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Research Article

Interior Gradient Estimates for Nonuniformly Parabolic Equations II

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We prove interior gradient estimates for a large class of parabolic equations in divergence form. Using some simple ideas, we prove these estimates for several types of equations that are not amenable to previous methods. In particular, we have no restrictions on the maximum eigenvalue of the coefficient matrix and we obtain interior gradient estimates for so-called false mean curvature equation.

1. Introduction

A key step in the study of second-order quasilinear parabolic equations is establishing suitable *a priori* estimates for any solution of the equation. This fact is the theme of many books on the subject [1–5] and our focus here is on one particular such estimate: a local pointwise gradient estimate for solutions of equations in divergence form:

$$u_t = \operatorname{div} A(X,u,Du) + B(X,u,Du). \tag{1.1}$$

The role of this divergence structure has been noted many times under varying hypotheses on the functions A and B (see, in particular [6, Sections VIII.4 and VIII.5], [3, Section V.4], [5, Section 11.5]). Our current interest is deriving this estimate using a surprising variant (detailed below) of standard methods. Although this variant seems, at first, to be a purely technical modification, we mention here two quite different types of estimates which follow from this variant and which appear to be new. First, we derive a local gradient estimate for a class of equations which includes the parabolic false mean curvature equation, that is, the equation with

$$A(X,z,p) = \exp\left(\frac{1}{2}(1+|p|^2)\right)p \tag{1.2}$$

and some conditions on B. Such an operator does not fall under the hypotheses from, for example, [3], and the present author has, previously, given an incorrect proof of this estimate [7, page 569] (we will point out the error later), and then in [5, Section 11.5, page 281] a correct but weaker version of the estimate. Second, we estimate the gradient of a solution to a large class of equations only in terms of the structure of the equation and a bound for the gradient of the initial function. (Ordinarily, a gradient estimate is given in terms of a maximum estimate for the solution, which, in turn, depends on some estimate on the boundary and initial data.) Such an estimate was first proved by Ecker for the parabolic prescribed mean curvature equation [8, Theorem 3.1], but we also show that such an estimate is valid for the parabolic p-Laplacian if $p < 2$, and this fact seems to be new. (In [9], a corresponding estimate was given for the L^q norm of the solution in terms of the L^q norm of the initial data, and this estimate can be used to infer a gradient estimate, but our goal here is to give an estimate directly.) This gradient estimate provides an interesting counterpoint to known results on these equations (see [6, Chapter XII] for a detailed description of these results). In particular, it is known that for $p > 2n/(n+1)$, solutions of this equation are bounded (and have Hölder continuous spatial derivatives) at any positive time for quite general initial data, in particular for L^1 initial data. On the other hand, [6, Section XII.13-(i)] provides an initial datum in L^1 for which the solution is unbounded for all sufficiently small positive time. Although the counterexample is described in all of $\mathbb{R}^n \times (0,\infty)$, it should be noted that it satisfies the boundary condition $u = 0$ on $\{|x| = 1, t > 0\}$, so the regularity of the solution is affected only by that of the initial datum. An important point for our comparison is that the solution becomes infinite only at $x = 0$ (for $t > 0$ as well) and the initial function is smooth except at $x = 0$. Our result shows that this is the only configuration in which the solution can be unbounded since we obtain a gradient estimate at any $x \neq 0$. Of course, the additional surprise is that our gradient estimate also applies to some equations with $p > 2n/(n+1)$.

The basic plan is to modify the Moser iteration technique [10] along the lines of Simon's estimate for elliptic equations [11]. Of course, this is the plan followed by the author before (especially [7]) but we add two important new twists. As in [12], we obtain an estimate that does not use an upper bound on the maximum eigenvalue of the matrix $\partial A/\partial p$. Such an approach is also useful in studying anisotropic problems (see [13, 14]) and we present the calculations for this case in [15]. In addition, we use a modified version of the Sobolev inequality from [11]. This inequality will allow us to prove some unusual estimates (in particular the estimates for parabolic p-Laplace equations) and also to use some more standard notations, in particular, we will use a^{ij} to denote the components of the matrix $\partial A/\partial p$; in [7, 11, 16], a^{ij} denoted the components of a slightly different matrix.

Following [11], we break the estimate into several steps. After giving some notation in Section 2, we prove an energy-type inequality in Section 3. We then present the Sobolev inequality in Section 4, and we use the energy inequality along with the Sobolev inequality in Section 5 to bound the maximum of the gradient in terms of an integral:

$$\int w(|Du|)^q Du \cdot A\, dX \tag{1.3}$$

for some function w and some exponent q, which we will detail in that section. This integral is estimated in Section 6 in terms of the integral of $Du \cdot A$, and this final integral is easily estimated; we will quote [5, Lemma 11.13]. Section 7 contains some examples, especially the false mean curvature equation, to illustrate our structure conditions. We also discuss some interesting variants of our estimate. In Section 8, we examine the application of our Sobolev inequality to some equations satisfying structure conditions depending on the maximum eigenvalue of $\partial A/\partial p$; the most important of such equations are the parabolic prescribed mean curvature equation and parabolic p-Laplacian with $p < 2$ described above. Finally, we look at parabolic equations with faster than exponential growth in Section 9; our method is only partially successful in dealing with such problems.

2. Notation

For the most part, we follow the notation in [5], so $X = (x,t)$ denotes a point in $\mathbb{R}^{n+1}$ with

$$|X| = \left(\sum_{i=1}^{n} (x^i)^2 + |t|\right)^{1/2}, \tag{2.1}$$

and, for $R > 0$, we write

$$\begin{aligned} Q(R) &= \{X \in \mathbb{R}^{n+1} : |x| < R,\, -R^2 < t < 0\}, \\ B(R) &= \{x \in \mathbb{R}^n : |x| < R\}. \end{aligned} \tag{2.2}$$

We also use $\mathcal{P}Q(R)$ to denote the parabolic boundary of $Q(R)$, that is, the set of X such that either

$$|x| = R, \qquad -R^2 \le t \le 0, \tag{2.3}$$

or

$$|x| < R, \qquad t = -R^2. \tag{2.4}$$

Moreover, we use N to denote n if $n > 2$ and an arbitrary constant greater than 2 if $n = 2$.

