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Existence of periodic solutions of a Liénard equation with a singularity of repulsive type

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Abstract

In this paper, the problem of positive periodic solutions is studied for the Liénard equation with a singularity of repulsive type,

$$x'' + f(x)x' - \frac{\alpha(t)}{x^\mu} = h(t),$$

where $f : (0, +\infty) \rightarrow R$ is continuous, α, h are continuous with T -periodic and $\alpha(t) \geq 0$ for all $t \in R$, $\mu \in (0, +\infty)$ is a constant. By means of a Manásevich-Mawhin's continuation theorem, a sufficient and necessary condition is obtained for the existence of positive T -periodic solutions of the equation. The interesting point is that the weak singularity of restoring force $\frac{\alpha(t)}{x^\mu}$ at $x = 0$ is allowed and f may have a singularity at $x = 0$.

Keywords: Liénard equation; Manásevich-Mawhin's continuation theorem; singularity; periodic solution

1 Introduction

In the past years, much attention from researchers in differential equations was paid to investigating the problem of periodic solutions for second order differential equations with singularities. This is due to the fact that the singularity has a significant background in applied sciences and physics (see [1–12] and the references therein). The first study on the periodic problem for second order singular differential equations seems to be the work of Nagumo [13] in 1943. After some work [14–16], the interest increased in this area with the pioneering paper of Lazer and Solimini [17]. They considered the existence of periodic solutions for the equation with a singularity of repulsive type,

$$x'' - \frac{1}{x^\alpha(t)} = h(t), \tag{1.1}$$

where $h : R \rightarrow R$ is continuous with T -periodic, $\alpha \in (0, +\infty)$ is a constant. For $\alpha \in [1, +\infty)$ (called the strong force condition), by using topological degree methods, they found that the necessary and sufficient condition for the existence of positive periodic solutions for equation (1.1) is

$$\bar{h} := \frac{1}{T} \int_0^T h(s) ds < 0.$$

For $\alpha \in (0,1)$ (weak singularity condition), they produced some examples of $h(t)$ with $\bar{h} < 0$ and such that equation (1.1) does not have any positive T -periodic solution. After that, the strong force condition $\alpha \geq 1$ was regarded as crucial assumption in [18–24]. By using some fixed point theorems in cones, the existence of periodic solutions has been widely studied recently for the conservative equation of repulsive type,

$$x'' + a(t)x - \frac{b(t)}{x^\alpha} = c(t), \tag{1.2}$$

where $a, b, c \in L^1[0, T]$ with $a(t) \geq 0, b(t) \geq 0$ for a.e. $t \in [0, T]$ and being positive in a set of positive measure [25–29]. Most of them focus on the case in which the singular nonlinearity was allowed to have a weak singularity ($\alpha \in (0,1)$). Compared with the study of conservative equations with weak singularities, the corresponding ones of the Liénard equation with a weak singularity of repulsive type is considerably neglected. We find that the strong singularity is needed in the most recent papers associated to singular Liénard equation of repulsive type [30–35]. For example, Jebelean and Mawhin in [11] considered the problem of the existence of positive periodic solutions for a p -Laplacian Liénard equation like

$$\left(|x'|^{p-2}x'\right)' + f(x)x' - \frac{\beta}{x^\mu} = h(t), \tag{1.3}$$

where $p > 1, \beta > 0, \mu > 0$ are constants, $f : [0, +\infty) \rightarrow \mathbb{R}$ is continuous, $h : \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic function with $h \in L^\infty[0, T]$. Under the condition of strong singularity $\mu \geq 1$, they found that the necessary and sufficient condition for the existence of positive periodic solutions for equation (1.3) is $\bar{h} < 0$. Wang in [32] further studied the existence of positive periodic solutions for a delay Liénard equation with a strong singularity ($\mu \in [1, +\infty)$) of repulsive type,

$$x'' + f(x)x' + a(t)x(t - \tau) - \frac{\beta}{x^\mu(t - \tau)} = h(t). \tag{1.4}$$

