $x'(t)$, we get

$$x''(t)x'(t) + \lambda a(t)x(t)x'(t) = \lambda \frac{x'(t)}{x(t)}. \tag{2.12}$$

Let $\tau \in [0,T]$ be as in (2.5). For any $\tau \le t \le T$, we integrate (2.12) on $[\tau,t]$ and get

$$
\begin{aligned}
\lambda \int_{x(\tau)}^{x(t)} \frac{1}{u}\,du &= \lambda \int_\tau^t \frac{x'(s)}{x(t)}\,ds \\
&= \frac{1}{2}x'(t)^2 - \frac{1}{2}x'(\tau)^2 + \lambda \int_\tau^t a(s)x(s)x'(s)\,ds.
\end{aligned} \tag{2.13}
$$

By (2.11) we have

$$
\begin{aligned}
&x'(t)^2 \le \lambda^2 M_2^2, \\
&\left|\int_\tau^t a(s)x(s)x'(s)\,ds\right| \le \lambda |a|_\infty M_1 M_2 T.
\end{aligned}
$$

With these inequalities we can derive from (2.13) that

$$\left|\int_{x(\tau)}^{x(t)} \frac{1}{u}\,du\right| \le M_2^2 + |a|_\infty M_1 M_2 T. \tag{2.14}$$

So, we know that there exists $M_3 > 0$ such that

$$x(t) \ge M_3, \quad \forall t \in [\tau, T], \tag{2.15}$$

since $\lim_{x\to 0^+} \int_1^x \frac{1}{u}\,du = +\infty$. The case $t \in [0,\tau]$ can be treated similarly.

Having in mind (2.5), (2.10), (2.11) and (2.15), we define

$$\Omega = \big\{x \in C_T^1 : E_1 < x(t) < E_2 \text{ and } |x'(t)| < E_3\ \forall t \in \mathbb{R}\big\}, \tag{2.16}$$

where $0 < E_1 < \min\{M_3, D_1\}$, $E_2 > \max\{M_1, D_2\}$ and $E_3 > M_2$. Then condition (i) of Lemma 2.1 is satisfied. For a constant $x \in \ker L$, $x > 0$, we have

$$\bar{g}(x) := \frac{1}{T}\int_0^T \left(a(t)x(t) - \frac{1}{x(t)}\right) dt.$$

Obviously, $\bar{g}(x) < 0$ for all $x \in (0, E_1)$, $\bar{g}(x) > 0$ for all $x > E_2$, so condition (ii) of Lemma 2.1 holds. Set

$$H(x,\mu) = \mu x + (1-\mu)\frac{1}{T}\int_0^T \left(a(t)x(t) - \frac{1}{x(t)}\right) dt,$$

we have $xH(x,\mu) > 0$. Thus $H(x,\mu)$ is a homotopic transformation and

$$\begin{aligned} \deg\{F, \Omega \cap \mathbb{R}, 0\} &= \deg\left\{\frac{1}{T}\int_0^T \left(a(t)x(t) - \frac{1}{x(t)}\right) dt, \Omega \cap \mathbb{R}, 0\right\} \\ &= \deg\{x, \Omega \cap \mathbb{R}, 0\} \neq 0. \end{aligned}$$

Thus assumption (iii) of Lemma 2.1 is also verified. Therefore (2.2) has at least one positive T-periodic solution. □

Next, we apply Theorem 2.1 to Brillouin electron beam focusing system (1.1). Equation (1.1) is of the form (2.2) with $a(t) = a(1 + \cos 2t)$.

Theorem 2.2 *If* $a \in (0, \frac{1}{2})$, *then* (1.1) *has at least one positive* 2π*-periodic solution.*

Proof If $a < \frac{1}{2}$, then

$$|a|_\infty = 2a < 1 = \frac{4\pi^2}{T^2},$$

i.e., $|a|_\infty < \frac{4\pi^2}{T^2}$ holds. Theorem 2.1 implies that (1.1) has at least one 2π-periodic positive solution. □

Finally, we present an example to illustrate our result.

Example 2.1 Consider the second order differential equation with singularity:

$$x''(t) + (1 + \cos t) = \frac{1}{x}. \tag{2.17}$$

It is clear that $T = \pi$, $a(t) = 1 + \cos t$. Obviously,

$$|a|_\infty = \max_{t\in[0,T]} |1 + \cos t| = 2 < 4 = \frac{4\pi^2}{\pi^2}.$$

Therefore, (2.17) has at least one π-periodic solution by application of Theorem 2.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

ZBC and SWY worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

Acknowledgements

ZBC and SWY would like to thank the referee for invaluable comments and insightful suggestions. This work was supported by the National Natural Science Foundation of China (No. 11501170), China Postdoctoral Science Foundation funded project (No. 2016M590886), Fundamental Research Funds for the Universities of Henan Province (NSFRF140142), Henan Polytechnic University Outstanding Youth Fund (J2015-02) and Henan Polytechnic University Doctor Fund (B2013-055).

Publisher's Note

Received: 13 February 2017 Accepted: 28 April 2017 Published online: 15 May 2017

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