We always assume that $u \in C^{2,1}(Q(R))$ for some $R > 0$ and we set

$$v = \left(1 + |Du|^2\right)^{1/2}, \qquad \nu = \frac{Du}{v}, \qquad g^{ij} = \delta^{ij} - \nu^i \nu^j. \tag{2.5}$$

We will also use this notation, without further comment, with p in place of Du to describe structural conditions on the functions A and B (and their derivatives). We also set

$$a^{ij} = \frac{\partial A^i}{\partial p_j}, \qquad \mathcal{C}^2 = a^{ij} g^{km} D_{ik}u D_{jm}u, \qquad \mathcal{E} = a^{ij} D_i v D_j v, \tag{2.6}$$

where we use the Einstein summation convention that repeated indices are summed from 1 to n. (Note that a^{ij}, $\mathscr{C}^2$, and $\mathscr{E}$ are not quite the same as in [7, 11, 16].)

We also define the oscillation of u over a set S by

$$\operatorname*{osc}_S u = \sup_S u - \inf_S u. \tag{2.7}$$

In addition, for parameters $\tau > 1$ and $r \in (0,R]$, we write $Q_\tau(r)$ and $q_\tau(r,t)$ for the subsets of $Q(r)$ and $B(r) \times \{t\}$, respectively, on which $v > \tau$.

3. The energy inequality

In this section, we prove an energy inequality, that is, an inequality which estimates integrals involving second spatial derivatives of u in terms of integrals involving only first derivatives. Before stating this inequality, we present some preliminary structure conditions. We suppose that there are matrices $[C_k^i]$ and $[D_k^i]$ such that D_k^i is differentiable with respect to (x,z,p) and

$$C_k^i + D_k^i = \frac{\partial A^i}{\partial z} p_k + \frac{\partial A^i}{\partial x^k} + B\delta_k^i. \tag{3.1}$$

For simplicity, we set

$$\mathscr{D}^{ij} = v^k \frac{\partial D_k^i}{\partial p_j}, \qquad \mathscr{F} = \left(p_i \frac{\partial D_k^i}{\partial z} + \frac{\partial D_k^i}{\partial x^i}\right) v^k. \tag{3.2}$$

Our structure conditions are stated in terms of these expressions. We assume that there are nonnegative constants $\tau_0 \geq 1$, β_1, and β_2 along with positive functions Λ_0, Λ_1, and Λ_2 such that

$$C_k^i g^{jk} \eta_{ij} \leq \beta_1 \Lambda_0^{1/2} \left(a^{ij} \eta_{ik} \eta_{jk}\right)^{1/2}, \tag{3.3a}$$

$$C_k^i v^k \xi_i \leq \beta_1 \Lambda_0^{1/2} \left(a^{ij} \xi_i \xi_j\right)^{1/2}, \tag{3.3b}$$

$$v \mathscr{D}^{ij} \eta_{ij} \leq \beta_1 \Lambda_0^{1/2} \left(a^{ij} \eta_{ik} \eta_{jk}\right)^{1/2}, \tag{3.3c}$$

$$v \mathscr{F} \leq \beta_1^2 \Lambda_0, \tag{3.3d}$$

$$v \left| v^k D_k^i - v^i B \right| \leq \beta_1 \Lambda_1, \tag{3.3e}$$

$$|A| g^{ij} \eta_{ij} \leq \beta_2 \Lambda_2^{1/2} \left(a^{ij} \eta_{ik} \eta_{jk}\right)^{1/2}, \tag{3.3f}$$

$$|A| v \cdot \xi \leq \beta_2 \Lambda_2^{1/2} \left(a^{ij} \xi_i \xi_j\right)^{1/2}, \tag{3.3g}$$

for all $n \times n$ matrices η, all n-vectors ξ, and all $(X,z,p) \in Q(R) \times \mathbb{R} \times \mathbb{R}^n$ such that $z = u(X)$ and $v > \tau_0$. Note that conditions (3.3a)–(3.3d) are exactly the same as [5, (11.41a–d)] (except for a slight variation in notation).

Our energy estimate is then a variant of [5, Lemma 11.10] (which in turn comes from [11, (2.11)]).

Lemma 3.1. *Let χ be an increasing, nonnegative Lipschitz function defined on $[\tau,\infty)$ for some $\tau \geq \tau_0$ and let ζ be a nonnegative $C^{2,1}(\overline{Q(R)})$ function which vanishes in a neighborhood of $\mathcal{P}Q(R)$. Suppose conditions (3.3) hold, and define*

$$\Xi(\sigma) = \int_\tau^\sigma (\xi-\tau)\chi(\xi)\,d\xi. \tag{3.4}$$

Then

$$\begin{aligned}
&\int_{q_\tau(R,s)} \Xi(v)\zeta^2 dx + \int_{Q_\tau(R)} \left[\left(1-\frac{\tau}{v}\right)\mathscr{C}^2+\mathscr{E}\right]\chi\zeta^2 dX \\
&\quad\leq 20\beta_1^2\int_{Q_\tau(R)} \Lambda_0\big((v-\tau)\chi'+\chi\big)\zeta^2 dX + 4\beta_1\int_{Q_\tau(R)}\Lambda_1\chi\zeta|D\zeta|\,dX \\
&\qquad + 4\int_{Q_\tau(R)}|A|\chi\big[\,|D^2\zeta|\,\zeta+|D\zeta|^2\big]v\,dX + 32\beta_2^2\int_{Q_\tau(R)}\Lambda_2\big((v-\tau)\chi'+\chi\big)|D\zeta|^2 dX \\
&\qquad + 4\int_{Q_\tau(R)}\Xi\zeta\zeta_t\,dX
\end{aligned} \tag{3.5}$$

for any $s\in(-R^2,0)$. (Here, and in what follows, the argument v from χ and Ξ is suppressed.)