Hakl, Torres and Zamora in [33] considered the periodic problem for the singular equation of repulsive type,

$$u''(t) + f(u(t))u'(t) + \varphi(t)(u(t))^\delta + g(u(t)) = 0, \tag{1.5}$$

where $\delta \in (0,1]$ is a constant, φ is a T -periodic function with $\varphi \in L([0, T], \mathbb{R}), f \in C((0, +\infty), \mathbb{R})$ may be singular at $x = 0, g \in C((0, +\infty), \mathbb{R})$ has a repulsive singularity at $x = 0$, i.e., $g(x) \rightarrow -\infty$ as $x \rightarrow 0^+$. By using Schauder’s fixed point theorem, some results on the existence of positive T -periodic solutions are obtained. However, a strong singularity, $\int_0^1 g(s) ds = -\infty$, is also required.

Now, the question is how to study the periodic problem of equation (1.5) under the condition of weak singularity $\int_0^1 g(s) ds > -\infty$. Motivated by this, the purpose of this paper is to investigate the existence of positive T -periodic solutions for Liénard equation with a singularity of repulsive type

$$x'' + f(x)x' - \frac{\alpha(t)}{x^\mu} = h(t), \tag{1.6}$$

where $f : (0, +\infty) \rightarrow R$ is continuous, α, h are continuous T -periodic functions with $\alpha(t) \geq 0$ for all $t \in R$ and $\alpha(t) \not\equiv 0, \mu \in (0, +\infty)$ is a constant. By using a continuation theorem established by Mawhin and Manásevich [36], some new results are obtained. The interesting point is that the weak singularity of restoring force term $\frac{\alpha(t)}{x^\mu}$ at $x = 0$ is allowed and f may have a singularity at $x = 0$. Furthermore, under the condition of $\int_0^1 f(s) ds = \infty$, a sufficient and necessary condition is obtained for the existence of positive T -periodic solutions of equation (1.6).

2 Preliminary lemmas

Let $C_T = x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t)$ for all $t \in \mathbb{R}$ with the norm defined by $|x|_\infty = \max_{t \in [0, T]} |x(t)|$. For any T -periodic solution $h(t)$ with $h \in C_T, h_+(t)$ and $h_-(t)$ is denoted by $\max\{(h(t), 0)\}$ and $-\min\{(h(t), 0)\}$, respectively, and $\bar{h} = \frac{1}{T} \int_0^T h(s) ds$. Clearly, $h(t) = h_+(t) - h_-(t)$ for all $t \in \mathbb{R}$, and $\bar{h} = \bar{h}_+ - \bar{h}_-$. Furthermore, $\|\varphi\|_p := (\int_0^T |\varphi(t)|^p dt)^{\frac{1}{p}}, p \in [1, +\infty), \varphi \in C_T$.

The following lemma is a corollary of Theorem 3.1 in [36].

Lemma 2.1 *Assume that there exist positive constants M_0, M_1 and M_2 with $0 < M_0 < M_1$, such that the following conditions hold:*

1. *for each $\lambda \in (0, 1]$, each possible positive T -periodic solution x to the equation*

$$u'' + \lambda f(u)u' - \lambda \frac{\alpha(t)}{u^\mu} = \lambda h(t),$$

satisfies the inequalities $M_0 < x(t) < M_1$ and $|x'(t)| < M_2$ for all $t \in [0, T]$;

2. *each possible solution $x \in (0, +\infty)$ to the equation*

$$\frac{\bar{\alpha}}{x^\mu} + \bar{h} = 0$$

satisfies the inequality $M_0 < x < M_1$;

3. *the inequality*

$$\left(\frac{\bar{\alpha}}{M_0^\mu} + \bar{h}\right) \left(\frac{\bar{\alpha}}{M_1^\mu} + \bar{h}\right) < 0$$

holds. Then equation (1.6) has at least one positive T -periodic solution $x(t)$ such that $M_0 < x(t) < M_1$ for all $t \in [0, T]$.

