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General decay for weak viscoelastic Kirchhoff plate equations with delay boundary conditions

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Abstract

We consider a weak viscoelastic Kirchhoff plate model with time-varying delay in the boundary. By using a suitable energy and Lyapunov function, we obtain a decay rate for the energy, which depends on the behavior of g and α.

Keywords: Kirchhoff plate; relaxation function; general decay; memory term; time-varying delay

1 Introduction

The equation which describes the small vibration of a thin homogeneous, isotropic plate of uniform thickness h is given by

$$\begin{cases} \rho h u_{tt} - \frac{\rho h^3}{12}\Delta u_{tt} + D(0)\Delta^2 u - \int_0^t D'(t-s)\Delta^2 u(s)\,ds = f, & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ \mathcal{B}_1 u - \mathcal{B}_1(\int_0^t D'(t-s)u(s)\,ds) = -\nu \cdot m, & \text{on } \Gamma_1 \times (0,\infty), \\ \mathcal{B}_2 u - \frac{h^2}{12}\frac{\partial u_{tt}}{\partial \nu} - \mathcal{B}_2(\int_0^t D'(t-s)u(s)\,ds) = -\frac{\partial \eta \cdot m}{\partial \eta}, & \text{on } \Gamma_1 \times (0,\infty), \end{cases} \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint. Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector to Γ, and let us denote by $\eta = (-\nu_2, \nu_1)$ the unit tangent vector positively oriented on Γ. The differential operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\mathcal{B}_1 u = \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

and the operators B_1 and B_2 are defined by

$$B_1 u = 2\nu_1\nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2},$$

$$B_2 u = (\nu_1^2 - \nu_2^2)\frac{\partial^2 u}{\partial x \partial y} + \nu_1\nu_2\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right).$$

The constants in the above equations have the following physical meanings: ρ is mass density, D is flexural rigidity, $\mu \in (0, \frac{1}{2})$ is Poisson's ratio, m is distribution of external force, $m \cdot \nu$ is a bending moment about the normal vector, $m \cdot \eta$ is a bending moment about the tangent vector and f is vertical loading on the faces of the plate. For simplicity, we assume that the bending moments about both the tangent and the normal vectors are zero. To simplify equation (1.1), we make the change of variable $t \to t\sqrt{D(0)/\rho h}$ in the time scale and we take $\gamma = h^2/12$, $g(t) = D'(t)$ for any $t > 0$; with these notations the initial boundary value problem (1.1) is equivalent to

$$\begin{cases} u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = 0, & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ \mathcal{B}_1 u - \mathcal{B}_1(\int_0^t g(t-s)u(s)\,ds) = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ \mathcal{B}_2 u - \gamma \frac{\partial u_{tt}}{\partial \nu} - \mathcal{B}_2(\int_0^t g(t-s)u(s)\,ds) = 0, & \text{on } \Gamma_1 \times (0,\infty). \end{cases} \tag{1.2}$$

Rivera *et al.* [1] showed exponential and polynomial decay of the solutions to viscoelastic plate equation (1.2). They considered a relaxation function satisfying

$$-c_0 g(t) \le g'(t) \le -c_1 g(t), \qquad 0 \le g''(t) \le c_2 g(t),$$

for some positive constant c_i, $i = 0, 1, 2$. The uniform stabilization of Kirchhoff plates with linear or nonlinear boundary feedback was investigated by several authors [2–6].

It is well known that delay effects often arise in many practical problems because these phenomena depend not only on the present state but also on the past history of the system. In recent years, the behavior of solutions for the PDEs with time delay effects has become an active area of research; see, for instance, [7–11] and the references therein. Datko *et al.* [9] proved that a small delay in a boundary control is a source of instability. To stabilize a system involving delay terms, additional control terms will be necessary. Nicaise and Pignotti [11] considered the following wave equation with a linear damping and delay term inside the domain:

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. It is also showed in the case $\mu_2 \ge \mu_1$ that there exists a sequence of arbitrary small (or large) delays such that instabilities occur. Moreover, the same results were proved when both the damping and the delay acted on the boundary. Kirane and Said-Houari [10] investigated the following linear viscoelastic wave equation with a linear damping and a delay term

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \tag{1.3}$$

where μ_1 and μ_2 are positive constants. They showed that its energy was exponentially decaying when $\mu_2 \le \mu_1$. Dai and Yang [7] improved the results of [10] under weaker conditions. They also obtained an exponential decay results for the energy of problem (1.3) in the case $\mu_1 = 0$. Furthermore, Nicaise and Pignotti [12] considered the following wave equation with time-dependent delay term:

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t\big(t - \tau(t)\big) = 0,$$

where $\tau(t) > 0$ is the time-varying delay, μ_1 and μ_2 are real numbers with $\mu_1 > 0$. They analyzed the exponential stability result under the condition

$$|\mu_2| < \sqrt{1-d}\mu_1, \tag{1.4}$$

where d is a constant such that $\tau'(t) \le d < 1$, $\forall t > 0$. Liu [13] investigated the viscoelastic wave equation (1.3) with time-varying delay term under condition (1.4).

The stability result of viscoelastic wave equations without time delay has been studied by many authors. Cavalcanti *et al.* [14] established an exponential rate of decay for a viscoelastic wave equation under the condition $-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t)$, $t \ge 0$, for some positive constant ξ_i, $i = 1, 2$. Later, this assumption was relaxed by several authors. Berrimi and Messaoudi [15] proved exponential and polynomial decay rates under the condition $g'(t) \le -\xi g^p(t)$, $t \ge 0$, $1 \le p < \frac{3}{2}$, for a positive constant ξ. Messaoudi [16] considered the following weak viscoelastic equation:

$$u_{tt} - \Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\,ds = 0, \tag{1.5}$$

where α and g are positive nonincreasing functions defined on $\mathbb{R}^+$. Under some assumptions on the relaxation function g and the potential α, the author obtained a general decay result which depends on the behavior of g and α. For more results on weak viscoelastic equations, we can refer to [17–19] and the references therein.

Recently, Yang [20] showed the existence and energy decay of solutions for the following Euler-Bernoulli equation with a delay:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0 \tag{1.6}$$

under some restrictions on μ_1 and μ_2. The author proved an exponential decay results for the energy in two cases ($\mu_1 \neq 0$ or $\mu_1 = 0$). Moreover, the stability of partial differential equations with time delay effects has been discussed by many authors [21–29].

