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A new kind of uniqueness theorems for inverse Sturm-Liouville problems

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Abstract

We prove Marchenko-type uniqueness theorems for inverse Sturm-Liouville problems. Moreover, we prove a generalization of Ambarzumyan's theorem.

Keywords: inverse problem; Sturm-Liouville operator; uniqueness theorem; Ambarzumyan theorem

1 Introduction

Let us denote by $L(q, \alpha, \beta)$ the Sturm-Liouville boundary value problem

$$-y'' + q(x)y = \mu y, \quad x \in (0, \pi), \mu \in \mathbb{C}, \quad (1.1)$$

$$y(0) \cot \alpha + y'(0) = 0, \quad \alpha \in (0, \pi), \quad (1.2)$$

$$y(\pi) \cot \beta + y'(\pi) = 0, \quad \beta \in (0, \pi), \quad (1.3)$$

where q is a real-valued, summable function, $q \in L^1_{\mathbb{R}}(0, \pi)$. At the same time, $L(q, \alpha, \beta)$ denotes the self-adjoint operator generated by problem (1.1)-(1.3) (see, e.g., [1–3]). It is known that under the above conditions the spectrum of operator $L(q, \alpha, \beta)$ is discrete and consists of real, simple eigenvalues (see, e.g., [2, 4]), which we denote by $\mu_n = \mu_n(q, \alpha, \beta)$, $n \geq 0$, emphasizing the dependence of μ_n on q, α and β . We assume that eigenvalues are enumerated in the increasing order, i.e.,

$$\mu_0(q, \alpha, \beta) < \mu_1(q, \alpha, \beta) < \cdots < \mu_n(q, \alpha, \beta) < \cdots$$

Let $\varphi(x, \mu)$ be a solution of equation (1.1), which satisfies the initial conditions

$$\varphi(0, \mu) = 1, \quad \varphi'(0, \mu) = -\cot \alpha. \quad (1.4)$$

The eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n \geq 0$, of $L(q, \alpha, \beta)$ are the solutions of equation

$$\varphi(\pi, \mu) \cot \beta + \varphi'(\pi, \mu) = 0.$$

It is easy to see that the functions $\varphi(x, \mu_n)$, $n \geq 0$, are the eigenfunctions corresponding to the eigenvalue μ_n . The squares of the L^2 -norm of these eigenfunctions

$$a_n = a_n(q, \alpha, \beta) := \int_0^\pi |\varphi(x, \mu_n)|^2 dx, \quad n \geq 0,$$

are called norming constants. The eigenvalues and norming constants are called spectral data (besides these, there are other quantities, which are also called spectral data). The inverse Sturm-Liouville problem is to reconstruct the quantities q, α, β by some spectral data.

Let $L = L(q, \alpha, \beta)$ and $L_0 = L(q_0, \alpha_0, \beta_0)$ be two operators. The following assertion is usually called the uniqueness theorem of Marchenko.^a

Theorem 1.1 (Marchenko [5]) *Let $q \in L^1_{\mathbb{R}}(0, \pi)$. If*

$$\mu_n(q, \alpha, \beta) = \mu_n(q_0, \alpha_0, \beta_0), \quad (1.5)$$

$$a_n(q, \alpha, \beta) = a_n(q_0, \alpha_0, \beta_0), \quad (1.6)$$

for all $n \geq 0$, then $\alpha = \alpha_0$, $\beta = \beta_0$ and $q(x) = q_0(x)$ almost everywhere.

One of the results of the present paper is the following theorem which, in some sense, is a generalization of Marchenko's uniqueness theorem.

Theorem 1.2 *Let $q' \in L^2_{\mathbb{R}}(0, \pi)$. If*

$$\mu_n(q, \alpha_0, \beta) = \mu_n(q_0, \alpha_0, \beta_0), \quad (1.7)$$

$$a_n(q, \alpha_0, \beta) \geq a_n(q_0, \alpha_0, \beta_0), \quad (1.8)$$

for all $n \geq 0$, then $\beta = \beta_0$ and $q(x) \equiv q_0(x)$.