Lemma 2.2 ([29]) *Let $x(t)$ be a continuously differentiable T -periodic function. Then, for any $\tau \in [0, T]$,*

$$\left(\int_0^T x^2(t) dt\right)^{\frac{1}{2}} \leq \frac{T}{\pi} \left(\int_0^T x'^2(t) dt\right)^{\frac{1}{2}} + \sqrt{T}|x(\tau)|.$$

In order to study the existence of positive periodic solutions to equation (1.6), we list the following assumptions:

(H₁) $\lim_{x \rightarrow 0^+} |\int_x^1 f(s) ds| = +\infty$.

(H₂) $(-\frac{\bar{\alpha}}{h})^{\frac{1}{\mu}} > T^{\frac{1}{2}} [\frac{T}{\pi} \|h\|_2 + (T^{\frac{1}{2}} (-\frac{\bar{\alpha}}{h})^{\frac{1}{\mu}} \|h\|_2)^{\frac{1}{2}}]$.

Remark 2.1 If $\bar{h} < 0$, then there are constants D_1 and D_2 with $0 < D_1 < D_2$ such that

$$\frac{\bar{\alpha}}{x^\mu} + \bar{h} > 0 \quad \forall x \in (0, D_1)$$

and

$$\frac{\bar{\alpha}}{x^\mu} + \bar{h} < 0 \quad \forall x \in (D_2, \infty).$$

Now, we embed equation (1.6) into the following equations family with a parameter $\lambda \in (0, 1]$,

$$x'' + \lambda f(x)x' - \lambda \frac{\alpha(t)}{x^\mu} = \lambda h(t), \quad \lambda \in (0, 1]. \tag{2.1}$$

3 Main results

Theorem 3.1 *If (H_1) holds, then equation (1.6) has a positive T -periodic solution if and only if $\bar{h} < 0$.*

Proof Suppose that equation (1.6) has a positive T -periodic solution $y(t)$, then

$$y'' + f(y)y' - \frac{\alpha(t)}{y^\mu} = h(t). \tag{3.1}$$

Integrating (3.1) on the interval $[0, T]$, and by using

$$\int_0^T y''(s) ds = \int_0^T f(y(s))y'(s) ds = 0,$$

we have

$$\int_0^T \frac{\alpha(s)}{y^\mu(s)} ds = -T\bar{h},$$

which together with the assumption of $\alpha(t) \geq 0$ and $y(t) > 0$ for all $t \in [0, T]$ gives a necessary condition for the existence of a positive T -periodic solution of equation (1.6): $\bar{h} < 0$.

Below, we will show the proof of sufficiency. In order to do it, suppose that $\bar{h} < 0$, and let u be an arbitrary positive T -periodic solution of (2.1). Then

$$u'' + \lambda f(u)u' - \lambda \frac{\alpha(t)}{u^\mu} = \lambda h(t), \quad \lambda \in (0, 1]. \tag{3.2}$$

Integrating (3.2) over the interval $[0, T]$, we have

$$\int_0^T \frac{\alpha(t)}{u^\mu} dt = - \int_0^T h(t) dt = -T\bar{h}. \tag{3.3}$$

Due to the fact that $\alpha(t)$ is non-negative, $\frac{1}{u_M^\mu} \int_0^T \alpha(t) dt \leq \int_0^T \frac{\alpha(t)}{u^\mu(t)} dt \leq \frac{1}{u_m^\mu} \int_0^T \alpha(t) dt$, where u_m, u_M are the global minimum and maximum, respectively, of u . Then there is a point $\eta \in [0, T]$ such that

$$\frac{1}{u^\mu(\eta)} \int_0^T \alpha(t) dt = -T\bar{h},$$

which results in

$$T\bar{\alpha} = -u^\mu(\eta)T\bar{h},$$

and then

$$u(\eta) = \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}}. \tag{3.4}$$