Then, a natural problem is what would happen when a delay term occurs in (1.2). Motivated by these results [16, 18, 20, 29], we consider a decay rate of the solutions for the following weak viscoelastic Kirchhoff plate equations (1.2) with time-varying delay in the boundary:

$$\begin{cases} u_{tt}(x,t) - \gamma\Delta u_{tt}(x,t) + \Delta^2 u(x,t) - \alpha(t)\int_0^t g(t-s)\Delta^2 u(x,s)\,ds = 0, \quad \text{in } \Omega \times (0,\infty), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \\ \mathcal{B}_1 u(x,t) - \mathcal{B}_1(\alpha(t)\int_0^t g(t-s)u(x,s)\,ds) = 0, \quad \text{on } \Gamma_1 \times (0,\infty), \\ \mathcal{B}_2 u(x,t) - \gamma\frac{\partial u_{tt}(x,t)}{\partial \nu} - \mathcal{B}_2(\alpha(t)\int_0^t g(t-s)u(x,s)\,ds) \\ \quad = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)), \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t) = f_0(x,t), \quad (x,t) \in \Gamma_1 \times [-\tau(0), 0), \end{cases} \tag{1.7}$$

where μ_1 is a positive constant, μ_2 is a real number, $\tau(t) > 0$ represents the time-varying delay, g and α are real functions satisfying some conditions to be specified later.

When the viscoelastic term is modulated by a time-dependent coefficient $\alpha(t)$, we prove an energy decay result of the solutions for weak viscoelastic Kirchhoff plate equations (1.7) in the case $\mu_1 \neq 0$ or $\mu_1 = 0$, respectively. In order to achieve this goal, we need a restriction of the size between the parameter μ_2 and the kernel g.

The paper is organized as follows. In Section 2, we give some preparations for our consideration and our main results. In Section 3, we study an energy decay of the solutions for problem (1.7). By introducing suitable energy and Lyapunov functionals, we obtain a decay estimate for the energy, which depends on the behavior of both α and g.

2 Preliminaries

In this section, we introduce some material needed in the proof of our result and state the main result. We denote

$$(u,v) = \int_\Omega uv\,dx, \qquad (u,v)_{\Gamma_1} = \int_{\Gamma_1} uv\,d\Gamma.$$

For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma_1)}$ by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_1}$, respectively. To study the existence of solution of system (1.7), we introduce the following spaces:

$$V = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_0\}, \qquad W = \left\{u \in H^2(\Omega) \;\Big|\; u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}.$$

Let us define the following bilinear symmetric form:

$$\begin{aligned} a(u,v) = \int_\Omega \bigg(& \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + \mu\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\right) \\ & + 2(1-\mu)\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y}\bigg)\,dx\,dy. \end{aligned}$$

Based on the integration by parts formula, a simple calculation yields

$$\left(\Delta^2 u, v\right) = a(u,v) + (\mathcal{B}_2 u, v)_{\Gamma_1} - \left(\mathcal{B}_1 u, \frac{\partial v}{\partial \nu}\right)_{\Gamma_1}.$$

Since $\Gamma_0 \neq \emptyset$, we know that $\sqrt{a(u,u)}$ is equivalent to the $H^2(\Omega)$ norm on W, that is,

$$c_0\|u\|^2_{H^2(\Omega)} \le a(u,u) \le \tilde{c}_0\|u\|^2_{H^2(\Omega)},$$

where c_0 and $\tilde{c}_0$ are generic positive constants. The Sobolev imbedding theorem and a trace estimate imply that for some positive constants C_p, C_s, $\tilde{C}_p$ and $\tilde{C}_s$,

$$\begin{aligned} & \|u\|^2 \le C_p a(u,u), \qquad \|\nabla u\|^2 \le C_s a(u,u), \\ & \|u\|^2_{\Gamma_1} \le \tilde{C}_p a(u,u) \quad \text{and} \quad \|u\|^2_{\Gamma_1} \le \tilde{C}_s\|\nabla u\|^2, \quad \forall u \in W. \end{aligned} \tag{2.1}$$

For the relaxation function g and the potential α, as in [16], we assume that

(H1) $g, \alpha : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying

$$\begin{aligned} &g(0) > 0, \quad l_0 := \int_0^\infty g(s)\,ds < \infty, \\ &\alpha(t) > 0, \quad 1 - 2\alpha(t)\int_0^t g(s)\,ds \geq l > 0, \quad \text{for } t \geq 0, \end{aligned} \tag{2.2}$$

and there exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\xi(t) > 0, \quad g'(t) \leq -\xi(t)g(t), \quad \text{for } t \geq 0 \tag{2.3}$$

and

$$\lim_{t\to\infty} \frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0. \tag{2.4}$$

Remark 2.1 Note that (H1) implies $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$.

Since the function g is continuous and positive, we obtain

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds := g_0 > 0 \tag{2.5}$$

for all $t \geq t_0 > 0$. This fact will be used subsequently in the proof of our main result.

As in [12], for the time-varying delay, we assume that $\tau \in W^{2,\infty}([0,T])$ for $T > 0$, and there exist positive constants τ_0, τ_1 and d satisfying

$$0 < \tau_0 \leq \tau(t) \leq \tau_1 \quad \text{and} \quad \tau'(t) \leq d < 1 \quad \text{for all } t > 0, \tag{2.6}$$

and that μ_1 and μ_2 satisfy

$$|\mu_2| < \sqrt{1-d}\,\mu_1. \tag{2.7}$$

Let us introduce the function as in [12]

$$z(x,\rho,t) = u_t\bigl(x, t - \tau(t)\rho\bigr), \quad x \in \Gamma_1, \rho \in (0,1), t > 0.$$

Then problem (1.7) is equivalent to

$$\begin{cases} u_{tt}(x,t) - \gamma\Delta u_{tt}(x,t) + \Delta^2 u(x,t) - \alpha(t)\int_0^t g(t-s)\Delta^2 u(x,s)\,ds = 0, \quad \text{in } \Omega \times (0,\infty), \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, \quad \text{in } \Gamma_1 \times (0,1) \times (0,\infty), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \\ \mathcal{B}_1 u(x,t) - \mathcal{B}_1(\alpha(t)\int_0^t g(t-s)u(x,s)\,ds) = 0, \quad \text{on } \Gamma_1 \times (0,\infty), \\ \mathcal{B}_2 u(x,t) - \gamma\frac{\partial u_{tt}(x,t)}{\partial \nu} - \mathcal{B}_2(\alpha(t)\int_0^t g(t-s)u(x,s)\,ds) \\ \quad = \mu_1 u_t(x,t) + \mu_2 z(x,1,t), \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ z(x,0,t) = u_t(x,t), \quad \text{on } \Gamma_1 \times (0,\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), \quad (x,\rho) \in \Gamma_1 \times (0,1). \end{cases} \tag{2.8}$$

We can prove the existence of weak solution by making use of the classical Faedo-Galerkin method. Then, using elliptic regularity and second order estimates, we can show the regularity of the solution. We state a well-posedness result without a proof here (see [1, 10, 12, 20]).