Proof. We begin just as in [5, Lemma 11.10]. Let θ be a vector-valued C^2 function which vanishes in a neighborhood of $\mathcal{P}Q(R)$, and set $Q = B(R)\times(-R^2,s)$. If we multiply the differential equation by $\operatorname{div}\theta$ and then integrate by parts, we obtain

$$\int_Q \big[-u_t D_k\theta^k + D_kA^iD_i\theta^k + BD_k\theta^k\big]dX = 0. \tag{3.6}$$

An easy approximation argument shows that this identity holds for any θ which is only Lipschitz (with respect to x only); in particular, we take

$$\theta = (v-\tau)_+\chi(v)\zeta^2\nu. \tag{3.7}$$

Just as in [5, pages 270-271], we see that

$$\int_Q -u_tD_k\theta^k dX = \int_{q_\tau(R,s)}\Xi\zeta^2 dx - 2\int_Q \Xi\zeta\zeta_t\,dX. \tag{3.8}$$

Next, we have

$$\begin{aligned}
\int_Q D_kA^iD_i\theta^k + BD_k\theta^k dX &= \int_Q \big[D_kA^i + B\delta_k^i\big]D_i\big((v-\tau)_+\chi\nu^k\big)\zeta^2 dX \\
&\quad + \int_Q D_kA^i(v-\tau)_+\chi\nu^kD_i(\zeta^2)\,dX \\
&\quad + \int_Q B(v-\tau)_+\chi\nu\cdot D(\zeta^2)\,dX.
\end{aligned} \tag{3.9}$$

The first integral is handled as usual. We set

$$\Psi = \begin{cases} (\nu-\tau)\chi' + \chi & \text{if } \nu > \tau, \\ 0 & \text{if } \nu \le \tau, \end{cases} \tag{3.10}$$

and we note that

$$D_i[(\nu-\tau)\chi\nu^k] = \Psi D_i\nu\nu^k + \left(1-\frac{\tau}{\nu}\right)_+ \chi g^{kj} D_{ij}u. \tag{3.11}$$

It follows that

$$\begin{aligned}[D_kA^i + B\delta^i_k]D_i((\nu-\tau)_+\chi\nu^k) = {} & \left(1-\frac{\tau}{\nu}\right)_+ \chi(\mathscr{C}^2 + C^i_k g^{kj}D_{ij}u) \\ & + \Psi(\mathscr{E} + \nu^k C^i_k D_i\nu) + D^i_k D_i[(\nu-\tau)_+\chi\nu^k].\end{aligned} \tag{3.12}$$

An integration by parts then yields

$$\begin{aligned}\int_Q D^i_k D_i[(\nu-\tau)_+\chi\nu^k]\zeta^2 dX = {} & -\int_Q (\mathscr{D}^{ij}D_{ij}u + \mathscr{F})[(\nu-\tau)_+\chi]\zeta^2 dX \\ & -2\int_Q D^i_k(\nu-\tau)_+\chi\nu^k\zeta D_i\zeta\, dX.\end{aligned} \tag{3.13}$$

For the second integral, we integrate by parts again (cf. the proof of [12, Lemma 2.3]):

$$\begin{aligned}&\int_Q D_kA^i(\nu-\tau)_+\chi\nu^k D_i(\zeta^2)dX \\ &\quad = -2\int_Q A^i D_k((\nu-\tau)_+\chi\nu^k)\zeta D_i\zeta\, dX - 2\int_Q A^i\chi(\nu-\tau)_+\nu^k[\zeta D_{ik}\zeta + D_i\zeta D_k\zeta]dX.\end{aligned} \tag{3.14}$$

To simplify the notation, we now set

$$\begin{aligned} I_1 &= \int_{q_\tau(R,s)} \Xi\zeta^2 dx, \\ I_2 &= \int_{Q_\tau(R)} \left[\left(1-\frac{\tau}{\nu}\right)\mathscr{C}^2\chi + \mathscr{E}(\chi'(\nu-\tau)+\chi)\right]\zeta^2 dX.\end{aligned} \tag{3.15}$$

Then

$$I_1 + I_2 = 2\int_Q \Xi\zeta\zeta_t dX + \sum_{j=3}^{10} I_j, \tag{3.16}$$

where

$$
\begin{aligned}
I_3 &= -\int_Q \left(1-\frac{\tau}{v}\right)_+ C_k^i \chi g^{kj} D_{ij}u\,dX,\\
I_4 &= -\int_Q \Psi C_k^i v^k D_i v \zeta^2\,dX,\\
I_5 &= \int_Q \mathcal{D}^{ij} D_{ij}u(v-\tau)_+\chi\zeta^2\,dX,\\
I_6 &= \int_Q \mathcal{F}(v-\tau)_+\chi\zeta^2\,dX,\\
I_7 &= 2\int_Q [D_k^i - B\delta_k^i](v-\tau)\chi v^k \zeta D_i\zeta\,dX,\\
I_8 &= 2\int_Q \left(1-\frac{\tau}{v}\right)_+ A^i D_i\zeta g^{kj} D_{kj}u\,dX,\\
I_9 &= 2\int_Q A^i(v-\tau)_+\chi v\cdot Dv\zeta D_i\zeta\,dX,\\
I_{10} &= 2\int_Q A^i\chi(v-\tau)_+ v^k[\zeta D_{ik}\zeta + D_i\zeta D_k\zeta]\,dX.
\end{aligned}
\tag{3.17}
$$

These terms are estimated as in [5, Lemma 11.10] using (3.3) and Cauchy's inequality. For the reader's convenience, we give a brief estimate of each integral.

First, from (3.3a), we have

$$I_3 \le \beta_1 \int_Q \Lambda_0^{1/2}\left(a^{ij}D_{ik}uD_{jk}u\right)^{1/2}\left(1-\frac{\tau}{v}\right)_+ \chi\zeta^2\,dX. \tag{3.18}$$

Since

$$a^{ij}D_{ik}uD_{jk}u = \mathcal{C}^2 + \mathcal{E} \tag{3.19}$$

and $\chi' \ge 0$, we have

$$a^{ij}D_{ik}uD_{jk}u\left(1-\frac{\tau}{v}\right)_+\chi \le \left(1-\frac{\tau}{v}\right)_+\chi\mathcal{C}^2 + \Psi\mathcal{E}. \tag{3.20}$$

Therefore, by Cauchy's inequality,

$$I_3 \le 3\beta_1^2\int_Q \Lambda_0\Psi\zeta^2\,dX + \frac{1}{12}I_2. \tag{3.21}$$

Similarly, since $\mathcal{C}^2 \ge 0$, we see from (3.3b) and Cauchy's inequality that

$$I_4 \le 3\beta_1^2\int_Q \Lambda_0\Psi\zeta^2\,dX + \frac{1}{12}I_2. \tag{3.22}$$

Next, we use (3.3c), (3.20), and Cauchy's inequality to obtain

$$I_5 \le 3\beta_1^2\int_Q \Lambda_0\Psi\zeta^2\,dX + \frac{1}{12}I_2. \tag{3.23}$$

Moreover, (3.3d) gives

$$I_6 \le \beta_1^2 \int_Q \Lambda_0 \Psi \zeta^2 dX, \tag{3.24}$$

and (3.3e) gives

$$I_7 \le 2\beta_1 \int_Q \Lambda_1 \chi (v-\tau)_+ \zeta |D\zeta| dX. \tag{3.25}$$

