Wang et al. *Boundary Value Problems* (2017) 2017:73
DOI 10.1186/s13661-017-0806-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH | Open Access

Complicated asymptotic behavior exponents for solutions of the evolution p-Laplacian equation with absorption

Liangwei Wang[1*], Jngxue Yin[2] and Yuqiu Wu[1]

*Correspondence:
wanglw08@163.com
[1]School of Mathematics and Statistics, Chongqing Three Gorges University, No. 666, Tian Xing Road, Wanzhou District, Chongqing, 404100, China
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate how the initial value belonging to spaces $W_\sigma(\mathbb{R}^N)$ ($0 < \sigma < N$) affects the complicated asymptotic behavior of solutions for the Cauchy problem of the evolution p-Laplacian equation with absorption. In fact, we reveal the fact that $\sigma = \frac{p}{q-p+1}$ is the critical exponent for the complicated asymptotic behavior of the solutions.

MSC: 35B40; 35K65

Keywords: complicated asymptotic behavior; evolution p-Laplacian equation; critical exponent; absorption

1 Introduction

In this paper, we study the complicated asymptotic behavior of solutions for the Cauchy problem of the evolution p-Laplacian equation with absorption

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda u^q = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $p > 2$, $q > p-1+\frac{p}{N}$, $N \geq 1$, $\lambda > 0$ and $\varphi(x) \in W_\sigma^+(\mathbb{R}^N)$, i.e., $\varphi(x) \geq 0$ and $\varphi \in W_\sigma(\mathbb{R}^N) \equiv \{\phi \in L^1_{\text{loc}}(\mathbb{R}^N); |x|^\sigma\phi(x) \in L^\infty(\mathbb{R}^N)\}$ with the norm $\|\varphi\|_{W_\sigma(\mathbb{R}^N)} = \||\cdot|^\sigma\varphi(\cdot)\|_{L^\infty(\mathbb{R}^N)}$.

For solutions of some evolution equations, different initial values may cause different asymptotic behaviors, see [1–5]. Consider Problem (1.1)-(1.2). If $\lambda = 0$ and the nonnegative initial value $u_0 \in L^1(\mathbb{R}^N)$, it is well known that the solutions $u(x,t)$ converge to the Barenblatt solution U_M in $L^1(\mathbb{R}^N)$ as $t \to \infty$ [6]. If $q > p-1+\frac{p}{N}$, $0 < \sigma < N$ and $\lambda = 1$, the initial value $u_0(x) \in W_\sigma^+(\mathbb{R}^N)$ and $\lim_{|x|\to\infty}|x|^\sigma u_0(x) = A$, then the solutions satisfy

$$t^{\frac{\sigma}{\sigma(p-2)+p}}\left|u(x,t) - w(x,t)\right| \to 0$$

uniformly on the cone $\{x \in \mathbb{R}^N; |x| \leq Ct^\beta\}$ as $t \to +\infty$, where $\beta = \frac{q-p+1}{p(q-1)}$ and $w(x,t) = (\frac{1}{q-1})^{\frac{1}{q-1}}$ if $0 < \sigma < \frac{p}{q-p+1}$, or $\beta = \frac{1}{\sigma(p-2)+p}$ and $w(x,t)$ is the solution of the Cauchy problem of the evolution p-Laplacian equation without absorption

$$\frac{\partial w}{\partial t} - \operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.3}$$

$$w(x,0) = w_0(x) = A|x|^{-\sigma} \quad \text{in } \mathbb{R}^N \tag{1.4}$$

if $\sigma = \frac{p}{q-p+1}$, or $\beta = \frac{1}{\sigma(p-2)+p}$ and $w(x,t)$ is the solution of Problem (1.1) with the initial value $w(x,0) = A|x|^{-\sigma}$ if $\frac{p}{q-p+1} < \sigma < N$, see details in [7]. If $\lambda = 0$ and the initial value belongs to the bounded function space, it was first founded by Vázquez and Zuazua [8] that there exists an initial value $u_0 \in L^\infty(\mathbb{R}^N)$ such that the rescaled solutions $u(t_n^{\frac{1}{p}}x, t_n)$ converge to different functions in the weak-star topology of $L^\infty(\mathbb{R}^N)$ along different sequences $t_n \to \infty$. This result means that the bounded function space $L^\infty(\mathbb{R}^N)$ provides the work space where complicated asymptotic behavior of solutions takes place.

Since then, much attention has been paid to studying the complicated asymptotic behavior of solutions for evolution equations [9–11]. For example, Cazenave et al. considered the Cauchy problem of the heat equation and got a series of important results about the complicated asymptotic behavior of the rescaled solutions $t^\mu(t^\beta x, t)$ $(\mu, \beta > 0)$ in papers [12–16]. In our previous papers [17–19], we investigated the complicated asymptotic behavior of solutions for the porous medium equation. One can find some other interesting results of the partial differential equations in [20–23].

Inspired by the above papers, in this paper, we try to find out how the initial value belonging to $W_\sigma(\mathbb{R}^N)$ with different σ affects the complicated asymptotic behavior for the solutions of Problem (1.1)-(1.2) with $\lambda = 1$. In fact, we find that if $0 < \sigma < \frac{p}{q-p+1}$, the complicated asymptotic behavior for the solutions of Problem (1.1)-(1.2) with the initial value $u_0 \in W_\sigma^+(\mathbb{R}^N)$ cannot happen. While if $\frac{p}{q-p+1} \le \sigma < N$, then the complicated asymptotic behavior for the solutions of Problem (1.1)-(1.2) with the initial value $u_0 \in W_\sigma^+(\mathbb{R}^N)$ may happen. In fact, if $\sigma = \frac{p}{q-p+1}$, there exists an initial value $u_0 \in B_M^{\sigma,+} \equiv \{f \in W_\sigma^+(\mathbb{R}^N); \|f\|_{W_\sigma^+(\mathbb{R}^N)} \le M\}$ such that for every $\phi \in B_M^{\sigma,+}$, there exists a sequence $\{t_n\}$ such that