Lemma 2.1 *Let* (2.6) *and* (2.7) *be satisfied and* g, α *satisfy* (H1). *If* $(u_0, u_1) \in W \times V$, $f_0 \in L^2(\Gamma_1 \times (0,1))$ *and* $T > 0$, *then there exists a unique weak solution* $(u, u_t) \in C([0,T]; W \times V)$ *of problem* (2.8). *Moreover, if* $(u_0, u_1) \in (W \cap H^4(\Omega)) \times (V \cap H^3(\Omega))$, $f_0 \in H^2(\Gamma_1 \times (0,1))$, *then the solution of* (2.8) *has the following regularity*:

$$u \in C^0\big([0,T]; W \cap H^4(\Omega)\big) \cap C^1\big([0,T]; V \cap H^3(\Omega)\big).$$

Inspired by [12, 16], we define a modified energy functional as

$$\begin{aligned} E(t) &:= \frac{1}{2}\|u_t\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{2}\left(1 - \alpha(t)\int_0^t g(s)\,ds\right)a(u,u) \\ &\quad + \frac{\alpha(t)}{2} g\Box\partial^2 u + \frac{\zeta}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\big\|u_t(s)\big\|_{\Gamma_1}^2\,ds, \end{aligned}$$

where ζ and λ are positive constants satisfying

$$\frac{|\mu_2|}{\sqrt{1-d}} < \zeta < 2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}} \quad \text{and} \quad \lambda < \frac{1}{\tau_1}\log\frac{\zeta\sqrt{1-d}}{|\mu_2|}. \tag{2.9}$$

Note that this choice of ζ is possible from assumption (2.7).

The main result of this paper is the following.

Theorem 2.1 *Let* (2.6) *be satisfied and* g, α *satisfy* (H1). *Assume that either one of the following two conditions holds*:

(i) $0 < |\mu_2| < \sqrt{1-d}\,\mu_1$,

(ii) $\mu_1 = 0$, $0 < |\mu_2| < \mu_0$ *and* $\alpha(t)\xi(t) > \xi_0$, $\forall t \geq t_1$.

Then there exist positive constants k *and* K *such that, for any solution of problem* (1.7), *the energy satisfies*

$$E(t) \leq K e^{-k\int_{t_2}^t \alpha(s)\xi(s)\,ds}, \quad \forall t \geq t_2, \tag{2.10}$$

where μ_0 *and* ξ_0 *are positive constants given by* (3.29) *and* (3.33), *respectively.*

3 General decay of solutions

In this section we show a general decay rate. To simplify calculation, in our analysis we introduce the following notation:

$$\begin{aligned} (g * u)(t) &= \int_0^t g(t-s)u(s)\,ds, \\ (g\Box u)(t) &:= \int_0^t g(t-s)\big\|u(t)-u(s)\big\|^2\,ds, \\ \big(g\Box\partial^2 u\big)(t) &:= \int_0^t g(t-s)a\big(u(t)-u(s), u(t)-u(s)\big)\,ds. \end{aligned}$$

We give some estimates related to the convolution operator. By the symmetry of $a(\cdot,\cdot)$ and direct calculations, we shall see that

$$\begin{aligned}\alpha(t)a(g*u,u_t) &= -\frac{\alpha(t)}{2}g(t)a(u,u)+\frac{\alpha(t)}{2}g'\Box\partial^2 u+\frac{\alpha'(t)}{2}g\Box\partial^2 u\\ &\quad -\frac{\alpha'(t)}{2}\left(\int_0^t g(s)\,ds\right)a(u,u)\\ &\quad -\frac{1}{2}\frac{d}{dt}\left[\alpha(t)g\Box\partial^2 u-\alpha(t)\left(\int_0^t g(s)\,ds\right)a(u,u)\right],\end{aligned}\tag{3.1}$$

and

$$a(g*u,u)\le 2\left(\int_0^t g(s)\,ds\right)a(u,u)+\frac{1}{4}g\Box\partial^2 u.\tag{3.2}$$

To prove our result, we need to introduce the following auxiliary functionals:

$$\begin{aligned}\Phi(t)&=\int_\Omega u_t u\,dx+\gamma\int_\Omega \nabla u_t\nabla u\,dx,\\ \Psi(t)&=-\int_\Omega u_t\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\ &\quad -\gamma\int_\Omega \nabla u_t\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx.\end{aligned}$$

We divide the proof of Theorem 2.1 into two cases as follows.

Case 1: $0<|\mu_2|<\sqrt{1-d}\mu_1$.

First, we consider the functional

$$L(t):=NE(t)+\epsilon_1\alpha(t)\Phi(t)+\epsilon_2\alpha(t)\Psi(t),\tag{3.3}$$

where ϵ_1 and ϵ_2 are positive constants. We easily get the following lemmas.