This kind of uniqueness theorem has not been considered before. The main difference between Theorems 1.1 and 1.2 is that we replace the equality in (1.6) with the inequality in (1.8). Note that we assume $q' \in L^2_{\mathbb{R}}(0, \pi)$ instead of general $q \in L^1_{\mathbb{R}}(0, \pi)$ since our proof is based on the results of Jodeit and Levitan (see [6]). And the parameter α of boundary condition is in advance fixed $\alpha = \alpha_0$.

Remark 1 Some analogues of Theorem 1.2 will be stated in the Appendix.

Historically, the first work in the theory of inverse spectral problems for Sturm-Liouville operators belongs to Ambarzumyan [7]. He proved that if the eigenvalues of Sturm-Liouville operator with Neumann boundary conditions are n^2 , then the potential q is 0 on $[0, \pi]$. It is known that the eigenvalues $\mu_n(0, \pi/2, \pi/2)$ of operator $L(0, \pi/2, \pi/2)$ are n^2 , $n \geq 0$. The classical Ambarzumyan theorem in our notations will be as follows.

Theorem 1.3 (Ambarzumyan [7]) *If $\mu_n(q, \pi/2, \pi/2) = \mu_n(0, \pi/2, \pi/2) = n^2$ for all $n \geq 0$, then $q(x) \equiv 0$.*

This was an exception as in general additional information was needed in order to reconstruct the potential q uniquely. There are many generalizations of Ambarzumyan's theorem in various directions, we mention several of them (see, e.g., [8–15] and the references therein).

Our generalization of Ambarzumyan's theorem is as follows.

Theorem 1.4 *Let $q' \in L^2_{\mathbb{R}}(0, \pi)$.*

If $\mu_n(q, \alpha, \pi - \alpha) = \mu_n(0, \alpha, \pi - \alpha)$ for all $n \geq 0$, then $q(x) \equiv 0$.

We think that Theorem 1.4 is a natural generalization, because we use only one spectrum to reconstruct the potential q without any additional conditions, as it is in the classical result.

2 Preliminaries

Two operators $L = L(q, \alpha, \beta)$ and $L_0 = L(q_0, \alpha_0, \beta_0)$ are called isospectral if they have the same spectra, i.e., $\mu_n(q, \alpha, \beta) = \mu_n(q_0, \alpha_0, \beta_0)$, $n \geq 0$. In what follows, if a certain symbol γ denotes an object related to L , then γ_0 (or γ^0 depending on situation) will denote a similar object related to L_0 .

The problem of describing all the operators L isospectral with L_0 first was considered by Trubowitz et al. (see [16–19]) for $q \in L^2_{\mathbb{R}}(0, \pi)$. The same problem was considered by Jodeit and Levitan in [6] for q such that $q' \in L^2_{\mathbb{R}}(0, \pi)$. For this aim the Gelfand-Levitan integral equation and transformation operators were used in [6]. They constructed the kernel $F(x, y)$ of the integral equation as follows. Let c_n , $n \geq 0$, be arbitrary real numbers converging to zero, as $n \rightarrow \infty$, so rapidly that the function

$$F(x, y) = \sum_{n=0}^{\infty} c_n \varphi_0(x, \mu_n^0) \varphi_0(y, \mu_n^0) \quad (2.1)$$

is continuous and all the second order partial derivatives are also continuous. The integral equation

$$K(x, y) + F(x, y) + \int_0^x K(x, t) F(t, y) dt = 0, \quad 0 \leq y \leq x \leq \pi, \quad (2.2)$$

is called Gelfand-Levitan integral equation.^b

They proved that if $1 + c_n \alpha_n^0 > 0$ for all $n \geq 0$, then the integral equation (2.2) has a unique solution $K(x, y)$ and the function