$$\lim_{t_n\to\infty} t_n^{\frac{\sigma}{\sigma(p-2)+p}} u\big(t_n^{\frac{1}{\sigma(p-2)+p}} x, t_n\big) = v(x,1)$$

uniformly on $\mathbb{R}^N$, where $v(x,t)$ is the solution of Problem (1.1) with the initial value $v(x,0) = \phi$; or if $\frac{p}{q-p+1} < \sigma < N$, then there exists an initial value $u_0 \in B_M^{\sigma,+}$ such that for every $\varphi \in B_M^{\sigma,+}$, there exists a sequence $\{t_n\}$ such that

$$\lim_{t_n\to\infty} t_n^{\frac{\sigma}{\sigma(p-2)+p}} u\big(t_n^{\frac{1}{\sigma(p-2)+p}} x, t_n\big) = w(x,1)$$

uniformly on $\mathbb{R}^N$, where $w(x,t)$ is the solution of Problem (1.3)-(1.4) with the initial value $w(x,0) = \varphi$. So, the complexity of asymptotic behavior of the solutions for $\frac{p}{q-p+1} \le \sigma < N$ occurs, according to Vázquez and Zuazua [8]. Therefore, we get that $\sigma = \frac{p}{q-p+1}$ is the critical exponent for the complexity of asymptotic behavior of solutions. For convenience, in the rest of this paper, we define $\gamma = \frac{1}{\sigma(p-2)+p}$ and put $\lambda = 1$ in (1.2).

The rest of this paper is organized as follows. In the next section, we give some concepts and lemmas. Section 3 is devoted to the study of the nonexistence of complexity for the asymptotic behavior of solutions. The complexity of asymptotic behavior for the solutions is considered for $\sigma = \frac{p}{q-p+1}$ in Section 4 and for $\frac{p}{q-p+1} < \sigma < N$ in Section 5, respectively.

2 Preliminaries

In this section, we first give some concepts as [24–26]. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $r > 0$, we define

$$|||f|||_r = \sup_{R \geq r} R^{-\frac{N(p-2)+p}{p-2}} \int_{\{|x| \leq R\}} \left|f(x)\right| dx.$$

The space X_0 is given by

$$X_0 \equiv \left\{\varphi \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right); |||\varphi|||_1 < \infty \text{ and } \lim_{r \to +\infty} |||\varphi|||_r = 0\right\}$$

with the norm $|||\cdot|||_1$. The existence and uniqueness of global weak solution for Problem (1.1)-(1.2) with the initial value $\varphi(x) \in X_0$ has been shown in [24, 25] and this solution satisfies

$$u(x,t) \in C^{\frac{\alpha}{2},\alpha}\left((0,\infty) \times \mathbb{R}^N\right) \tag{2.1}$$

for some $\alpha > 0$. Note that for $0 < \sigma < N$,

$$W_\sigma\left(\mathbb{R}^N\right) \subset X_0.$$

So we can define an operator $T(t) : W_\sigma(\mathbb{R}^N) \to C(\mathbb{R}^N)$ as

$$T(t)u_0(x) = u(x,t), \tag{2.2}$$

where $u(x,t)$ is the solution of Problem (1.1)-(1.2) with the initial value $u_0(x)$.

Lemma 2.1 ([24, 26]) *For $w_0 \in X_0$, there exists a unique global weak solution $w(x,t)$ of Problem* (1.3)-(1.4). *Moreover, the evolution p-Laplacian equation generates a bounded semigroup in X_0 given by*

$$S(t) : w_0 \to w(x,t).$$

If $1 \leq q \leq \infty$ and $w_0 \in L^q(\mathbb{R}^N) \subset X_0$, then $S(t)$ is a contraction bounded semigroup in $L^q(\mathbb{R}^N)$.

The following two lemmas appeared in [27] to study the chaotic dynamic systems in the evolution p-Laplacian equation. Let us write $\Omega(t) = \{x \in \mathbb{R}^N; w(x,t) > 0\}$, and let $d(x, \Omega(t))$ be the distance from x to $\Omega(t)$.

Lemma 2.2 (Propagation speed estimate [27]) *Suppose $0 < \sigma < N$. If $w(x,t)$ is the weak solution of Problem* (1.3)-(1.4) *with the initial value $w_0 \in W^+_\sigma(\mathbb{R}^N)$, then for $0 \leq t_1 < t_2 < \infty$, we have*

$$\Omega(t_2) \subset \Omega_{\rho(t_2-t_1)}(t_1),$$

where $\Omega_{\rho(t_2-t_1)}(t_1) \equiv \{x \in \mathbb{R}^N; d(x, \Omega(t_1)) < \rho(t_2 - t_1)\}$ and $\rho(t_2 - t_1) = C(t_2 - t_1)^{\frac{1}{\sigma(p-2)+p}} \times \|u_0\|^{\frac{p-2}{\gamma}}_{W_\sigma(\mathbb{R}^N)}$.

The following lemma concerns the decay estimate of the solutions.

Lemma 2.3 (Space-time decay estimate [27]) *Let* $0 < \sigma < N$. *If* $u_0 \in W_\sigma^+(\mathbb{R}^N)$, *then for* $t > 0$ *and* $x \in \mathbb{R}^N$,

$$S(t)u_0(x) \le C\big(t^{\frac{2}{\gamma}} + |x|^2\big)^{-\sigma}. \tag{2.3}$$

3 Nonexistence of complexity: $0 < \sigma < \frac{p}{q-p+1}$

In this section, we consider the nonexistence of complicated asymptotic behavior for the solutions of Problem (1.1)-(1.2) with the initial value $u_0 \in W_\sigma^+(\mathbb{R}^N)$. The ideas of the proof of the following lemma come from [1, 2, 7], we give it here for completeness.