From (3.3f) and Cauchy's inequality, we infer that

$$I_8 \le 8\beta_2^2 \int_Q \Lambda_2 \Psi |D\zeta|^2 dX + \frac{1}{8} I_2, \tag{3.26}$$

and, finally, (3.3g), (3.20), and Cauchy's inequality imply that

$$I_9 \le 8\beta_2^2 \int_Q \Lambda_2 \Psi |D\zeta|^2 dX + \frac{1}{8} I_2. \tag{3.27}$$

It follows that

$$\begin{aligned} I_1 + I_2 \le 2\int_Q \Xi \zeta \zeta_t dX + 10\beta_1^2 \int_Q \Lambda_0 \Psi \zeta^2 dX + 2\beta_1 \int_Q \Lambda_1 \chi \zeta |D\zeta| dX \\ + 16\beta_2 \int_Q \Lambda_2 \Psi |D\zeta|^2 dX + 2\int_Q |A| \chi [\, |D^2\zeta| \zeta + |D\zeta|^2] dX + \frac{1}{2} I_2. \end{aligned} \tag{3.28}$$

Then (3.5) follows from this inequality by simple algebra. □

In Section 6, we will need a sharper version of this lemma. To obtain this version, we note that (3.3d) is only needed to estimate the positive part of $\mathcal{F}$, so (3.5) also holds with an additional term of

$$-\int_{Q(R)} \mathcal{F}_- \chi (v-\tau)_+ \zeta^2 dX \tag{3.29}$$

on the right-hand side.

4. The Sobolev inequality

We now present our modified Sobolev inequality, which is an easy consequence of [17, Theorem 2.1]; however, for notational reasons (in particular the use of n and m), we quote a consequence of this theorem (see [5, Corollary 11.9]).

Lemma 4.1. *Let $n \ge 2$, and let $g \in L^\infty(Q(R))$ be nonnegative. Set $H^i = D_j(g^{ij})$ and $\kappa = (N+2)/N$. Then*

$$\begin{aligned} \int_{Q(R)} |h|^{2\kappa} g^{2/N} dX \le C(N) \left(\sup_{s\in(-R^2,0)} \int_{B(R)} |h(x,s)|^2 g(x,s) dx \right)^{2/N} \\ \times \left(\int_{Q(R)} [g^{ij} D_i h D_j h + h^2 |H|^2] dX \right)^{n/N} \left(\int_{Q(R)} h^2 dX \right)^{(N-n)/N} \end{aligned} \tag{4.1}$$

for any $h \in C(\overline{Q(R)})$ that vanishes on $\{|x| = R\}$ and which is uniformly Lipschitz with respect to x.

Proof. Let us set $m = n+1$ and $U = B(R)$. We define $\nu^{n+1} = -1/v$ and extend the definition $g^{ij} = \delta^{ij} - \nu^i\nu^j$ for i and j in $\{1,\dots,m\}$. With $d\mu = dx$, it is easy to check that all the hypotheses of [5, Corollary 11.9] are satisfied, and this corollary gives

$$\begin{aligned}\int_U |h|^{2\kappa} g^{2/N}\,dx \le C(N)&\left(\int_U |h|^2 g\,dx\right)^{2/N} \\ &\times\left(\int_U [g^{ij}D_ihD_jh + h^2|H|^2]dx\right)^{n/N}\left(\int_U h^2 dx\right)^{(N-n)/N}\end{aligned} \tag{4.2}$$

for each $t \in (-R^2, 0)$. (In this equation, all functions are evaluated at (x,t).) The proof is completed as in [5, Theorem 6.9]: note that

$$\int_U h^2 g\,dx \le \sup_{s\in(-R^2,0)} \int_U h(x,s)^2 g(x,s)dx, \tag{4.3}$$

integrate the resulting inequality with respect to t, and then apply Hölder's inequality if $n = 2$. □

Note that the vector H is not quite the usual mean curvature vector. For later reference, we observe that

$$v^2|H|^2 \le C(n)[g^{ij}D_{ik}ug^{km}D_{jm}u + g^{ij}D_ivD_jvv]. \tag{4.4}$$

5. Estimate of the maximum in terms of an integral

From our energy inequality and the Sobolev inequality, we can now reduce our pointwise estimate of $|Du|$ to an integral estimate of a suitable quantity. For this reduction, we introduce three positive $C^1[\tau_0,\infty)$ functions w, λ, and Λ. In addition to their smoothness, the functions w, λ, and Λ obey the following monotonicity properties:

$$w \text{ is increasing}, \tag{5.1a}$$

$$\xi^{-\beta}w(\xi) \text{ is a decreasing function of } \xi, \tag{5.1b}$$

$$w(\xi)^{-\beta}\left(\frac{(\Lambda(\xi)/\lambda(\xi))^{N/2}}{\xi^2}\right) \text{ is an increasing function of } \xi, \tag{5.1c}$$

$$\xi^{-\beta}\left(\frac{\Lambda(\xi)}{\lambda(\xi)}\right)^{N/2} \text{ is a decreasing function of } \xi \tag{5.1d}$$

for some nonnegative constant β. We also assume that

$$\Lambda_0 \le v\Lambda, \tag{5.2a}$$

$$\Lambda_1 \le v\Lambda, \qquad \Lambda_2 \le v\Lambda, \tag{5.2b}$$

$$\lambda \le \Lambda, \tag{5.2c}$$

$$1 \le \Lambda, \tag{5.2d}$$

and that

$$|A| \le \beta_2 \Lambda. \tag{5.3}$$

Finally, we assume that

$$\lambda\left(1+\left(\frac{\nu\lambda'}{\lambda}\right)^2\right)g^{ij}\xi_i\xi_j \le \nu a^{ij}\xi_i\xi_j, \tag{5.4}$$

where (as before) we suppress the argument ν from λ, Λ, and their derivatives. These hypotheses imply a pointwise estimate for the gradient in terms of an integral.

Lemma 5.1. *Suppose that conditions (3.3), (5.1), (5.2), (5.3), and (5.4) hold. Then there is a constant $c_1(n,\beta,\beta_1 R,\beta_2)$ such that*

$$\sup_{Q_\tau(R/2)}\left(1-\frac{\tau}{\nu}\right)^{N+2} w \le c_1 R^{-n-2}\int_{Q_\tau(R)} w\left(\frac{\Lambda}{\lambda}\right)^{N/2}\frac{\Lambda}{\nu}\,dX. \tag{5.5}$$

Proof. The proof is essentially the same as that of [5, Lemma 11.11], so we only give a sketch.