Lemma 3.1 *For $N>0$ large enough, there exist positive constants C_1 and C_2 such that*

$$C_1E(t)\le L(t)\le C_2E(t),\quad \forall t\ge 0.\tag{3.4}$$

Proof By applying Young's inequality, the Cauchy-Schwarz inequality, (2.1) and (2.2), we clearly have

$$\begin{aligned}\big|\Phi(t)\big|&\le\frac{1}{2}\|u_t\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2+\frac{C_p+\gamma C_s}{2l}\left(1-\alpha(t)\int_0^t g(s)\,ds\right)a(u,u),\\ \big|\Psi(t)\big|&\le\frac{1}{2}\|u_t\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2+\frac{(C_p+\gamma C_s)l_0}{2}g\Box\partial^2 u,\end{aligned}$$

which gives us

$$\begin{aligned}\left|L(t) - NE(t)\right| &\leq \frac{\alpha(0)}{2}(\epsilon_1 + \epsilon_2)\|u_t\|^2 + \frac{\gamma\alpha(0)}{2}(\epsilon_1 + \epsilon_2)\|\nabla u_t\|^2 \\ &\quad + \frac{\epsilon_2(C_p + \gamma C_s)l_0}{2}\alpha(t)g\Box\partial^2 u \\ &\quad + \frac{\epsilon_1(C_p + \gamma C_s)\alpha(0)}{2l}\left(1 - \alpha(t)\int_0^t g(s)\,ds\right)a(u,u) \\ &\leq C_3 E(t),\end{aligned}$$

where $C_3 = \max\{\alpha(0)(\epsilon_1+\epsilon_2), \epsilon_2(C_p+\gamma C_s)l_0, \frac{\epsilon_1(C_p+\gamma C_s)\alpha(0)}{l}\}$. Choosing $N > 0$ large, we complete the proof of Lemma 3.1. □

Lemma 3.2 *Let* (2.6) *and* (2.7) *be satisfied and* g, α *satisfy* (H1). *Then, for all regular solutions of problem* (1.7), *there exist positive constants* α_0 *and* α_1 *satisfying*

$$\begin{aligned}E'(t) &\leq -\alpha_0\|u_t\|^2_{\Gamma_1} - \alpha_1\left\|u_t\bigl(t-\tau(t)\bigr)\right\|^2_{\Gamma_1} + \frac{\alpha(t)}{2}g'\Box\partial^2 u - \frac{\alpha'(t)}{2}\left(\int_0^t g(s)\,ds\right)a(u,u) \\ &\quad - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left\|u_t(s)\right\|^2_{\Gamma_1}\,ds.\end{aligned} \tag{3.5}$$

Proof Multiplying (1.7) by $u_t(t)$, we get the identity

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{2}a(u,u)\right\} \\ &\quad = -\mu_1\|u_t\|^2_{\Gamma_1} - \mu_2\bigl(u_t\bigl(t-\tau(t)\bigr), u_t\bigr)_{\Gamma_1} + \alpha(t)a(g*u, u_t).\end{aligned} \tag{3.6}$$

Applying (3.1) to (3.6), we have

$$\begin{aligned}E'(t) &= -\mu_1\|u_t\|^2_{\Gamma_1} - \mu_2\bigl(u_t\bigl(t-\tau(t)\bigr), u_t\bigr)_{\Gamma_1} \\ &\quad - \frac{\alpha(t)}{2}g(t)a(u,u) + \frac{\alpha(t)}{2}g'\Box\partial^2 u + \frac{\alpha'(t)}{2}g\Box\partial^2 u \\ &\quad - \frac{\alpha'(t)}{2}\left(\int_0^t g(s)\,ds\right)a(u,u) + \frac{\zeta}{2}\|u_t\|^2_{\Gamma_1} \\ &\quad - \frac{\zeta}{2}e^{-\lambda\tau(t)}\bigl(1-\tau'(t)\bigr)\left\|u_t\bigl(t-\tau(t)\bigr)\right\|^2_{\Gamma_1} - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left\|u_t(s)\right\|^2_{\Gamma_1}\,ds.\end{aligned} \tag{3.7}$$

From Young's inequality, we obtain

$$-\mu_2\bigl(u_t\bigl(t-\tau(t)\bigr), u_t\bigr)_{\Gamma_1} \leq \frac{|\mu_2|}{2\sqrt{1-d}}\|u_t\|^2_{\Gamma_1} + \frac{|\mu_2|\sqrt{1-d}}{2}\left\|u_t\bigl(t-\tau(t)\bigr)\right\|^2_{\Gamma_1}. \tag{3.8}$$

By (2.6), we get

$$-\frac{\zeta}{2}e^{-\lambda\tau(t)}\bigl(1-\tau'(t)\bigr)\left\|u_t\bigl(t-\tau(t)\bigr)\right\|^2_{\Gamma_1} \leq -\frac{\zeta(1-d)}{2e^{\lambda\tau_1}}\left\|u_t\bigl(t-\tau(t)\bigr)\right\|^2_{\Gamma_1}. \tag{3.9}$$

Combining (3.7)-(3.9) and (H1), we have

$$\begin{aligned} E'(t) \le & -\left(\mu_1 - \frac{\zeta}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\right)\|u_t\|_{\Gamma_1}^2 - \left(\frac{\zeta(1-d)}{2e^{\lambda\tau_1}} - \frac{|\mu_2|\sqrt{1-d}}{2}\right)\left\|u_t\big(t-\tau(t)\big)\right\|_{\Gamma_1}^2 \\ & + \frac{\alpha(t)}{2} g'\Box\partial^2 u - \frac{\alpha'(t)}{2}\left(\int_0^t g(s)\,ds\right)a(u,u) - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left\|u_t(s)\right\|_{\Gamma_1}^2\,ds. \end{aligned}$$

By using condition (2.9), we obtain

$$\alpha_0 := \mu_1 - \frac{\zeta}{2} - \frac{|\mu_2|}{2\sqrt{1-d}} > 0 \quad \text{and} \quad \alpha_1 := \frac{\zeta(1-d)}{2e^{\lambda\tau_1}} - \frac{|\mu_2|\sqrt{1-d}}{2} > 0,$$

which implies the desired inequality (3.5). The proof is now complete. □

Lemma 3.3 *Under assumption* (2.2), *the functional* Φ *satisfies the estimate*

$$\begin{aligned} \Phi'(t) \le & -\frac{l}{2}a(u,u) + \|u_t\|^2 + \gamma\|\nabla u_t\|^2 \\ & + \frac{\alpha(t)}{4} g\Box\partial^2 u + \frac{\mu_1^2\tilde{C}_p}{l}\|u_t\|_{\Gamma_1}^2 + \frac{\mu_2^2\tilde{C}_p}{l}\left\|u_t\big(t-\tau(t)\big)\right\|_{\Gamma_1}^2. \end{aligned} \tag{3.10}$$

Proof By using (1.7) and (3.2), we get

$$\begin{aligned} \Phi'(t) = & \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - a(u,u) + \alpha(t)a(g*u,u) \\ & - \mu_1(u_t,u)_{\Gamma_1} - \mu_2\big(u_t\big(t-\tau(t)\big),u\big)_{\Gamma_1} \\ \le & \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - \left(1 - 2\alpha(t)\int_0^t g(s)\,ds\right)a(u,u) + \frac{\alpha(t)}{4}g\Box\partial^2 u \\ & - \mu_1(u_t,u)_{\Gamma_1} - \mu_2\big(u_t\big(t-\tau(t)\big),u\big)_{\Gamma_1}. \end{aligned} \tag{3.11}$$