$$\varphi(x, \mu) = \varphi_0(x, \mu) + \int_0^x K(x, t) \varphi_0(t, \mu) dt$$

is a solution of the differential equation (1.1), with potential function

$$q(x) = q_0(x) + 2 \frac{d}{dx} K(x, x), \quad (2.3)$$

and $\varphi(x, \mu)$ satisfies the initial conditions

$$\varphi(0, \mu) = 1, \quad \varphi'(0, \mu) = -\cot \alpha,$$

where

$$\cot \alpha = \cot \alpha_0 + \sum_{n=0}^{\infty} c_n. \quad (2.4)$$

It means that the function $\varphi(x, \mu)$ satisfies the boundary condition (1.2) for all $\mu \in \mathbb{C}$.

Find $\beta \in (0, \pi)$ such that $\mu_n(q, \alpha, \beta) = \mu_n(q_0, \alpha_0, \beta_0)$ for all $n \geq 0$, i.e., $\varphi(x, \mu)$ should satisfy, at the point $x = \pi$, the boundary condition (1.3)

$$\varphi(\pi, \mu_n^0) \cot \beta + \varphi'(\pi, \mu_n^0) = 0$$

for this $\beta \in (0, \pi)$. Such β (in [6]) is being defined from the following relation

$$\cot \beta = \cot \beta_0 + \sum_{n=0}^{\infty} \frac{c_n \varphi_0^2(\pi, \mu_n^0)}{1 + c_n a_n^0}. \quad (2.5)$$

Thus Jodeit and Levitan showed that each admissible sequence $\{c_n\}_{n=0}^{\infty}$ generates an isospectral operator $L(q, \alpha, \beta)$, where q, α and β are given by formulae (2.3), (2.4) and (2.5), respectively. In this way they obtained all the potentials q , with $q' \in L^2(0, \pi)$, having a given spectrum $\mu_n^0 = \mu_n(q_0, \alpha_0, \beta_0)$, $n \geq 0$.

3 Proof of Theorem 1.2

Consider operators $L_0 = L(q_0, \alpha_0, \beta_0)$ and $L = L(q, \alpha, \beta)$ with the set of norming constants $a_n^0 = a_n(q_0, \alpha_0, \beta_0)$ and $a_n = a_n(q, \alpha, \beta)$, $n \geq 0$, respectively. It is known (see, e.g., [6]) that in this case the kernel $F(x, y)$ of the integral equation (2.2) is

$$F(x, y) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right) \varphi_0(x, \mu_n^0) \varphi_0(y, \mu_n^0). \quad (3.1)$$

Since by the condition of Theorem 1.2 the operators L and L_0 are isospectral, then formulae (2.3)-(2.5) hold. If we compare kernels (2.1) and (3.1), we will refer that $c_n = \frac{1}{a_n} - \frac{1}{a_n^0}$. So formulae (2.4) and (2.5) will become

$$\cot \alpha = \cot \alpha_0 + \sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right), \quad (3.2)$$

$$\cot \beta = \cot \beta_0 + \sum_{n=0}^{\infty} (a_n^0 - a_n) \frac{\varphi_0^2(\pi, \mu_n^0)}{(a_n^0)^2}. \quad (3.3)$$

Thus, we have all the operators $L(q, \alpha, \beta)$ isospectral with $L(q_0, \alpha_0, \beta_0)$.

We supposed that $\alpha = \alpha_0$, then by formula (3.2) we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right) = 0. \quad (3.4)$$

Since $a_n \geq a_n^0$ for all $n \geq 0$, thus from equation (3.4) it refers that $a_n = a_n^0$ for all $n \geq 0$. Thus, from Marchenko's uniqueness theorem 1.1 we obtain $q(x) \equiv q_0(x)$ and $\beta = \beta_0$.

This completes the proof.