Lemma 3.1 *Let*

$$q > p - 1 + \frac{p}{N}, \qquad 0 < \sigma < \frac{p}{q-p+1},$$

and let

$$\varphi \in W_\sigma^+\big(\mathbb{R}^N\big).$$

If $u(x,t)$ *is the solution of Problem* (1.1)-(1.2) *with the initial value* $u_0(x) = \varphi(x)$, *then*

$$\lim_{t\to\infty} \sup_{|x|\le Ct^{\frac{q-p+1}{p(q-1)}}} t^{\frac{1}{q-1}} u(x,t) \le \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}. \tag{3.1}$$

Proof Let

$$u_k(x,t) = k^{\frac{p}{q-p+1}} u\big(kx, k^{\frac{p(q-1)}{q-p+1}} t\big), \quad k > 0.$$

So, for every $k > 0$, $u_k(x,t)$ is a solution of Problem (1.1)-(1.2) with the initial value

$$u_k(x,0) = \varphi_k(x) = k^{\frac{p}{q-p+1}} \varphi(kx), \quad k > 0.$$

Since $\overline{u(x,t)} = (\frac{1}{q-1})^{\frac{1}{q-1}} t^{-\frac{1}{q-1}}$ is a solution of equation (1.1), it follows from the comparison principle that for every $(x,t) \in (0,+\infty) \times \mathbb{R}^N$,

$$u_k(x,t) \le \overline{u(x,t)}.$$

This uniform upper bound implies that the sequence $\{u_k\}$ is equicontinuous on compact subsets of $(0,+\infty) \times \mathbb{R}^N$. So we can extract a convergent subsequence $\{u_{k'}\}$ such that

$$u_{k'}(x,t) \overset{k'\to\infty}{\longrightarrow} U(x,t) \le \overline{u(x,t)}$$

on compact subsets of $(0,+\infty)\times\mathbb{R}^N$. Therefore, for every $C>0$, putting $t=1$, $kx=x'$ and $k^{\frac{p(q-1)}{q-p+1}}=t'$, we obtain, omitting the primes,

$$\lim_{t\to\infty} t^{\frac{1}{q-1}} \sup_{\{|x|\le Ct^{\frac{q-p+1}{p(q-1)}}\}} u(x,t) \le \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}.$$

The proof of this lemma is complete. □

Theorem 3.2 *Suppose*

$$q>p-1+\frac{p}{N} \quad \textit{and} \quad 0<\sigma<\frac{p}{q-p+1}.$$

Let $u(x,t)$ be the solutions of Problem (1.1)-(1.2) *with the initial value $u_0\in W_\sigma^+(\mathbb{R}^N)$. Then the complexity cannot occur in the asymptotic behavior of the rescaled solutions $t^{\frac{\sigma}{\gamma}}u(t^{\frac{1}{\gamma}}x,t)$ in $L^\infty(\mathbb{R}^N)$. In other words, if there exists a function $\phi\in W_\sigma(\mathbb{R}^N)$ and a sequence $t_n\to\infty$ such that*

$$\lim_{t_n\to\infty} t_n^{\frac{\sigma}{\gamma}} u\big(t_n^{\frac{1}{\gamma}}x,t_n\big)=\varphi(x) \tag{3.2}$$

uniformly on $\mathbb{R}^N$, then

$$\varphi(x)\equiv 0.$$

Proof Suppose that (3.2) holds for some $\varphi(x)\not\equiv 0$. And, without loss of generality, we assume that for some $x_0\in\mathbb{R}^N$,

$$\varphi(x_0)>0. \tag{3.3}$$

It follows from (3.2) that there exists an integer n_1 such that if $n\ge n_1$, then

$$t_n^{\frac{\sigma}{\gamma}} u\big(t_n^{\frac{1}{\gamma}}x_0,t_n\big)\ge\frac{1}{2}\varphi(x_0). \tag{3.4}$$

Note that $u_0\in W_\sigma^+(\mathbb{R}^N)$. By using Lemma 3.1, we obtain that

$$\lim_{t\to\infty} t^{\frac{1}{q-1}}u(x,t)\le\left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} \tag{3.5}$$

uniformly on the sets $\{x\in\mathbb{R}^N;|x|\le Ct^{\frac{q-p+1}{p(q-1)}}\}$ for $C>0$. It follows from (3.3) and the fact $\frac{\sigma}{\gamma}-\frac{1}{q-1}<0$ that there exists an integer n_2 such that if $n\ge n_2$, then

$$t_n^{\frac{\sigma}{\gamma}-\frac{1}{q-1}}\left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}<\frac{1}{2}\varphi(x_0).$$

So, from (3.5), we have

$$t_n^{\frac{\sigma}{\gamma}}u(x,t_n)=t_n^{\frac{\sigma}{\gamma}-\frac{1}{q-1}}t^{\frac{1}{q-1}}u(x,t_n)<\frac{1}{2}\varphi(x_0) \quad \text{for } x\in\big\{y;|y|\le Ct_n^{\frac{q-p+1}{p(q-1)}}\big\}. \tag{3.6}$$

Taking $n = \max(n_1, n_2)$, and then letting $C = 2|x_0|t_n^{\frac{1}{\gamma}-\frac{q-p+1}{p(q-1)}}$ and $x = t_n^{\frac{1}{\gamma}}x_0$, we have

$$x \in \{y; |y| \le Ct_n^{\frac{q-p+1}{p(q-1)}}\}.$$

Thus we deduce from (3.4) and (3.6) that

$$\frac{1}{2}\varphi(x_0) \le t_n^{\frac{\sigma}{\gamma}} u(t_n^{\frac{1}{\gamma}}x_0, t_n) < \frac{1}{2}\varphi(x_0).$$

So we get a contradiction. Therefore, (3.2) cannot hold for $\varphi(x) \not\equiv 0$. This means that if (3.2) holds with $0 < \sigma < \frac{p}{q-p+1}$, then

$$\varphi(x) \equiv 0 \quad \text{for } x \in \mathbb{R}^N,$$

and the proof is complete. □

4 Complexity: $\sigma = \frac{p}{q-p+1}$

To give the result about the complicated asymptotic behavior of solutions, we need introduce some concepts. For $0 < \sigma < N$ and $M > 0$, the convex closed set $B_M^{\sigma,+}$ is defined as

$$B_M^{\sigma,+} \equiv \{\varphi \in W_\sigma^+(\mathbb{R}^N) \cap C(\mathbb{R}^N); \|\varphi\|_{W_\sigma^+(\mathbb{R}^N)} \le M\}.$$