From Young's inequality and (2.1), we see that, for any $\eta > 0$,

$$-\mu_1(u_t,u)_{\Gamma_1} \le \frac{\eta\tilde{C}_p}{2}a(u,u) + \frac{\mu_1^2}{2\eta}\|u_t\|_{\Gamma_1}^2, \tag{3.12}$$

$$-\mu_2\big(u_t\big(t-\tau(t)\big),u\big)_{\Gamma_1} \le \frac{\eta\tilde{C}_p}{2}a(u,u) + \frac{\mu_2^2}{2\eta}\left\|u_t\big(t-\tau(t)\big)\right\|_{\Gamma_1}^2. \tag{3.13}$$

Combining (2.2), (3.12) and (3.13) with (3.11) and choosing $\eta = \frac{l}{2\tilde{C}_p}$, we have (3.10). □

Lemma 3.4 *Under assumption* (2.2), *the functional* Ψ *satisfies the estimate*

$$\begin{aligned} \Psi'(t) \le & -\left(\int_0^t g(s)\,ds - \delta\right)\|u_t\|^2 - \gamma\left(\int_0^t g(s)\,ds - \delta\right)\|\nabla u_t\|^2 \\ & + \delta\left(1 + \left(\frac{1-l}{2}\right)^2\right)a(u,u) + \delta\|u_t\|_{\Gamma_1}^2 + \delta\left\|u_t\big(t-\tau(t)\big)\right\|_{\Gamma_1}^2 \\ & + \left(\alpha(t) + \frac{1}{2\delta} + \frac{\mu_1^2\tilde{C}_p}{4\delta} + \frac{\mu_2^2\tilde{C}_p}{4\delta}\right)\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u \\ & - \frac{g(0)(C_p + \gamma C_s)}{4\delta}g'\Box\partial^2 u. \end{aligned} \tag{3.14}$$

Proof Similarly, we find that

$$\begin{aligned}
\Psi'(t) = {} & \int_0^t g(t-s)a\big(u(t)-u(s),u(t)\big)\,ds \\
& - \alpha(t)\int_0^t g(t-s)a\left(u(t)-u(s),\int_0^t g(t-\tau)u(\tau)\,d\tau\right)ds \\
& + \mu_1\int_0^t g(t-s)\big(u(t)-u(s),u_t(t)\big)_{\Gamma_1}\,ds \\
& + \mu_2\int_0^t g(t-s)\big(u(t)-u(s),u_t\big(t-\tau(t)\big)\big)_{\Gamma_1}\,ds \\
& - \gamma\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s),\nabla u_t(t)\big)\,ds \\
& - \int_0^t g'(t-s)\big(u(t)-u(s),u_t(t)\big)\,ds \\
& - \gamma\left(\int_0^t g(s)\,ds\right)\|\nabla u_t\|^2 - \left(\int_0^t g(s)\,ds\right)\|u_t\|^2 \\
:= {} & I_1+\cdots+I_6-\gamma\left(\int_0^t g(s)\,ds\right)\|\nabla u_t\|^2-\left(\int_0^t g(s)\,ds\right)\|u_t\|^2.
\end{aligned} \tag{3.15}$$

Now, we estimate the terms on the right-hand side of (3.15). Young's inequality, (2.1) and (2.2) give that

$$\begin{aligned}
|I_1| & \le \delta a(u,u)+\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u, \\
|I_2| & \le \alpha(t)\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u+\alpha(t)\int_0^t g(t-s)\int_0^t g(t-\tau)a\big(u(t)-u(s),u(t)\big)\,d\tau\,ds \\
& \le \delta\left(\frac{1-l}{2}\right)^2 a(u,u)+\left(\alpha(t)+\frac{1}{4\delta}\right)\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u, \\
|I_3| & \le \delta\|u_t\|_{\Gamma_1}^2+\frac{\mu_1^2\tilde{C}_p}{4\delta}\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u, \\
|I_4| & \le \delta\big\|u_t\big(t-\tau(t)\big)\big\|_{\Gamma_1}^2+\frac{\mu_2^2\tilde{C}_p}{4\delta}\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u, \\
|I_5| & \le \gamma\delta\|\nabla u_t\|^2+\frac{\gamma}{4\delta}\int_\Omega\left(\int_0^t g'(t-s)\big|\nabla u(t)-\nabla u(s)\big|\,ds\right)^2 dx \\
& \le \gamma\delta\|\nabla u_t\|^2-\frac{g(0)\gamma C_s}{4\delta}g'\Box\partial^2 u, \\
|I_6| & \le \delta\|u_t\|^2+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g'(t-s)\big|u(t)-u(s)\big|\,ds\right)^2 dx \le \delta\|u_t\|^2-\frac{g(0)C_p}{4\delta}g'\Box\partial^2 u,
\end{aligned}$$

where $\delta > 0$. From the above estimates, we obtain (3.14). □

Lemma 3.5 *For $t_0 > 0$ and sufficiently large $N > 0$, there exist $k_1 > 0$, $k_2 > 0$ and $t_1 \ge t_0$ such that*

$$L'(t) \le -k_1\alpha(t)E(t)+k_2\alpha(t)g\Box\partial^2 u, \quad \forall t \ge t_1, \tag{3.16}$$

where k_1 and k_2 depend on g_0.