Remark 2 From equation (3.4) it follows that the condition $a_n \geq a_n^0$ can be changed with $a_n \leq a_n^0$. From relation (3.3) it follows that we can assume $\beta = \beta_0$ instead of $\alpha = \alpha_0$ with the condition $a_n \geq a_n^0$ or $a_n \leq a_n^0$, and then we will also obtain $q(x) \equiv q_0(x)$ and $\alpha = \alpha_0$.

4 Proof of Theorem 1.4

Consider an operator $L(q, \alpha, \pi - \alpha)$ and an even operator^c $L(0, \alpha, \pi - \alpha)$.

Levinson proved [20] (see also [21]) that an operator L is even if and only if

$$\varphi(\pi, \mu_n) = (-1)^n, \quad n \geq 0. \quad (4.1)$$

The condition of the theorem means that the operator $L(q, \alpha, \pi - \alpha)$ is isospectral with $L(0, \alpha, \pi - \alpha)$. Since the method of Jodeit and Levitan has described all the isospectral operators for a potential function q with $q' \in L^2(0, \pi)$, then there exists a sequence $\{c_n\}_{n=0}^\infty$ such that $1 + c_n a_n^0 > 0$ for all $n \geq 0$, $\{c_n\}_{n=0}^\infty$ has the properties described in Section 2, and formulae (2.3)-(2.5) hold for operators $L(q, \alpha, \pi - \alpha)$ and $L(0, \alpha, \pi - \alpha)$.

Therefore, taking into account that $q_0(x) \equiv 0$, $\alpha_0 = \alpha$, $\beta_0 = \beta = \pi - \alpha$ and (4.1), relations (2.3)-(2.5), which connect these two operators, will become

$$q(x) = 2 \frac{d}{dx} K(x, x), \quad (4.2)$$

$$\sum_{n=0}^{\infty} c_n = 0, \quad (4.3)$$

$$\sum_{n=0}^{\infty} \frac{c_n}{1 + c_n a_n^0} = 0. \quad (4.4)$$

If we subtract (4.3) from (4.4), we will obtain

$$\sum_{n=0}^{\infty} \frac{c_n^2 a_n^0}{1 + c_n a_n^0} = 0. \quad (4.5)$$

Since $1 + c_n a_n^0 > 0$ and $a_n^0 > 0$ for all $n \geq 0$, then from equation (4.5) we obtain that $c_n = 0$, $n \geq 0$. Thus, from equations (2.1), (2.2) and (4.2) it follows that $q(x) \equiv 0$.

Remark 3 We will get the classical Ambarzumyan theorem if we take $\alpha = \pi/2$.

Appendix: Analogues of Theorem 1.2

Consider the $L(q, \alpha, \beta)$ problem. Let $\psi(x, \mu)$ be a solution of equation (1.1), which satisfies the initial conditions

$$\psi(\pi, \mu) = 1, \quad \psi'(\pi, \mu) = -\cot \beta. \quad (A.1)$$

The eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n \geq 0$, are the solutions of the equation

$$\Phi(\mu) := \varphi(\pi, \mu) \cot \beta + \varphi'(\pi, \mu) = 0,$$

or of the equation

$$\Psi(\mu) := \psi(0, \mu) \cot \alpha + \psi'(0, \mu) = 0.$$

$\Phi(\mu)$ and $\Psi(\mu)$ are called characteristic functions for the operator $L(q, \alpha, \beta)$. In [22] it is proved that characteristic functions and their derivatives are uniquely determined only from their zeros, i.e., from eigenvalues $\{\mu_n\}_{n=0}^{\infty}$. It is easy to see that the functions $\psi(x, \mu_n)$, $n \geq 0$, are the eigenfunctions corresponding to the eigenvalue μ_n . The squares of the L^2 -norm of these eigenfunctions

$$b_n = b_n(q, \alpha, \beta) := \int_0^\pi |\psi(x, \mu_n)|^2 dx, \quad n \geq 0,$$

are called norming constants.