For $\lambda > 0$, $0 < \sigma < N$, $\varphi(x) \in L^1(\mathbb{R}^N)$, we define

$$D_\lambda^\sigma \varphi(x) = \lambda^{\frac{2\sigma}{\gamma}} \varphi(\lambda^{\frac{2}{\gamma}}x).$$

For $\sigma = \frac{p}{q-p+1}$, it follows from this definition and (2.2) that the following commutative relation holds [28]:

$$D_\lambda^\sigma[T(\lambda^2 t)u_0] = T(t)[D_\lambda^\sigma u_0]. \tag{4.1}$$

Now we give the result that for $\sigma = \frac{p}{q-p+1}$, the complicated asymptotic behavior for the solutions of Problem (1.1)-(1.2) with the initial value $u_0 \in W_\sigma^+(\mathbb{R}^N)$ can happen.

Theorem 4.1 *Suppose $q > p + \frac{p}{N}$ and $M > 0$. Let*

$$\sigma = \frac{p}{q-p+1}.$$

Then there exists a function $u_0 \in B_M^{\sigma,+}$ such that for every $\varphi \in B_M^{\sigma,+}$, there exists a sequence $t_n \to \infty$ satisfying

$$\lim_{n\to\infty} t_n^{\frac{\sigma}{\gamma}} u(t_n^{\frac{1}{\gamma}}x, t_n) = T(1)\varphi(x) \tag{4.2}$$

uniformly on $\mathbb{R}^N$, where $u(x,t)$ is the solution of Problem (1.1)-(1.2) *with the initial value $u_0(x)$.*

Proof By the definition of $B_M^{\sigma,+}$, there exists a countable set $F = \{\phi_i; \phi_i \in B_M^{\sigma,+} \bigcap L^1(\mathbb{R}^N), i = 1, 2, \ldots\}$ such that for every $\epsilon > 0$ and $\varphi \in B_M^{\sigma,+}$, there exists a function $\phi_\epsilon \in F$ satisfying

$$\|\phi_\epsilon - \varphi\|_{L^\infty(\mathbb{R}^N)} < \epsilon. \tag{4.3}$$

Therefore, there exists a sequence $\{\varphi_j\}_{j\geq 1} \subset F$ such that

I. For every $\phi_i \in F$, there exists a subsequence $\{\varphi_{i_k}\}_{k\geq 1}$ of the sequence $\{\varphi_j\}_{j\geq 1}$ such that

$$\varphi_{j_k}(x) = \phi_j \quad \text{for all } k \geq 1; \tag{4.4}$$

II. There exists a constant $C > 0$ such that

$$\max\bigl(\|\varphi_j\|_{L^\infty(\mathbb{R}^N)}, \|\varphi_j\|_{L^1(\mathbb{R}^N)}\bigr) \leq Cj \quad \text{for } j \geq 1. \tag{4.5}$$

Note that $\frac{p}{q-p+1} = \sigma < N$. So the following inequality holds:

$$N\gamma - \sigma\bigl(N(p-2) + p\bigr) > 0.$$

Let

$$\lambda_j = \begin{cases} 2, & j = 1, \\ \max(j^{\frac{3\gamma}{N\gamma-\sigma[N(p-2)+p]}}\lambda_{j-1}, (2^j \lambda_{j-1}^{\frac{2}{\gamma}} + C2^j M^{\frac{p-2}{\gamma}})^{\frac{\gamma}{2}}), & j > 1. \end{cases} \tag{4.6}$$

Now we can follow the methods given in [9, 10, 12] to construct an initial value by

$$u_0(x) == \sum_{j=1}^{\infty} D^{\sigma}_{\lambda_j^{-1}}\bigl[\chi_j(x)\varphi_j(x)\bigr] = u_n + v_n + w_n, \tag{4.7}$$

where

$$\begin{aligned} &u_n = \sum_{j=1}^{n-1} D^{\sigma}_{\lambda_j^{-1}}\bigl[\chi_j(x)\varphi_j(x)\bigr], \qquad v_n = D^{\sigma}_{\lambda_n^{-1}}\bigl[\chi_n(x)\varphi_n(x)\bigr], \\ &w_n = \sum_{j=n+1}^{\infty} D^{\sigma}_{\lambda_j^{-1}}\bigl[\chi_j(x)\varphi_j(x)\bigr], \end{aligned} \tag{4.8}$$

and $\chi_j(x)$ is the cut-off function defined on $\{x \in \mathbb{R}^N; 2^{-j} < |x| < 2^j\}$ relatively to $\{x \in \mathbb{R}^N; 2^{-j+1} < |x| < 2^{j-1}\}$. Note first that if $\varphi \in B_M^{\sigma,+}$, then

$$\|\varphi\|_{W_\sigma(\mathbb{R}^N)} \leq M$$

and

$$0 \leq \varphi \in C\bigl(\mathbb{R}^N\bigr).$$

By (4.6) and (4.7), we have

$$\|u_0\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{W_\sigma(\mathbb{R}^N)} \leq \sup_{j\geq 1}\bigl\|\lambda_j^{-\frac{2\sigma}{\gamma}}\chi_j\bigl(x/\lambda_j^{\frac{2}{\gamma}}\bigr)\varphi_j\bigl(x/\lambda_j^{\frac{2}{\gamma}}\bigr)\bigr\|_{W_\sigma(\mathbb{R}^N)} \leq M,$$

so

$$u_0 \in B_M^{\sigma,+}.$$

It follows from (4.1) that

$$D_{\lambda_n}^{\sigma}\big[T(\lambda_n^2 t)u_0\big] = T(t)\big[D_{\lambda_n}^{\sigma}u_0\big] = T(t)\big[D_{\lambda_n}^{\sigma}u_n + D_{\lambda_n}^{\sigma}v_n + D_{\lambda_n}^{\sigma}w_n\big].$$