First, for $q \ge 1+\beta$ a parameter at our disposal, we set

$$\chi = \left(\frac{\Lambda}{\lambda}\right)^{N/2} w^q\left[\left(1-\frac{\tau}{\nu}\right)_+\right]^{(N+2)(q-1)}\nu^{-2}. \tag{5.6}$$

Then conditions (5.1a), (5.1c) imply that χ is increasing while conditions (5.1b), (5.1d) imply that $\Psi \le C(\beta)q^2\chi$. Now let ζ be as in Lemma 3.1, and note that we can take ζ so that $|D\zeta| \le C/R$, $|D^2\zeta| + |\zeta_t| \le C/R^2$, and $0 \le \zeta \le 1$ in $Q(R)$. It then follows from Lemma 3.1 with $\zeta^{(N+2)q-N}$ in place of ζ^2 that

$$\begin{aligned}\sup_{t\in(-R^2,0)}\int_{q_\tau(R,t)}\Xi(\nu)\zeta^2\,dx + \int_{Q_\tau(R)}\left[\left(1-\frac{\tau}{\nu}\right)\mathscr{C}^2+\mathscr{E}\right]\chi\zeta^2\,dX \\ \le C(\beta,\beta_1 R,\beta_2)\frac{q^2}{R^2}\int_{Q_\tau(R)}\chi\Lambda\zeta^{(N+2)(q-1)}\nu\,dX\end{aligned} \tag{5.7}$$

by taking (5.2a), (5.2b), (5.2c), and (5.3) into account and observing that

$$\Xi(\nu) \le \frac{1}{2}\chi(\nu)(\nu-\tau)^2 \tag{5.8}$$

(because χ is increasing).

Now we define h by the equation

$$h^2 = \chi\lambda\left(1-\frac{\tau}{\nu}\right)^2\nu\zeta^{(N+2)q-N}, \tag{5.9}$$

so

$$g^{ij}D_ihD_jh \le Cq^2\zeta^{(N+2)(q-1)}\frac{\chi\lambda}{\nu}\left(\zeta^2\left(1+\left(\frac{\nu\lambda'}{\lambda}\right)^2\right)g^{ij}D_i\nu D_j\nu + \frac{\nu^2}{R^2}\right). \tag{5.10}$$

In addition, from conditions (5.1b), (5.1d), we infer that

$$\Xi(v) \geq \frac{1}{2+C(\beta)q}\chi(v)(v-\tau)^2. \tag{5.11}$$

It then follows from (4.4) and (5.4) that

$$\sup_{t\in(-R^2,0)} \int_{B(R)\times\{t\}} h^2\frac{v}{\lambda}dx + \int_{Q(R)} [g^{ij}D_ihD_jh + h^2|H|^2]dX \leq Cq^4R^{-2}\int_{Q_\tau(R)} \chi\Lambda v\zeta^{(N+2)(q-1)}dX. \tag{5.12}$$

Lemma 4.1, with $g = v/\lambda$, then yields

$$\left(\int_\Sigma \overline{w}^{\kappa q}d\mu\right)^{1/\kappa} \leq Cq^4\int_\Sigma \overline{w}^q d\mu \tag{5.13}$$

for $\overline{w} = \zeta(1-\tau/v)_+w$,

$$\begin{aligned} \Sigma &= \{X \in Q_\tau(R) : \zeta(X) > 0, v > \tau\}, \\ d\mu &= \left(\frac{\Lambda}{\lambda}\right)^{N/2}\frac{\Lambda}{v}\left[\left(1-\frac{\tau}{v}\right)\zeta\right]^{-N-2}dX. \end{aligned} \tag{5.14}$$

A standard iteration argument (based on [10]) completes the proof. □

If we assume further that there are nonnegative constants β_3 and β_4 such that

$$w\left(\frac{\Lambda}{\lambda}\right)^{N/2}\Lambda \leq \beta_3 w^{\beta_4+1}Du\cdot A \tag{5.15}$$

(see [11, (1.5)] or [5, (11.50)]), then we have reduced the pointwise estimate to an estimate of the integral

$$\int v^q Du\cdot A\,dX \tag{5.16}$$

for $q = \beta_4$, and we estimate this integral in the next section. (Note that if $\beta_4 = 0$, this estimate is particularly simple.)

6. Estimate of the integral

We now examine the integral (5.16), and we provide an estimate specifically for the case $w = v$. To this end, we make some basic assumptions relating the sizes of A, B, and $Du\cdot A$:

$$v|A| \leq \beta_5 Du\cdot A, \tag{6.1a}$$

$$B \leq \beta_6 Du\cdot A. \tag{6.1b}$$

We also use a variant of (3.3e): we assume that there is a decreasing function ε such that

$$-v^kD_k^iv_i \leq \varepsilon(v)Du\cdot A. \tag{6.2}$$

Next, we suppose that the functions Λ_0, Λ_1, and Λ_2 can be estimated suitably in terms of $Du \cdot A$:

$$\Lambda_0 \le \varepsilon(v)^2 v^2 Du \cdot A, \tag{6.3a}$$

for the same function ε as in (6.2),

$$\Lambda_1 \le v Du \cdot A, \tag{6.3b}$$

$$\Lambda_2 \le Du \cdot A. \tag{6.3c}$$

Finally, we assume that

$$v \le \beta_7 Du \cdot A. \tag{6.4}$$

Under these hypotheses, we obtain an estimate for (5.16) provided that ε can be made sufficiently small when v is large.