Proof By using (2.1) and Young's inequality, we get

$$\begin{aligned}\epsilon_1\alpha'(t)\Phi(t)+\epsilon_2\alpha'(t)\Psi(t) &\le -\alpha'(t)\|u_t\|^2-\alpha'(t)\gamma\|\nabla u_t\|^2-\frac{\alpha'(t)\epsilon_1^2}{2}(C_p+\gamma C_s)a(u,u)\\ &\quad -\frac{\alpha'(t)\epsilon_2^2}{2}(C_p+\gamma C_s)\left(\int_0^t g(s)\,ds\right)g\Box\partial^2 u. \qquad (3.17)\end{aligned}$$

Then, using (2.5), (3.3), (3.5), (3.10), (3.14) and (3.17), we have

$$\begin{aligned}L'(t) &\le -\alpha(t)\left((g_0-\delta)\epsilon_2-\epsilon_1+\frac{\alpha'(t)}{\alpha(t)}\right)\|u_t\|^2-\gamma\alpha(t)\left((g_0-\delta)\epsilon_2-\epsilon_1+\frac{\alpha'(t)}{\alpha(t)}\right)\|\nabla u_t\|^2\\ &\quad -\alpha(t)\left\{\frac{l\epsilon_1}{2}-\left(1+\left(\frac{1-l}{2}\right)^2\right)\delta\epsilon_2\right.\\ &\quad \left.+\frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\,ds\right)+\frac{\alpha'(t)\epsilon_1^2}{2\alpha(t)}(C_p+\gamma C_s)\right\}a(u,u)\\ &\quad -\alpha(t)\left(\frac{\alpha_0 N}{\alpha(0)}-\delta\epsilon_2-\frac{\mu_1^2\tilde{C}_p\epsilon_1}{l}\right)\|u_t\|_{\Gamma_1}^2\\ &\quad -\alpha(t)\left(\frac{\alpha_1 N}{\alpha(0)}-\delta\epsilon_2-\frac{\mu_2^2\tilde{C}_p\epsilon_1}{l}\right)\left\|u_t\left(t-\tau(t)\right)\right\|_{\Gamma_1}^2\\ &\quad +\alpha(t)\left[\frac{\epsilon_1\alpha(t)}{4}\right.\\ &\quad \left.+\left\{\epsilon_2\left(\alpha(t)+\frac{2+(\mu_1^2+\mu_2^2)\tilde{C}_p}{4\delta}\right)-\frac{\alpha'(t)\epsilon_2^2}{2\alpha(t)}(C_p+\gamma C_s)\right\}\left(\int_0^t g(s)\,ds\right)\right]g\Box\partial^2 u\\ &\quad +\alpha(t)\left(\frac{N}{2}-\frac{g(0)\epsilon_2}{4\delta}(C_p+\gamma C_s)\right)g'\Box\partial^2 u-\frac{\lambda\zeta N}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\|u_t(s)\|_{\Gamma_1}^2\,ds.\end{aligned}$$

We first choose $\delta>0$ so small that

$$\delta<\min\left\{\frac{g_0}{2},\frac{lg_0}{2(4+(1-l)^2)}\right\}.$$

Then, we obtain

$$g_0-\delta>\frac{1}{2}g_0 \quad\text{and}\quad \frac{2\delta}{l}\left(1+\left(\frac{1-l}{2}\right)^2\right)<\frac{1}{4}g_0.$$

Hence δ is fixed, the choice of any two positive constants ϵ_1 and ϵ_2 satisfying

$$\frac{g_0}{4}\epsilon_2<\epsilon_1<\frac{g_0}{2}\epsilon_2$$

will make

$$(g_0-\delta)\epsilon_2-\epsilon_1>0 \quad\text{and}\quad \frac{l\epsilon_1}{2}-\left(1+\left(\frac{1-l}{2}\right)^2\right)\delta\epsilon_2>0.$$

As long as δ, ϵ_1 and ϵ_2 are fixed, we take N large enough such that

$$\frac{\alpha_0 N}{\alpha(0)}-\delta\epsilon_2-\frac{\mu_1^2\tilde{C}_p\epsilon_1}{l}>0, \qquad \frac{\alpha_1 N}{\alpha(0)}-\delta\epsilon_2-\frac{\mu_2^2\tilde{C}_p\epsilon_1}{l}>0$$

and

$$\frac{N}{2} - \frac{g(0)\epsilon_2}{4\delta}(C_p + \gamma C_s) > 0.$$

Since $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$, we can take $t_1 \geq t_0$ sufficiently large so that

$$(g_0 - \delta)\epsilon_2 - \epsilon_1 + \frac{\alpha'(t)}{\alpha(t)} > 0,$$
$$\frac{l\epsilon_1}{2} - \left(1 + \left(\frac{1-l}{2}\right)^2\right)\delta\epsilon_2 + \frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\,ds\right) + \frac{\alpha'(t)\epsilon_1^2}{2\alpha(t)}(C_p + \gamma C_s) > 0.$$

Therefore, we get (3.16) for some positive constants k_1 and k_2 depending on g_0. □

Multiplying (3.16) by $\xi(t)$ and using (2.3) and (3.5), we find that

$$\begin{aligned} \xi(t)L'(t) &\leq -k_1\alpha(t)\xi(t)E(t) + k_2\alpha(t)\xi(t)g\Box\partial^2 u \\ &\leq -k_1\alpha(t)\xi(t)E(t) - k_2\alpha(t)g'\Box\partial^2 u \\ &\leq -k_1\alpha(t)\xi(t)E(t) - k_2\left(2E'(t) + \alpha'(t)\left(\int_0^t g(s)\,ds\right)a(u,u)\right), \quad \forall t \geq t_1. \end{aligned}$$

From $\xi'(t) \leq 0$, (2.2) and the definition of $E(t)$, we obtain

$$\begin{aligned} \big(\xi(t)L(t) + 2k_2E(t)\big)' &\leq -k_1\alpha(t)\xi(t)E(t) - k_2\alpha'(t)\left(\int_0^t g(s)\,ds\right)a(u,u) \\ &\leq -\left(k_1 + \frac{2k_2\alpha'(t)}{l\alpha(t)\xi(t)}\left(\int_0^t g(s)\,ds\right)\right)\alpha(t)\xi(t)E(t), \quad \forall t \geq t_1. \end{aligned}$$

By (2.4), we can choose $t_2 \geq t_1$ such that $k_1 + \frac{2k_2\alpha'(t)}{l\alpha(t)\xi(t)}(\int_0^t g(s)\,ds) > 0$ for $t \geq t_2$. Let $\mathcal{L}(t) = \xi(t)L(t) + 2k_2E(t)$, then from (3.4) we can see that $\mathcal{L}(t)$ is equivalent to $E(t)$. Then we deduce that

$$\mathcal{L}'(t) \leq -k\alpha(t)\xi(t)\mathcal{L}(t), \quad \forall t \geq t_2,$$

for some positive constant k depending on g_0, α and ξ. Integrating this over (t_2, t), we get

$$\mathcal{L}(t) \leq \mathcal{L}(t_2)e^{-k\int_{t_2}^t \alpha(s)\xi(s)\,ds}, \quad \forall t \geq t_2.$$

Using the equivalence of $\mathcal{L}(t)$ and $E(t)$ again, we have

$$E(t) \leq Ke^{-k\int_{t_2}^t \alpha(s)\xi(s)\,ds}, \quad \forall t \geq t_2,$$

for some positive constant K depending on the initial data.