Multiplying (3.2) with $u(t)$, and integrating it over the interval $[0, T]$, we obtain

$$\int_0^T u''(t)u(t) dt - \lambda \int_0^T \frac{\alpha(t)u(t)}{u^\mu(t)} dt = \lambda \int_0^T h(t)u(t) dt.$$

By using $\int_0^T u''(t)u(t) dt = -\int_0^T |u'(t)|^2 dt$, we have

$$\int_0^T |u'(t)|^2 dt + \lambda \int_0^T \alpha(t)u^{1-\mu}(t) dt = -\lambda \int_0^T h(t)u(t) dt,$$

which together with the fact of $\alpha(t) \geq 0$ for all $t \in [0, T]$ and $\alpha(t) \not\equiv 0$ gives

$$\int_0^T |u'(t)|^2 dt < \int_0^T |h(t)|u(t) dt \leq \left(\int_0^T h^2(t) dt\right)^{\frac{1}{2}} \left(\int_0^T u^2(t) dt\right)^{\frac{1}{2}}. \tag{3.5}$$

By Lemma 2.2, we have

$$\left(\int_0^T u^2(t) dt\right)^{\frac{1}{2}} < \frac{T}{\pi} \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} + \sqrt{T}u(\eta).$$

It follows from (3.4) that

$$\left(\int_0^T u^2(t) dt\right)^{\frac{1}{2}} < \frac{T}{\pi} \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} + \sqrt{T} \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}}.$$

Substituting it into (3.5), we have

$$\begin{aligned} \int_0^T |u'(t)|^2 dt &< \left(\int_0^T h^2(t) dt\right)^{\frac{1}{2}} \left[\frac{T}{\pi} \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} + \sqrt{T} \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} \right] \\ &= \frac{T}{\pi} \left(\int_0^T h^2(t) dt\right)^{\frac{1}{2}} \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} + T^{\frac{1}{2}} \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} \left(\int_0^T h^2(t) dt\right)^{\frac{1}{2}}, \end{aligned}$$

by the inequality $X^2 - AX - B < 0$, we can obtain $X < A + B^{\frac{1}{2}}$. Let $X = \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}}$, $A = \frac{T}{\pi} \left(\int_0^T h^2(t) dt\right)^{\frac{1}{2}}$, $B = T^{\frac{1}{2}} \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} \left(\int_0^T h^2(t) dt\right)^{\frac{1}{2}}$, we have

$$\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} < \frac{T}{\pi} \|h\|_2 + T^{\frac{1}{4}} \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{2\mu}} \|h\|_2^{\frac{1}{2}} := \rho_1. \tag{3.6}$$

Combined (3.4) with (3.6), we have

$$u(t) \leq u(\eta) + \sqrt{T} \|u'\|_2 < \left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} + T^{\frac{1}{2}} \rho_1 := M. \tag{3.7}$$

Let t_1, t_2 be the maximum point and the minimum point of $u(t)$ on $[0, T]$, respectively, then

$$\int_{t_1}^{t_2} u''(t) dt + \lambda \int_{t_1}^{t_2} f(u(t))u'(t) dt - \lambda \int_{t_1}^{t_2} \frac{\alpha(t)}{u^\mu(t)} dt = \lambda \int_{t_1}^{t_2} h(t) dt,$$

which together with $u'(t_1) = u'(t_2) = 0$ yields

$$F(u(t_2)) - F(u(t_1)) = \int_{t_1}^{t_2} \frac{\alpha(t)}{u^\mu(t)} dt + \int_{t_1}^{t_2} h(t) dt,$$

where $F(x) = \int_1^x f(s) ds$, and then

$$|F(u(t_2))| \leq |F(u(t_1))| + \int_0^T \frac{\alpha(t)}{u^\mu(t)} dt + \int_0^T |h(t)| dt.$$