Lemma 6.1. *Suppose conditions (3.3), (6.1), (6.2), (6.3), and (6.4) are satisfied. Let $q > 0$ and set $\omega = \operatorname{osc}_{Q(R)} u$, $E = \exp(\beta_6 \omega)$, $\Sigma = 1 + \beta_7 \omega / R$, and $q_* = \max\{q, 2\}$. If there is a constant τ_1 greater than $\max\{\tau, 2\}$ such that*

$$8\omega\varepsilon(\tau_1) + \left[10\beta_2 q_*^3 + 640\beta_2 q_*^4 + 1280\left(\beta_1 q_* \omega \varepsilon(\tau_1)\right)^2 + 80 q_*^2\right] E \beta_1^2 q_* \omega^2 \varepsilon(\tau_1)^2 \le 1, \tag{6.5}$$

then there is a constant C determined only by $\beta_1 \omega \varepsilon(\tau_1)$, β_2, β_5, and q such that

$$\int_{Q_\tau(R)} v^q Du \cdot A \, dX \le C \left(\tau_1 + \Sigma E \frac{\omega}{R}\right)^q \int_{Q_\tau(2R)} Du \cdot A \, dX. \tag{6.6}$$

Proof. Suppose first that $q \ge 2$. Our proof in this case is a modification of the proof of [11, Lemma 2]. First, we set

$$G(\sigma) = \begin{cases} \sigma^q - q\tau^{q-1}\sigma + (q-1)\tau^q & \text{if } \sigma > \tau, \\ 0 & \text{if } \sigma \le \tau, \end{cases} \tag{6.7}$$

and we observe that $G'(\sigma) = q(\sigma^{q-1} - \tau^{q-1})_+$. Hence

$$0 \le G(\sigma) \le \frac{1}{2} G'(\sigma)\sigma, \tag{6.8a}$$

$$G'(\sigma) \le q\sigma^{q-1}\left(1 - \frac{\tau}{\sigma}\right)_+, \tag{6.8b}$$

$$G(\sigma) \le \sigma^q \left(1 - \frac{\tau}{\sigma}\right)_+, \tag{6.8c}$$

$$0 \le G''(\sigma) \le q^2 \sigma^{q-2}. \tag{6.8d}$$

In what follows, we suppress the argument v from G and its derivatives. Next, we set

$$F(z) = \frac{1}{\beta_6^2}\left(\beta_6 z \exp(\beta_6 z) + 1 - \exp(\beta_6 z)\right) \tag{6.9}$$

and $F_1(z) = F(z)\exp(-\beta_6 z)$. We note that $F_1(0) = F_1'(0) = 0$ and $F_1''(z) \le 1$ for $z \ge 0$, so $F_1(z) \le (1/2)z^2$ for $z \ge 0$. It follows that for z replaced by $\overline{u} = u - \inf_{Q(R)} u$, F satisfies the properties

$$0 \le F \le \frac{1}{2}\omega^2 \overline{E} \le \frac{1}{2}\omega^2 E, \tag{6.10a}$$

$$0 \le F' \le \omega \overline{E} \le \omega E, \tag{6.10b}$$

$$0 \le F'' \le (1 + \beta_6 \omega) E, \tag{6.10c}$$

$$F'' - \beta_6 F' = \overline{E}, \tag{6.10d}$$

where $\overline{E} = \exp(\beta_6 \overline{u})$.

We also define

$$\zeta(X) = \left[\left(1 - \frac{|x|^2}{4R^2}\right)_+\right]^2 \left(1 + \frac{t}{4R^2}\right), \tag{6.11}$$

so that

$$|D\zeta| \le \frac{1}{2R}, \qquad |D^2\zeta| \le \frac{1}{R^2}, \qquad |\zeta_t| \le \frac{1}{4R^2}. \tag{6.12}$$

Now, we set

$$I' = -\int_{Q(2R)} \zeta^{2q} F'' G Du \cdot A\, dX = -\int_{Q(2R)} [\zeta^{2q} G A] \cdot D(F') dX, \tag{6.13}$$

and an integration by parts gives

$$I' = I_1 + I_2 + I_3 \tag{6.14}$$

with

$$\begin{aligned} I_1 &= \int_{Q(2R)} \zeta^{2q} F' G \operatorname{div} A\, dX, \\ I_2 &= \int_{Q(2R)} \zeta^{2q} F' G' A \cdot Dv\, dX, \\ I_3 &= 2q \int_{Q(2R)} \zeta^{2q-1} F' G A \cdot D\zeta\, dX. \end{aligned} \tag{6.15}$$

The estimate for I_1 is, in the present situation, the most complex. First, we use the differential equation to see that $I_1 = I_4 + I_5$ with

$$\begin{aligned} I_4 &= \int_{Q(2R)} \zeta^{2q} F' G u_t dX = \int_{Q(2R)} \zeta^{2q} F_t G\, dX, \\ I_5 &= -\int_{Q(2R)} \zeta^{2q} F' G B\, dX. \end{aligned} \tag{6.16}$$

To estimate I_4, we need some further integration by parts which is easily justified if A, B, and u are smoother than we have assumed. The justification under our current hypotheses is to let (u_m) be a sequence of C^∞ functions which converge in $C^{2,1}$ to u. Writing $v_m = (1+|Du_m|^2)^{1/2}$ and G_m for $G(v_m)$, we have

$$\begin{aligned}\int_{Q(2R)} \zeta^{2q} F_t G_m dX &= -\int_{Q(2R)} (\zeta^{2q} G_m)_t F\, dX + \int_{B(R)\times\{0\}} \zeta^{2q} F G_m dx \\ &= -2q \int_{Q(2R)} \zeta^{2q-1} G_m F \zeta_t dX \\ &\quad - \int_{Q(2R)} \zeta^{2q} G_m'(v_m)_t F\, dX + \int_{B(R)\times\{0\}} \zeta^{2q} F G_m dx.\end{aligned} \tag{6.17}$$

But $(v_m)_t = v_m^k D_k(u_m)_t$, so

$$-\int_{Q(2R)} \zeta^{2q} G_m'(v_m)_t F\, dX = \int_{Q(2R)} D_k[\zeta^{2q} G_m' v_m^k F](u_m)_t dX. \tag{6.18}$$

Sending $m \to \infty$ then gives

$$I_4 = -2q \int_{Q(2R)} \zeta^{2q-1} F G \zeta_t dX + \int_{Q(2R)} D_k[\zeta^{2q} F G' v^k] u_t dX + \int_{B(2R)\times\{0\}} \zeta^{2q} F G\, dx. \tag{6.19}$$

Then we use the differential equation again to conclude that

$$I_4 = -2q \int_{Q(2R)} \zeta^{2q-1} F G \zeta_t dX + \int_{Q(2R)} D_k[\zeta^{2q} F G' v_k](D_i A^i + B)\, dX + \int_{B(2R)\times\{0\}} \zeta^{2q} F G\, dx, \tag{6.20}$$

and another integration by parts (as in Lemma 3.1) gives us

$$\int_{Q(2R)} D_k[\zeta^{2q} F G' v_k] D_i A^i dX = \int_{Q(2R)} D_i[\zeta^{2q} F G' v_k] D_k A^i dX. \tag{6.21}$$