Since all the eigenvalues of $L(q, \alpha, \beta)$ are simple, then there exist constants $\kappa_n = \kappa_n(q, \alpha, \beta)$, $n \geq 0$, such that

$$\varphi(x, \mu_n) = \kappa_n \psi(x, \mu_n). \quad (\text{A.2})$$

The theorem of uniqueness of Harutyunyan (see [21]) states the following.

Theorem A.1 *If*

$$\mu_n(q, \alpha, \beta) = \mu_n(q_0, \alpha_0, \beta_0),$$

$$\kappa_n(q, \alpha, \beta) = \kappa_n(q_0, \alpha_0, \beta_0),$$

for all $n \geq 0$, then $\alpha = \alpha_0$, $\beta = \beta_0$ and $q(x) = q_0(x)$ almost everywhere.

From (1.4), (A.1) and (A.2) it follows

$$\kappa_n = \varphi(\pi, \mu_n) = \psi^{-1}(0, \mu_n). \quad (\text{A.3})$$

There is a relationship between norming constants and characteristic functions (see, e.g., [6, 21])

$$a_n = |\varphi(\pi, \mu_n)| |\dot{\Phi}(\mu_n)|, \quad (\text{A.4})$$

$$b_n = |\psi(0, \mu_n)| |\dot{\Psi}(\mu_n)|, \quad (\text{A.5})$$

where the dot over Φ (or over Ψ) denotes the derivative of $\Phi(\mu)$ with respect to μ . From equations (A.3) and (A.4) we obtain

$$a_n = |\kappa_n| |\dot{\Phi}(\mu_n)|. \quad (\text{A.6})$$

Consider two isospectral operators $L(q, \alpha, \beta)$ and $L(q_0, \alpha_0, \beta_0)$. The following formulae, analogues to (2.4) and (2.5), can be obtained for κ_n :

$$\cot \alpha = \cot \alpha_0 + \sum_{n=0}^{\infty} \frac{1}{|\dot{\Phi}(\mu_n^0)|} \left(\frac{1}{|\kappa_n|} - \frac{1}{|\kappa_n^0|} \right), \quad (\text{A.7})$$

$$\cot \beta = \cot \beta_0 + \sum_{n=0}^{\infty} \frac{|\kappa_n^0| - |\kappa_n|}{|\dot{\Phi}(\mu_n^0)|}. \quad (\text{A.8})$$

From Theorem A.1 and formulae (2.3), (A.7), (A.8), a new statement, similar to Theorem 1.2, can be proven for κ_n as follows.

Theorem A.2 *Let $q' \in L^2_{\mathbb{R}}(0, \pi)$. If*

$$\begin{aligned} \mu_n(q, \alpha_0, \beta) &= \mu_n(q_0, \alpha_0, \beta_0), \\ |\kappa_n(q, \alpha_0, \beta)| &\geq |\kappa_n(q_0, \alpha_0, \beta_0)|, \end{aligned}$$

for all $n \geq 0$, then $\beta = \beta_0$ and $q(x) \equiv q_0(x)$.

Remark 4 Instead of $\alpha = \alpha_0$, we can fix $\beta = \beta_0$ and/or replace the inequality sign (' \geq ') with less than or equal sign (' \leq '). Even so, the result is valid. Similar theorems can be proven for $\varphi(\pi, \mu_n)$.

Remark 5 Since the uniqueness theorem of Marchenko is also true for norming constants b_n , taking into consideration relations (A.2), (A.3) and (A.5), analogues to Theorem 1.2 can be proven for $\psi(0, \mu_n)$ and b_n .

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Competing interests

The author declares that he has no competing interests.

Author's contributions

The author read and approved the final manuscript.

Endnotes

- ^a The theorem of Marchenko is more general, see e.g. [5, 23–25].
- ^b Here $F(x, y)$ is a kernel of integral equation (2.2), where x is a parameter, $F(x, y)$ is a known function and $K(x, y)$ is an unknown function, as functions of y .
- ^c A problem $L(q, \alpha, \beta)$ is said to be even if $q(x) = q(\pi - x)$ and $\alpha + \beta = \pi$.

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