We thus conclude from the definition of λ_j, comparison principle [29] and Lemma 2.2 that

$$\operatorname{supp}\big(T(1)\big[D_n^{\sigma}(w_n)\big]\big) \subset \big\{x \in \mathbb{R}^N; |x| > 2^n + CM^{\frac{p-2}{\gamma}}\big\}$$

and

$$\operatorname{supp}\big(T(1)\big[D_n^{\sigma}(v_n + u_n)\big]\big) \subset \big\{x \in \mathbb{R}^N; |x| < 2^n + CM^{\frac{p-2}{\gamma}}\big\},$$

so

$$\operatorname{supp}\big(T(1)\big[D_n^{\sigma}(w_n)\big]\big) \cap \operatorname{supp}\big(T(1)\big[D_n^{\sigma}(v_n + u_n)\big]\big) = \emptyset,$$

hence

$$T(1)\big[D_{\lambda_n}^{\sigma}u_n + D_{\lambda_n}^{\sigma}v_n + D_{\lambda_n}^{\sigma}w_n\big] = T(1)\big[D_{\lambda_n}^{\sigma}u_n + D_{\lambda_n}^{\sigma}v_n\big] + T(1)\big[D_{\lambda_n}^{\sigma}w_n\big].$$

The same result holds for $0 < t < 1$,

$$T(t)\big[D_{\lambda_n}^{\sigma}u_n + D_{\lambda_n}^{\sigma}v_n + D_{\lambda_n}^{\sigma}w_n\big] = T(t)\big[D_{\lambda_n}^{\sigma}u_n + D_{\lambda_n}^{\sigma}v_n\big] + T(t)\big[D_{\lambda_n}^{\sigma}w_n\big].$$

From the comparison principle [25, 29], Lemma 2.1, (4.5) and (4.6), we have

$$\begin{aligned}\big\|T(1)\big[D_{\lambda_n}^{\sigma}w_n\big]\big\|_{L^\infty(\mathbb{R}^N)} &\le \big\|S(1)\big[D_{\lambda_n}^{\sigma}w_n\big]\big\|_{L^\infty(\mathbb{R}^N)} \le C\big\|D_{\lambda_n}^{\sigma}w_n\big\|_{L^\infty(\mathbb{R}^N)} \\ &\le C\lambda_n^{\frac{2\sigma}{\gamma}}\lambda_{n+1}^{-\frac{2\sigma}{\gamma}}\sum_{i=n+1}^{\infty}2^{-i}i \le C2^{-n} \to 0 \end{aligned} \tag{4.9}$$

as $n \to \infty$. For every $\phi \in F$, it follows from (4.4) and (4.8) that there exists a sequence $\{\varphi_{n_k}\}_{k\ge1}$ such that if

$$x \in E_k \equiv \big\{y \in \mathbb{R}^N; 2^{-n_k+1} < |y| < 2^{n_k-1}\big\},$$

then

$$D_{\lambda_{n_k}}^{\sigma}u_{n_k}(x) = D_{\lambda_{n_k}}^{\sigma}\big[D_{\lambda_{n_k}^{-1}}^{\sigma}\chi_{n_k}\varphi_{n_k}\big](x) = \chi_{n_k}(x)\varphi_{n_k}(x) = \phi(x). \tag{4.10}$$

By the L^1-contraction principle [25, 26], we conclude from (4.6) and (4.10) that

$$\begin{aligned}
&\left\| T(1/2)\big[D^{\sigma}_{\lambda_n}(u_{n_k}+v_{n_k})\big] - T\left(\frac{1}{2}\right)\phi\right\|_{L^1(\mathbb{R}^N)} \\
&\quad\le \big\|\big[D^{\sigma}_{\lambda_{n_k}}(u_{n_k}+v_{n_k})\big]-\phi\big\|_{L^1(\mathbb{R}^N)} \\
&\quad= C\big\|D^{\sigma}_{\lambda_{n_k}}u_{n_k}\big\|_{L^1(\mathbb{R}^N)} + C\|\phi\|_{L^1(\mathbb{R}^N\setminus E_{n_k})} \\
&\quad\le Cn_k\left(\frac{\lambda_{n_k-1}}{\lambda_{n_k}}\right)^{\frac{2\sigma}{\gamma}(N(q-p+1)+p)} + C\|\phi\|_{L^1(\mathbb{R}^N\setminus E_{n_k})} \\
&\quad\le C{n_k}^{-2} + C\|\phi\|_{L^1(\mathbb{R}^N\setminus E_{n_k})} \to 0
\end{aligned} \tag{4.11}$$

as $k \to \infty$. Note that

$$\begin{aligned}
&\left\| T\left(\frac{1}{2}\right)\big[D^{\sigma}_{\lambda_n}(u_{n_k}+v_{n_k})\big]\right\|_{L^\infty(\mathbb{R}^N)} \\
&\quad\le \left\| S\left(\frac{1}{2}\right)D^{\frac{2\sigma}{\gamma},\beta}_{\lambda_{n_k}}[u_{n_k-1}+v_{n_k}]\right\|_{L^\infty} \\
&\quad\le C\big\|D^{\sigma}_{\lambda_{n_k}}[u_{n_k-1}+v_{n_k}]\big\|^{\frac{2}{N(m-1)+2}}_{L^1(\mathbb{R}^N)} \le C\big(\|\phi\|_{L^1}\big).
\end{aligned} \tag{4.12}$$

Thus we deduce from (4.11) and (4.12) that there exists a subsequence, which we still denote as $T(\frac{1}{2})D^{\sigma}_{\lambda_{n_k}}[u_{n_k-1}+v_{n_k}]$, which satisfies

$$T\left(\frac{1}{2}\right)D^{\sigma}_{\lambda_{n_k}}[u_{n_k-1}+v_{n_k}] \xrightarrow{w*} T\left(\frac{1}{2}\right)\phi \quad \text{in } L^\infty\big(\mathbb{R}^N\big) \text{ as } k\to\infty.$$

Therefore, the regularity of the solutions (see (2.1)) indicates that

$$T(1)D^{\sigma}_{\lambda_{n_k}}[u_{n_k-1}+v_{n_k}] \xrightarrow{k\to\infty} T(1)\phi \quad \text{in } L^\infty_{\text{loc}}\big(\mathbb{R}^N\big). \tag{4.13}$$