Case 2: $\mu_1 = 0$, $|\mu_2| > 0$.

First, we define the Lyapunov function

$$F(t) := E(t) + \varepsilon_1\alpha(t)\Phi(t) + \varepsilon_2\alpha(t)\Psi(t), \tag{3.18}$$

where ε_1 and ε_2 are positive constants.

Similar to Case 1, from Lemma 3.1, we can obtain, for $\varepsilon_1, \varepsilon_2 > 0$ small enough,

$$\beta_1 E(t) \leq F(t) \leq \beta_2 E(t), \quad \forall t \geq 0, \tag{3.19}$$

where β_1 and β_2 are positive constants. From Lemma 3.2, we get

$$\begin{aligned} E'(t) \leq {} & \left(\frac{\zeta}{2} + \frac{|\mu_2|}{2\sqrt{1-d}}\right)\|u_t\|_{\Gamma_1}^2 + \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\zeta(1-d)}{2e^{\lambda\tau_1}}\right)\big\|u_t\big(t-\tau(t)\big)\big\|_{\Gamma_1}^2 \\ & + \frac{\alpha(t)}{2} g'\Box\partial^2 u - \frac{\alpha'(t)}{2}\left(\int_0^t g(s)\,ds\right) a(u,u) \\ & - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\big\|u_t(s)\big\|_{\Gamma_1}^2\,ds. \end{aligned} \tag{3.20}$$

Similar to Lemmas 3.3 and 3.4, we have

$$\Phi'(t) \leq -\frac{l}{2}a(u,u) + C_0\left(\|u_t\|^2 + \gamma\|\nabla u_t\|^2 + \big\|u_t\big(t-\tau(t)\big)\big\|_{\Gamma_1}^2\right) + \frac{\alpha(t)}{4} g\Box\partial^2 u, \tag{3.21}$$

where $C_0 = \max\{1, \frac{\mu_2^2\tilde{C}_p}{2l}\}$ and

$$\begin{aligned} \Psi'(t) \leq {} & -(g_0-\delta)\|u_t\|^2 - \gamma(g_0-\delta)\|\nabla u_t\|^2 \\ & + \delta\left(1 + \left(\frac{1-l}{2}\right)^2\right) a(u,u) + \delta\big\|u_t\big(t-\tau(t)\big)\big\|_{\Gamma_1}^2 \\ & + \left(\alpha(t) + \frac{1}{2\delta} + \frac{\mu_2^2\tilde{C}_p}{4\delta}\right)\left(\int_0^t g(s)\,ds\right) g\Box\partial^2 u - \frac{g(0)(C_p+\gamma C_s)}{4\delta} g'\Box\partial^2 u, \end{aligned} \tag{3.22}$$

respectively. By (3.18), (3.20)-(3.22) and (2.1), we obtain

$$\begin{aligned} F'(t) \leq {} & \alpha(t)\left(\varepsilon_1 C_0 - (g_0-\delta)\varepsilon_2 - \frac{\alpha'(t)}{\alpha(t)}\right)\|u_t\|^2 \\ & + \gamma\alpha(t)\left(\varepsilon_1 C_0 - (g_0-\delta)\varepsilon_2 + \frac{\tilde{C}_s}{\alpha(0)\gamma}\left(\frac{\zeta}{2} + \frac{|\mu_2|}{2\sqrt{1-d}}\right) - \frac{\alpha'(t)}{\alpha(t)}\right)\|\nabla u_t\|^2 \\ & + \alpha(t)\left\{\left(1 + \left(\frac{1-l}{2}\right)^2\right)\delta\varepsilon_2 - \frac{l\varepsilon_1}{2}\right. \\ & \left. - \frac{\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\,ds\right) - \frac{\alpha'(t)\varepsilon_1^2}{2\alpha(t)}(C_p + \gamma C_s)\right\} a(u,u) \\ & + \alpha(t)\left(\varepsilon_1 C_0 + \delta\varepsilon_2 + \frac{(1-d)}{\alpha(0)}\left(\frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\zeta}{2e^{\lambda\tau_1}}\right)\right)\big\|u_t\big(t-\tau(t)\big)\big\|_{\Gamma_1}^2 \\ & + \alpha(t)\left[\frac{\varepsilon_1\alpha(t)}{4}\right. \\ & \left. + \left\{\varepsilon_2\left(\alpha(t) + \frac{2+\mu_2^2\tilde{C}_p}{4\delta}\right) - \frac{\alpha'(t)\varepsilon_2^2}{2\alpha(t)}(C_p+\gamma C_s)\right\}\left(\int_0^t g(s)\,ds\right)\right] g\Box\partial^2 u \\ & + \alpha(t)\left(\frac{1}{2} - \frac{g(0)\varepsilon_2}{4\delta}(C_p+\gamma C_s)\right) g'\Box\partial^2 u - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\big\|u_t(s)\big\|_{\Gamma_1}^2\,ds. \end{aligned}$$

Now, we choose $\delta > 0$ small enough such that

$$\delta < \min\left\{\frac{g_0}{4}, \frac{lg_0}{2C_0(4 + (1-l)^2)}\right\}. \tag{3.23}$$

As long as δ is fixed, we take ε_2 such that

$$0 < \varepsilon_2 < \frac{2\delta}{g(0)(C_p + \gamma C_s)}.$$

Then we get

$$\frac{1}{2} - \frac{g(0)\varepsilon_2}{4\delta}(C_p + \gamma C_s) > 0. \tag{3.24}$$

From the choice of δ, we have

$$\frac{g_0}{4C_0} < \frac{g_0 - 2\delta}{2C_0}.$$

Then we select ε_1 such that

$$\frac{g_0\varepsilon_2}{4C_0} < \varepsilon_1 < \frac{(g_0 - 2\delta)\varepsilon_2}{2C_0}. \tag{3.25}$$

By (3.23) and (3.25), we obtain

$$\left(1 + \left(\frac{1-l}{2}\right)^2\right)\delta\varepsilon_2 - \frac{l\varepsilon_1}{2} < 0 \tag{3.26}$$

and

$$0 < \varepsilon_1 C_0 + \delta\varepsilon_2 < (g_0 - \delta)\varepsilon_2 - \varepsilon_1 C_0. \tag{3.27}$$