It follows from (3.4) and (3.7) that

$$\begin{aligned} |F(u(t_2))| &\leq \max_{\left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} \leq z \leq M} |F(z)| - T\bar{h} + T|\bar{h}| \\ &\leq \max_{\left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} \leq z \leq M} |F(z)| + 2T\bar{h}_-. \end{aligned} \tag{3.8}$$

It is easy to see from (H_1) that there is a constant $\gamma_0 > 0$, such that

$$|F(z)| = \left| \int_1^z f(s) ds \right| > \max_{\left(-\frac{\bar{\alpha}}{\bar{h}}\right)^{\frac{1}{\mu}} \leq z \leq M} |F(z)| + 2T\bar{h}_- \quad \text{for } z \in (0, \gamma_0].$$

By (3.8), we have

$$u(t_2) > \gamma_0,$$

and then

$$u(t) > \gamma_0 \quad \text{for all } t \in [0, T]. \tag{3.9}$$

Next, if u attains its maximum over $[0, T]$ at $t_1 \in [0, T]$, then $u'(t_1) = 0$ and we see from (3.2) that

$$u'(t) = \lambda \int_{t_1}^t \left[-f(u)u' + \frac{\alpha(t)}{u^\mu} + h(t) \right] dt \quad \text{for all } t \in [t_1, t_1 + T].$$

Therefore,

$$\begin{aligned} |u'(t)| &\leq \lambda |F(u(t)) - F(u(t_1))| + \lambda \int_{t_1}^{t_1+T} \frac{\alpha(t)}{u^\mu(t)} dt + \lambda \int_{t_1}^{t_1+T} h_+(s) ds \\ &\leq 2\lambda \max_{\gamma_0 \leq u \leq M} |F(u)| + \lambda \int_0^T \frac{\alpha(t)}{u^\mu(t)} dt + T\bar{h}_+. \end{aligned} \tag{3.10}$$

Substituting (3.3) into (3.10), we have

$$|u'(t)| \leq 2 \max_{\gamma_0 \leq u \leq M} |F(u)| - T\bar{h} + T\bar{h}_+ := M^* \quad \text{for all } t \in [0, T]$$

and then

$$|u'(t)| \leq M^* \quad \text{for all } t \in [0, T]. \tag{3.11}$$

From (3.7), (3.9), (3.11) and Remark 2.1, we can choose $M_0 := \min\{\gamma_0, D_1\}$ where D_1 is determined in Remark 2.1, $M_1 = M$ and $M_2 = M^*$ such that all the conditions of Lemma 2.1 are satisfied. Thus, by using Lemma 2.1, we see that equation (1.6) has at least one positive T -periodic solution. The proof is complete. \square

Theorem 3.2 *Suppose that $\bar{h} < 0$ and (H_2) holds. Then equation (1.6) has at least one positive T -periodic solution.*

Proof Suppose that u is an arbitrary positive T -periodic solution of equation (2.1), then

$$u'' + \lambda f(u)u' - \lambda \frac{\alpha(t)}{u^\mu} = \lambda h(t), \quad \lambda \in (0, 1]. \tag{3.12}$$

Similar to the proof of (3.4) and (3.6), we find that there is a point $\xi \in [0, T]$ such that

$$u(\xi) = \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} \tag{3.13}$$

and

$$\left(\int_0^T |u'(t)|^2 dt \right)^{\frac{1}{2}} < \frac{T}{\pi} \|h\|_2 + T^{\frac{1}{4}} \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{2\mu}} \|h\|_2^{\frac{1}{2}} := \rho_1. \tag{3.14}$$