Note that

$$D^{\sigma}_{\lambda_{n_k}}[u_{n_k}+v_{n_k}], \quad \phi\in B^{\sigma,+}_M.$$

So, for every $\varepsilon > 0$, we obtain from Lemma 2.2 and the comparison principle that there exists $k_1 > 0$ such that if $|x| \ge 2^{n_{k1}}$, then

$$T(1)\big[D^{\sigma}_{\lambda_{n_k}}u_{n_k}+v_{n_k}\big](x) \le S(1)\big[D^{\sigma}_{\lambda_{n_k}}u_{n_k}+v_{n_k}\big](x) < \frac{\varepsilon}{3} \tag{4.14}$$

and

$$T(1)\phi(x) \le S(1)\phi(x) < \frac{\varepsilon}{3}. \tag{4.15}$$

Therefore, from (4.13), (4.14) and (4.15), we get

$$T(1)\big[D^{\sigma}_{\lambda_{n_k}}u_{n_k}+v_{n_k}\big](x) \xrightarrow{k\to\infty} T(1)\phi(x)$$

uniformly on $\mathbb{R}^N$. Combining this with (4.9), we get that for every $\phi \in F$, there exists a sequence $n_k \to \infty$ as $k \to \infty$ such that

$$D^{\sigma}_{\lambda_{n_k}}\left[T\left(\lambda^2_{n_k}\right)u_0\right] \xrightarrow{k\to\infty} T(1)\phi \tag{4.16}$$

uniformly on $\mathbb{R}^N$. Taking $t_k = \lambda_{n_k}^{\frac{1}{2}}$ in (4.16), we can conclude from (4.3) that (4.2) holds, and the proof is complete. □

5 Complexity: $\frac{p}{q-p+1} < \sigma < N$

In this section, we investigate the complicated asymptotic behavior of solutions for Problem (1.1)-(1.2) with the initial value $u_0 \in W_\sigma(\mathbb{R}^N)$ ($\frac{p}{q-p+1} < \sigma < N$).

Theorem 5.1 *Suppose $q > p + \frac{p}{N}$ and $M > 0$. Let*

$$\frac{p}{q-p+1} < \sigma < N.$$

Then there exists a function $u_0 \in B_M^{\sigma,+}$ such that for every $\phi \in B_M^{\sigma,+}$, there exists a sequence $t_n \to \infty$ satisfying

$$\lim_{n\to\infty} t_n^{\frac{\sigma}{\gamma}} u\left(t_n^{\frac{1}{\gamma}} x, t_n\right) = S(1)\phi(x) \tag{5.1}$$

uniformly on $\mathbb{R}^N$, where $u(x,t)$ is the solution of Problem (1.1)-(1.2) *with the initial value u_0.*

Proof In our previous paper [27], we have obtained the result that there exists a function $u_0 \in B_M^{\sigma,+}$ such that for every $\phi \in B_M^{\sigma,+}$, there exists a sequence $t_n \to \infty$ satisfying

$$\lim_{n\to\infty} t_n^{\frac{\sigma}{\gamma}} w\left(t_n^{\frac{1}{\gamma}} x, t_n\right) = S(1)\phi(x) \tag{5.2}$$

uniformly on $\mathbb{R}^N$, where $w(x,t)$ is the solution of Problem (1.3)-(1.4) with the initial value $w_0(x) = u_0(x)$. To get Theorem 5.1, we only need to prove that if $u_0(x) = \varphi(x) \in W_\sigma^+(\mathbb{R}^N)$, then for every sequence $t_n \to \infty$, the following limit holds:

$$\lim_{t_n\to\infty} t_n^{\frac{\sigma}{p+\sigma(p-2)}} \left|u\left(t_n^{\frac{1}{p+\sigma(p-2)}} x, t_n\right) - w\left(t_n^{\frac{1}{p+\sigma(p-2)}} x, t_n\right)\right| = 0 \tag{5.3}$$

uniformly on $\mathbb{R}^N$. The ideas of the following proof come from [1, 2, 7].

Without loss of generality, assuming that $\|\varphi\|_{W_\sigma(\mathbb{R}^N)} \le M$, we consider the following problem:

$$\begin{aligned}
&\frac{\partial V}{\partial t} - \operatorname{div}\left(|\nabla V|^{p-2}\nabla V\right) = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\
&V(x,0) = M|x|^{-\sigma} \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\end{aligned}$$

Then we define the functions

$$w_k(x,t) = k^\sigma w\left(kx, k^\gamma t\right), \qquad u_k(x,t) = k^\sigma u\left(kx, k^\gamma t\right)$$

and

$$V_k(x,t) = k^\sigma V\big(kx, k^\gamma t\big).$$

It follows from the comparison principle that

$$V(x,t) = V_k(x,t) \geq w_k(x,t) \geq u_k(x,t).$$

Therefore

$$u_k(x,t) \leq w_k(x,t) \leq CV_k\left(x, t + \frac{1}{k^\gamma}\right), \quad k > 0.$$

It is well known that

$$V(x,t) = t^{-\frac{\sigma}{\gamma}} f\left(\frac{|x|}{t^{\frac{1}{\gamma}}}\right),$$

where $f(x)$ is the positive solution of the equation

$$f''(\eta) + \left(\frac{n-1}{\eta} + \frac{\eta}{\gamma}\right) f'(\eta) + \frac{\sigma}{\gamma} f(\eta) = 0.$$