Now, we add a restriction condition on γ, that is, we suppose that

$$\frac{\tilde{C}_s}{1-d} < \gamma. \tag{3.28}$$

Note that $e^{\lambda\tau_1} \to 1$ as $\lambda \to 0$. Hence, if we take λ small enough, and from (3.27) and (3.28), there exists a positive constant ζ such that

$$\frac{2\alpha(0)e^{\lambda\tau_1}}{1-d}(\varepsilon_1 C_0 + \delta\varepsilon_2) < \zeta < \frac{2\alpha(0)\gamma}{\tilde{C}_s}\big((g_0 - \delta)\varepsilon_2 - \varepsilon_1 C_0\big).$$

And then, we see that

$$\frac{\zeta}{e^{\lambda\tau_1}} - \frac{2\alpha(0)}{1-d}(\varepsilon_1 C_0 + \delta\varepsilon_2) > 0$$

and

$$\frac{2\alpha(0)\gamma}{\tilde{C}_s}\big((g_0 - \delta)\varepsilon_2 - \varepsilon_1 C_0\big) - \zeta > 0.$$

If we choose $|\mu_2| > 0$ such that

$$|\mu_2| < \sqrt{1-d}\left(\min\left\{\frac{\zeta}{e^{\lambda\tau_1}} - \frac{2\alpha(0)}{1-d}(\varepsilon_1 C_0 + \delta\varepsilon_2), \frac{2\alpha(0)\gamma}{\tilde{C}_s}\big((g_0-\delta)\varepsilon_2 - \varepsilon_1 C_0\big) - \zeta\right\}\right)$$
$$=: \mu_0, \tag{3.29}$$

where μ_0 depends on g_0, we find that

$$\varepsilon_1 C_0 + \delta\varepsilon_2 + \frac{(1-d)}{\alpha(0)}\left(\frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\zeta}{2e^{\lambda\tau_1}}\right) < 0 \tag{3.30}$$

and

$$\varepsilon_1 C_0 - (g_0-\delta)\varepsilon_2 + \frac{\tilde{C}_s}{\alpha(0)\gamma}\left(\frac{\zeta}{2} + \frac{|\mu_2|}{2\sqrt{1-d}}\right) < 0. \tag{3.31}$$

Consequently, from Remark 2.1, (3.24), (3.26), (3.27), (3.30) and (3.31), there exist two positive constants k_3 and k_4 such that, for $t_1 \geq t_0$,

$$F'(t) \leq -k_3\alpha(t)E(t) + k_4\alpha(t)g\Box\partial^2 u, \quad \forall t \geq t_1, \tag{3.32}$$

where k_3 and k_4 depend on g_0. Multiplying (3.32) by $\xi(t)$ and using (2.3), (3.20), (3.31) and the definition of $E(t)$, we get, for $t \geq t_1$,

$$\begin{aligned}
\xi(t)F'(t) &\leq -k_3\alpha(t)\xi(t)E(t) - k_4\alpha(t)g'\Box\partial^2 u \\
&\leq -k_3\alpha(t)\xi(t)E(t) - 2k_4E'(t) + 2k_4\gamma\alpha(0)\big((g_0-\delta)\varepsilon_2 - \varepsilon_1 C_0\big)\|\nabla u_t\|^2 \\
&\quad - k_4\alpha'(t)\left(\int_0^t g(s)\,ds\right)a(u,u) \\
&\leq -k_3\alpha(t)\xi(t)E(t) - 2k_4E'(t) + 4k_4\alpha(0)\big((g_0-\delta)\varepsilon_2 - \varepsilon_1 C_0\big)E(t) \\
&\quad - \frac{2k_4\alpha'(t)}{l}\left(\int_0^t g(s)\,ds\right)E(t).
\end{aligned}$$

By $\xi'(t) \leq 0$, we have, for $t \geq t_1$,

$$\begin{aligned}
&\big(\xi(t)F(t) + 2k_4E(t)\big)' \\
&\quad \leq -\left(k_3 - \frac{4k_4\alpha(0)}{\alpha(t)\xi(t)}\big((g_0-\delta)\varepsilon_2 - \varepsilon_1 C_0\big) + \frac{2k_4\alpha'(t)}{l\alpha(t)\xi(t)}\left(\int_0^t g(s)\,ds\right)\right)\alpha(t)\xi(t)E(t).
\end{aligned}$$

Now, we add a restriction condition on α and ξ, that is, we assume that

$$\alpha(t)\xi(t) > \frac{4k_4\alpha(0)}{k_3}\big((g_0-\delta)\varepsilon_2 - \varepsilon_1 C_0\big) := \xi_0, \quad \forall t \geq t_1. \tag{3.33}$$

From (2.4), we can take $t_2 \geq t_1$ such that $k_3 - \frac{4k_4\alpha(0)}{\alpha(t)\xi(t)}((g_0-\delta)\varepsilon_2 - \varepsilon_1 C_0) + \frac{2k_4\alpha'(t)}{l\alpha(t)\xi(t)}(\int_0^t g(s)\,ds) > 0$ for $t \geq t_2$. Hence, there exists a positive constant k such that

$$\mathcal{F}'(t) \leq -k\alpha(t)\xi(t)\mathcal{F}(t), \quad \forall t \geq t_2,$$

where $\mathcal{F}(t) = \xi(t)F(t) + 2k_4E(t)$. From (3.19), we can see that $\mathcal{F}(t)$ is equivalent to $E(t)$. Integrating this over (t_2, t) and using the equivalence of $\mathcal{F}(t)$ and $E(t)$ again, we obtain (2.10). Then, we complete the proof.

Example If g decays exponentially, $\xi(t) = a$ and $\alpha(t) = \frac{b}{1+t} + c$, then (2.10) gives us

$$E(t) \le Ke^{-k(ab\ln(1+t)+act)},$$

where $a, b, c > 0$.

4 Conclusions

In the present paper, we consider a decay rate of the solutions for weak viscoelastic Kirchhoff plate equations with time-varying delay in the boundary. By introducing suitable energy and Lyapunov functions, we obtain a decay estimate for the energy, which depends on the behavior of both α and g. On the other hand, different from the previous literature, we use the memory term instead of the damping term to control the delay term.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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