It follows that $I_4 = I_6 + I_7 + I_8 + I_9$ with

$$\begin{aligned} I_6 &= \int_{Q(2R)} D_i[\zeta^{2q} F G' v_k] D_k A^i dX, \\ I_7 &= \int_{Q(2R)} D_i[\zeta^{2q} F G' v_k] \delta_k^i B\, dX, \\ I_8 &= -2q \int_{Q(2R)} \zeta^{2q-1} F G \zeta_t dX, \\ I_9 &= \int_{B(2R)\times\{0\}} \zeta^{2q} F G\, dx. \end{aligned} \tag{6.22}$$

Next, we write

$$I_6 = \int_{Q(2R)} D_i(\zeta^{2q}F)G'\nu_k D_k A^i dX + \int_{Q(2R)} \zeta^{2q} F D_i(G'\nu_k) D_k A^i dX, \tag{6.23}$$

and another integration by parts yields

$$\begin{aligned}\int_{Q(2R)} D_i(\zeta^{2q}F)G'\nu_k D_k A^i dX = &-\int_{Q(2R)} D_{ik}(\zeta^{2q}F)G'\nu_k A^i dX \\ &-\int_{Q(2R)} D_i(\zeta^{2q}F)D_k(G'\nu^k)A^i dX,\end{aligned} \tag{6.24}$$

so

$$I_6 = \sum_{j=10}^{19} I_j, \tag{6.25}$$

with

$$\begin{aligned}
I_{10} &= -2q\int_{Q(2R)} \zeta^{2q-2}FG'\nu^k[\zeta D_{ik}\zeta + (2q-1)D_i\zeta D_k\zeta]A^i dX,\\
I_{11} &= -2q\int_{Q(2R)} \zeta^{2q-1}F'G'[(\nu\cdot D\zeta)(Du\cdot A) + (\nu\cdot Du)(D\zeta\cdot A)]dX,\\
I_{12} &= -I_2,\\
I_{13} &= -\int_{Q(2R)} \zeta^{2q}F''G'\nu\cdot Du Du\cdot A\, dX,\\
I_{14} &= -2q\int_{Q(2R)} \zeta^{2q-1}FG''\nu\cdot D\nu D\zeta\cdot A\, dX,\\
I_{15} &= -2q\int_{Q(2R)} \zeta^{2q-1}FG'\frac{1}{\nu}g^{kj}D_{jk}uD\zeta\cdot A\, dX,\\
I_{16} &= -\int_{Q(2R)} \zeta^{2q}F'G''\nu\cdot D\nu Du\cdot A\, dX,\\
I_{17} &= -\int_{Q(2R)} \zeta^{2q}F'G'\frac{1}{\nu}g^{kj}D_{jk}uDu\cdot A\, dX,\\
I_{18} &= \int_{Q(2R)} \zeta^{2q}FG''\nu_k a^{ij}D_{jk}uD_i\nu\, dX + \int_{Q(2R)} \zeta^{2q}F\frac{G'}{\nu}g^{km}D_{im}ua^{ij}D_{jk}u\, dX,\\
I_{19} &= \int_{Q(2R)} \zeta^{2q}FG''\nu_k D_i\nu\left(\frac{\partial A^i}{\partial z}D_k u + \frac{\partial A^i}{\partial x^k}\right)dX\\
&\quad + \int_{Q(2R)} \zeta^{2q}FG'\frac{1}{\nu}g^{kj}D_{ij}u\left(\frac{\partial A^i}{\partial z}D_k u + \frac{\partial A^i}{\partial x^k}\right)dX.
\end{aligned} \tag{6.26}$$

We now combine some of these integrals:

$$\begin{aligned} I_5+I_7+I_{19} &= \int_{Q(2R)} \zeta^{2q} F D_i(G'\nu_k)(C^i_k + D^i_k)\,dX \\ &\quad + \int_{Q(2R)} D_k(\zeta^{2q}F)G'\nu^k B\,dX - \int_{Q(2R)} \zeta^{2q}F' GB\,dX \\ &= I_{20}+I_{21}+I_{22}+I_{23}+I_{24}+I_{25}+I_{26}, \end{aligned} \tag{6.27}$$

with

$$\begin{aligned} I_{20} &= \int_{Q(2R)} \zeta^{2q} FG' \frac{1}{\nu} C^i_k g^{kj} D_{ij}u\,dX, \\ I_{21} &= \int_{Q(2R)} \zeta^{2q} FG''\nu^k C^i_k D_i \nu\,dX, \\ I_{22} &= -\int_{Q(2R)} \zeta^{2q} F'G'\nu^k D^i_k D_i u\,dX, \\ I_{23} &= -2q\int_{Q(2R)} \zeta^{2q-1} FG'[\nu^k D^i_k - \nu_i B]D_i\zeta\,dX, \\ I_{24} &= -\int_{Q(2R)} \zeta^{2q} FG'\mathcal{D}^{ij} D_{ij}u\,dX, \\ I_{25} &= -\int_{Q(2R)} \zeta^{2q} FG'\mathcal{F}\,dX, \\ I_{26} &= \int_{Q(2R)} \zeta^{2q} F'(G'\nu\cdot Du - G)B\,dX. \end{aligned} \tag{6.28}$$

It follows that

$$\int_{Q(2R)} \zeta^{2q} F''(G'\nu\cdot Du - G)Du\cdot A\,dX = I' - I_{13} = I_3 + \sum_{j=8}^{11} I_j + \sum_{j=14}^{18} I_j + \sum_{j=20}^{26} I_j. \tag{6.29}$$

If we assume now that $\tau \geq 2$, we have $\nu\cdot Du \geq (3/4)\nu$ for $\nu \geq \tau$, and hence (6.8a) implies that

$$G'\nu\cdot Du - G \geq \frac{1}{4}G'\nu = \frac{q}{4}(\nu^q - \tau^{q-1}\nu)_+. \tag{6.30}$$

From (6.1b) and (6.10d), we then conclude that

$$\begin{aligned} \frac{q}{4}\int_{Q(2R)} \zeta^{2q}\overline{E}(\nu^q - \tau^{q-1}\nu)_+ Du\cdot A\,dX &\leq \int_{Q(2R)} \zeta^{2q}\overline{E}[G'\nu\cdot Du - G]Du\cdot A\,dX \\ &\leq I_3 + \sum_{j=8}^{11} I_j + \sum_{j=14}^{18} I_j + \sum_{j=20}^{25} I_j. \end{aligned} \tag{6.31}$$