As in [7], there exists a constant $C > 0$ such that if $k > 0$, $x \in \mathbb{R}^N$, $t \geq \tau > 0$, then

$$V_k(x,t) \leq C\tau^{-\frac{\sigma}{\gamma}},$$

and

$$\lim_{\eta \to \infty} \eta^{\frac{\sigma}{\gamma}} f(\eta) = M.$$

From these, we can get that

$$\int_0^\tau \int_{B_1} V(x,t)\,dx\,dt \leq C\tau \tag{5.4}$$

and

$$\int_0^\tau \int_{B_1} V^q(x,t)\,dx\,dt \leq C\tau + C \begin{cases} \tau^{\frac{N-\sigma q+\gamma}{\gamma}} & \text{if } N - \sigma q + \gamma > 0, N \neq \sigma q, \\ \tau \log \frac{1}{\tau} & \text{if } N = \sigma q, \\ \log(1 + k^\gamma \tau) & \text{if } N - \sigma q + \gamma = 0, \\ k^{-N+\sigma q-\gamma} & \text{if } N - \sigma q + \gamma < 0, \end{cases} \tag{5.5}$$

where $k^\gamma \tau \geq 1$. Let $\xi \in C^\infty(Q_T)$ which vanishes at large x and at $t = T$, then u_k and w_k satisfy the integral identity

$$\iint_{Q_T} \left[\xi_t(w_k - u_k) - \frac{1}{\gamma} k^{-\alpha} \xi u_k^q\right] dx\,dt + \iint_{Q_T} a^{ij} \frac{\partial(w_k - u_k)}{\partial x_i} \frac{\partial \xi}{\partial x_j}\,dx\,dt = 0, \tag{5.6}$$

where

$$\alpha = \sigma(q-p+1) - p > \frac{p}{q-p+1}(q-p+1) - p = 0$$

and

$$\begin{aligned} a_k^{i,j}(x,t) = \delta_{ij} \cdot \int_0^1 \left|s\nabla u_k + (1-s)\nabla w_k\right|^{p-2} ds \\ + (p-2)\int_0^1 \left|s\nabla u_k + (1-s)\nabla w_k\right|^{p-4} \bigl(su_k + (1-s)w_k\bigr)_{x_i} \bigl(su_k + (1-s)w_k\bigr)_{x_j} ds. \end{aligned}$$

Note that $\{w_k\}$, $\{u_k\}$ are uniformly bounded on any compact subsets of $Q_T \setminus \{(0,0)\}$, and that $\{\nabla w_k\}$, $\{\nabla u_k\}$ are Hölder continuous on any compact subsets of Q_T, see [25]. Then there exist subsequences $\{v_{k_\ell}\}$ of $\{w_k\}$ and $\{u_{k_\ell}\}$ of $\{u_k\}$, and two functions $w'(x,t), u'(x,t) \in C(Q_T) \cap L^1_{\mathrm{loc}}(0,T; W^1_{\mathrm{loc}}(\mathbb{R}^N))$ such that

$$\begin{aligned} &w_{k_\ell}(x,t) \to w'(x,t), \qquad u_{k_\ell}(x,t) \to u'(x,t), \\ &\nabla w_{k_\ell}(x,t) \to \nabla w'(x,t), \qquad \nabla u_{k_\ell}(x,t) \to \nabla u'(x,t), \end{aligned}$$

in $C(\mathbb{K})$ as $k_\ell \to \infty$, where $\mathbb{K}$ is a compact subset of S_T. So, letting $k = k_\ell \to +\infty$ in (5.6) and applying (5.4), (5.5), we have

$$\iint_{Q_T} \xi_t\bigl(w' - u'\bigr)\,dx\,dt + \iint_{Q_T} a^{ij} \frac{\partial(w'-u')}{\partial x_i} \frac{\partial \xi}{\partial x_j}\,dx\,dt = 0, \tag{5.7}$$

where

$$\begin{aligned} a^{i,j}(x,t) = \delta_{ij} \cdot \int_0^1 \left|s\nabla u' + (1-s)\nabla w'\right|^{p-2} ds \\ = (p-2)\int_0^1 \left|s\nabla u' + (1-s)\nabla w'\right|^{p-4} \bigl(su_k + (1-s)w'\bigr)_{x_i} \bigl(su' + (1-s)w'\bigr)_{x_j} ds. \end{aligned}$$

Applying the existence and uniqueness theorem [25, 26] to (5.7), we obtain that

$$u'(x,t) - w'(x,t) = 0 \quad \text{a.e. on } Q_T,$$

hence the entire sequence

$$u_k(\cdot,t) - w_k(\cdot,t) \to 0 \tag{5.8}$$

uniformly on any compact subset of $\mathbb{R}^N$ as $k \to \infty$. Put $t = 1$ and $k = t_n^{\frac{1}{\gamma}}$ in (5.8), then

$$t_n^{\frac{\sigma}{\gamma}} \left|u\bigl(t_n^{\frac{1}{\gamma}} \cdot, t_n\bigr) - w\bigl(t_n^{\frac{1}{\gamma}} \cdot, t_n\bigr)\right| \to 0 \tag{5.9}$$

uniformly on any compact subset of $\mathbb{R}^N$ as $t_n \to \infty$. Note that $0 < \frac{p}{q-p+1} < \sigma < N$. It now follows from Lemma 3.1 that

$$t_n^{\frac{\sigma}{\gamma}} V\bigl(t_n^{\frac{1}{\gamma}} x, t_n\bigr) \le C\bigl(1 + |x|^2\bigr)^{-\frac{\sigma}{2}}$$

for all $t_n > 0$ and all $x \in \mathbb{R}^N$. Then, for every $\epsilon > 0$, there exists $R > 0$ such that

$$\left\| t_n^{\frac{\sigma}{\gamma}} V\left(t_n^{\frac{1}{\gamma}} \cdot, t\right) \right\|_{L^\infty(\mathbb{R}^N \setminus B_R)} < \epsilon.$$

Using the comparison principle, we obtain that

$$\begin{aligned} \left\| t_n^{\frac{\sigma}{\gamma}} u\left(t_n^{\frac{1}{\gamma}} \cdot, t\right) \right\|_{L^\infty(\mathbb{R}^N \setminus B_R)} &\le \left\| t_n^{\frac{\sigma}{\gamma}} w\left(t_n^{\frac{1}{\gamma}} \cdot, t\right) \right\|_{L^\infty(\mathbb{R}^N \setminus B_R)} \\ &\le \left\| t_n^{\frac{\sigma}{\gamma}} V\left(t_n^{\frac{1}{\gamma}} \cdot, t\right) \right\|_{L^\infty(\mathbb{R}^N \setminus B_R)} < \epsilon. \end{aligned} \tag{5.10}$$

Therefore, (5.9) and (5.10) indicate that (5.3) holds. Combining this with (5.2), we can get (5.1), and the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Chongqing Three Gorges University, No. 666, Tian Xing Road, Wanzhou District, Chongqing, 404100, China. [2]School of Mathematical Sciences, South China Normal University, No. 55, West of Zhong Shan Road, Tianhe District, Guangzhou, 510631, China.