In view of the inequality

$$\begin{aligned} u(t) &= u(\xi) + \int_\xi^t u'(s) ds \leq u(\xi) + \int_\xi^{\xi+T} |u'(s)| ds \\ &= u(\xi) + \int_0^T |u'(s)| ds, \quad t \in [\xi, \xi + T], \end{aligned}$$

and by using (3.13), together with Schwarz inequality, we have

$$\max_{t \in [0, T]} u(t) \leq \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} + T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds \right)^{\frac{1}{2}} \tag{3.15}$$

and

$$\min_{t \in [0, T]} u(t) \geq u(\xi) - \left| \int_{\xi}^t u'(s) ds \right| \geq \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} - T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds \right)^{\frac{1}{2}}. \tag{3.16}$$

Substituting (3.14) into (3.15) and (3.16), respectively, we have

$$\max_{t \in [0, T]} u(t) \leq \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} + T^{\frac{1}{2}} \left[\frac{T}{\pi} \|h\|_2 + T^{\frac{1}{4}} \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{2\mu}} \|h\|_2^{\frac{1}{2}} \right] \tag{3.17}$$

and

$$\min_{t \in [0, T]} u(t) \geq \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} - T^{\frac{1}{2}} \left[\frac{T}{\pi} \|h\|_2 + T^{\frac{1}{4}} \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{2\mu}} \|h\|_2^{\frac{1}{2}} \right]. \tag{3.18}$$

The rest of the proof works almost analogously to the corresponding ones of Theorem 3.1. □

Example 3.1 Considering the following equation:

$$x''(t) + \frac{x'(t)}{x^2(t)} - \frac{\sin^2 t}{x^{\frac{2}{3}}} = -1 + \cos t. \tag{3.19}$$

Corresponding to equation (1.6), we have $f(x) = \frac{1}{x^2}$, $\mu = \frac{2}{3}$, $\alpha(t) = \sin^2 t$, $h(t) = -1 + \cos t$, and then $T = 2\pi$, $\int_0^1 f(s) ds = +\infty$. This implies that assumption (H₁) holds. Since $\bar{h} = -1 < 0$, by using Theorem 3.1, we find that equation (3.19) has at least one positive 2π -periodic solution.

Example 3.2 Considering the following equation:

$$x''(t) + \frac{x'(t)}{x^{\frac{1}{2}}(t)} - \frac{\sin^2 8t}{x^{\frac{3}{4}}} = -\cos^2 8t. \tag{3.20}$$

Corresponding to equation (1.6), f can be regarded as $f(x) = \frac{1}{x^{\frac{1}{2}}}$, $\mu = \frac{3}{4}$, $\alpha(t) = \sin^2 8t$ and $h(t) = -\cos^2 8t$. Since $\int_0^1 f(s) ds = 2$, it follows that assumption (H₁) does not hold. This implies that Theorem 3.1 cannot be used to study the existence of periodic solutions to (3.20). But, by simple calculating, we can verify that

$$-\frac{\bar{h}}{\bar{\alpha}} = 1, \quad \|h\|_2^2 = \frac{3T}{8},$$

where $T = \frac{\pi}{8}$, and then

$$\left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} - T^{\frac{1}{2}} \left[\frac{T}{\pi} \|h\|_2 + \left(T^{\frac{1}{2}} \left(-\frac{\bar{\alpha}}{h} \right)^{\frac{1}{\mu}} \|h\|_2 \right)^{\frac{1}{2}} \right] = 1 - \left(\frac{\sqrt{6}\pi}{256} + \frac{\sqrt[4]{6}\pi}{16} \right) > 0,$$

which implies that assumption (H₂) holds. Thus, by using Theorem 3.2, we find that (3.20) has at least one positive $\frac{\pi}{8}$ -periodic solution.

Remark The above two examples can neither be studied by using the results in [31, 32, 34] and [35], since $f(x)$ in (3.19) and in (3.20) are all singular at $x = 0$, nor be studied by using the results in [33], since the restoring force terms of $\frac{\sin^2 t}{x^{\frac{2}{3}}}$ in (3.19) and $\frac{\sin^2 8t}{x^{\frac{2}{4}}}$ in (3.20) have weak singularities at $x = 0$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors read and approved the final manuscript.

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