Acknowledgements
This research was supported by the National Natural Science Foundation of China (11071099 and 11371153), Chongqing Fundamental and Frontier Research Project (cstc2016jcyjA0596) and the Chongqing Municipal Commission of Education (KJ1401003, KJ1601009), Innovation Team Building at Institutions of Higher Education in Chongqing (CXTDX201601035).

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 December 2016 Accepted: 8 May 2017 Published online: 19 May 2017

References
1. Kamin, S, Peletier, LA: Large time behaviour of solutions of the heat equation with absorption. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **12**, 393-408 (1985)
2. Kamin, S, Peletier, LA: Large time behaviour of solutions of the porous medium equation with absorption. Isr. J. Math. **55**(2), 129-146 (1986)
3. Herraiz, L: Asymptotic behaviour of solutions of some semilinear parabolic problems. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **16**, 49-105 (1999)
4. Lee, K, Petrosyan, A, Vazquez, JL: Large-time geometric properties of solutions of the evolution *p*-Laplacian equation equation. J. Differ. Equ. **229**, 389-411 (2006)
5. Iagar, RG, Vázquez, JL: Asymptotic analysis of the *p*-Laplacian flow in an exterior domain. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **26**(2), 497-520 (2009)
6. Kamin, S, Vázquez, JL: Fundamental solutions and asymptotic behaviour for the p-Laplacian equation. Rev. Mat. Iberoam. **4**, 339-354 (1988)
7. Zhao, JN: The asymptotic behavior of solutions of a quasilinear degenerate parabolic equations. J. Differ. Equ. **102**, 35-52 (1993)
8. Vázquez, JL, Zuazua, E: Complexity of large time behaviour of evolution equations with bounded data. Chin. Ann. Math., Ser. B **23**, 293-310 (2002)
9. Cazenave, T, Dickstein, F, Weissler, FB: Chaotic behavior of solutions of the Navier-Stokes system in $\mathbb{R}^N$. Adv. Differ. Equ. **10**, 361-398 (2005)
10. Vázquez, JL, Winkler, M: Highly time-oscillating solutions for very fast diffusion equations. J. Evol. Equ. **11**(3), 725-742 (2011)
11. Wang, LW, Yin, JX: Critical exponent for the asymptotic behavior of rescaled solutions to the porous medium equation. Electron. J. Differ. Equ. **10**, 1 (2017)
12. Cazenave, T, Dickstein, F, Weissler, FB: Universal solutions of a nonlinear heat equation on $\mathbb{R}^N$. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **2**, 77-117 (2003)
13. Cazenave, T, Dickstein, F, Weissler, FB: A solution of the constant coefficient heat equation on $\mathbb{R}$ with exceptional asymptotic behavior: an explicit construction. J. Math. Pures Appl. **85**(1), 119-150 (2006)
14. Cazenave, T, Dickstein, F, Weissler, FB: Nonparabolic asymptotic limits of solutions of the heat equation on $\mathbb{R}^N$. J. Dyn. Differ. Equ. **19**, 789-818 (2007)

15. Cazenave, T, Dickstein, F, Weissler, FB: Multi-scale multi-profile global solutions of parabolic equations in $\mathbb{R}^N$. Discrete Contin. Dyn. Syst., Ser. S **5**(3), 449-472 (2012)
16. Mouajria, H, Tayachi, S, Weissler, FB: The heat semigroup on sectorial domains, highly singular initial values and applications. J. Evol. Equ. **16**(2), 341-364 (2016)
17. Yin, JX, Wang, LW, Huang, R: Complexity of asymptotic behavior of solutions for the porous medium equation with absorption. Acta Math. Sci. Ser. B Engl. Ed. **6**(30B), 1865-1880 (2010)
18. Yin, JX, Wang, LW, Huang, R: Complexity of asymptotic behavior of the porous medium equation in $\mathbb{R}^N$. J. Evol. Equ. **11**, 429-455 (2011)
19. Wang, LW, Yin, JX, Jin, CH: ω-Limit sets for porous medium equation with initial data in some weighted spaces. Discrete Contin. Dyn. Syst., Ser. B **18**(1), 223-236 (2013)
20. Bougherara, B, Giacomoni, J: Existence of mild solutions for a singular parabolic equation and stabilization. Adv. Nonlinear Anal. **4**(2), 123-134 (2015)
21. Ghergu, M, Rădulescu, V: Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term. J. Math. Anal. Appl. **333**(1), 265-273 (2007)
22. Papageorgiou, N, Rădulescu, V, Repovs, D: Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions. Adv. Nonlinear Anal. **6**(2), 199-235 (2017)
23. Mihăilescu, M, Rădulescu, V: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. A, Math. Phys. Eng. Sci. **462**(462), 2625-2641 (2006)
24. DiBenedetto, E, Herrero, MA: On the Cauchy problem and initial traces for a degenerate parabolic equation. Trans. Am. Math. Soc. **314**(1), 187-224 (1989)
25. DiBenedetto, E: Degenerate Parabolic Equations. Springer, New York (1993)
26. Vázquez, JL: The Porous Medium Equation: Mathematical Theory. Oxford Mathematical Monographs. Clarendon, Oxford (2007)
27. Wang, LW, Yin, JX: Chaotic dynamical system in evolution p-Laplacian equation. Preprint (2016)
28. Yin, JX, Wang, LW, Huang, R: Classifications of Γ-limit set of solutions for evolution p-Laplacian equation. Preprint (2016)
29. Pucci, P, Serrin, J: The Maximum Principle. Progress in Nonlinear Differential Equations and Their Applications, vol. 73. Birkhäuser, Basel (2007)