We are now ready to estimate the right-hand side of this inequality, one term at a time. First, we define the measure μ by

$$\mu(S) = \int_{Q_\tau(2R)\cap S} Du \cdot A\,dX, \tag{6.32}$$

so, for any function f, we have

$$\int_S f\,d\mu = \int_{Q_\tau(2R)\cap S} f(X)Du \cdot A\,dX, \tag{6.33}$$

then

$$I_3 \le \beta_5 qE\frac{\omega}{R}\int_{Q(2R)} (\zeta^2 v)^{q-1}d\mu \tag{6.34}$$

by (6.1a), (6.8c), and (6.10b);

$$I_8 \le \frac{1}{2}\beta_7 qE\frac{\omega^2}{R^2}\int_{Q(2R)} (\zeta^2 v)^{q-1}d\mu \tag{6.35}$$

by (6.4), (6.8c), and (6.10a);

$$I_9 \le \frac{1}{2}E\omega^2\int_{B(R)\times\{0\}} \zeta^{2q}G(v)dx \tag{6.36}$$

by (6.10a);

$$I_{10} \le q^3\beta_5 E\frac{\omega^2}{R^2}\int_{Q(2R)} (\zeta^2 v)^{q-2}d\mu \tag{6.37}$$

by (6.1a), (6.8b), and (6.10a);

$$I_{11} \le 2\beta_5 q^2 E\frac{\omega}{R}\int_{Q(2R)} (\zeta^2 v)^{q-1}d\mu \tag{6.38}$$

by (6.1a), (6.8b), (6.10b), and the observation that $\beta_5 \ge 1$. Next

$$I_{14} \le \frac{1}{2}\beta_2 q^3 E\frac{\omega^2}{R}\left(\int_{Q_\tau(2R)} \zeta^{2q}v^{q-2}\mathscr{E}\,dX\right)^{1/2}\left(\int_{Q(2R)} (\zeta^2 v)^{q-2}d\mu\right)^{1/2} \tag{6.39}$$

by (3.3g), (6.3c), (6.8d), and (6.10a);

$$I_{15} \le \frac{1}{2}\beta_2 q^2 E\frac{\omega^2}{R}\left(\int_{Q_\tau(2R)} \zeta^{2q}v^{q-2}\left[\left(1-\frac{\tau}{v}\right)\mathscr{C}^2+\mathscr{E}\right]dX\right)^{1/2}\left(\int_{Q(2R)} (\zeta^2 v)^{q-2}d\mu\right)^{1/2} \tag{6.40}$$

by (3.3f), (6.3c), (6.8b), and (6.10a);

$$I_{16} \le \beta_2 q^2\omega\left(E\int_{Q_\tau(2R)} \zeta^{2q}v^{q-2}\mathscr{E}\,dX\right)^{1/2}\left(\int_{Q(2R)} \overline{E}(\zeta^2 v)^q d\mu\right)^{1/2} \tag{6.41}$$

by (3.3g), (6.3c), (6.8d), and (6.10b);

$$I_{17} \leq \beta_2 q\omega\left(E\int_{Q_\tau(2R)} \zeta^{2q}\nu^{q-2}\left[\left(1-\frac{\tau}{\nu}\right)\mathscr{C}^2+\mathscr{E}\right]dX\right)^{1/2}\left(\int_{Q(2R)} \overline{E}(\zeta^2\nu)^q d\mu\right)^{1/2} \tag{6.42}$$

by (3.3f), (6.3c), (6.8b), and (6.10b);

$$I_{18} \leq \frac{1}{2}q^2 E\omega^2 \int_{Q_\tau(2R)} \zeta^{2q}\nu^{q-2}\left[\left(1-\frac{\tau}{\nu}\right)\mathscr{C}^2+\mathscr{E}\right]dX \tag{6.43}$$

by (6.8b), (6.8d), and (6.10b);

$$I_{20} \leq \frac{1}{2}\beta_1 q\omega^2\varepsilon(\tau)\left(E\int_{Q_\tau(2R)} \zeta^{2q}\nu^{q-2}\left[\left(1-\frac{\tau}{\nu}\right)\mathscr{C}^2+\mathscr{E}\right]dX\right)^{1/2}\left(\int_{Q(2R)} \overline{E}(\zeta^2\nu)^q d\mu\right)^{1/2} \tag{6.44}$$

by (3.3a), (6.3a), (6.8a), and (6.10b);

$$I_{21} \leq \frac{1}{2}\beta_1 q^2\omega^2\varepsilon(\tau)\left(E\int_{Q_\tau(2R)} \zeta^{2q}\nu^{q-2}\mathscr{E}\,dX\right)^{1/2}\left(\int_{Q(2R)} \overline{E}(\zeta^2\nu)^q d\mu\right)^{1/2} \tag{6.45}$$

by (3.3b), (6.3a), (6.8d), and (6.10a);

$$I_{22} \leq \omega\varepsilon(\tau)q\int_{Q(2R)} \overline{E}(\zeta^2\nu)^q d\mu \tag{6.46}$$

by (6.2), (6.8b), and (6.10b);

$$I_{23} \leq \frac{1}{2}\beta_1 q^2 E\frac{\omega^2}{R}\int_{Q(2R)} (\zeta^2\nu)^{q-1} d\mu \tag{6.47}$$

by (3.3e), (6.3b), (6.8b), and (6.10a);

$$I_{24} \leq \frac{1}{2}\beta_1 q\omega^2\varepsilon(\tau)\left(E\int_{Q_\tau(2R)} \zeta^{2q}\nu^{q-2}\left[\left(1-\frac{\tau}{\nu}\right)\mathscr{C}^2+\mathscr{E}\right]dX\right)^{1/2}\left(\int_{Q(2R)} \overline{E}(\zeta^2\nu)^q d\mu\right)^{1/2} \tag{6.48}$$

by (3.3c), (6.3a), (6.8b), and (6.10a);

$$I_{25} \leq \frac{1}{2}qE\omega^2\int_{Q(2R)} \mathscr{F}_-\zeta^{2q}\nu^{q-2}(\nu-\tau)_+ dX \tag{6.49}$$

by (6.8b) and (6.10a).

Combining all these estimates and using Cauchy's inequality, we find that

$$\begin{aligned}
&\frac{q}{4}\int_{Q(2R)}\overline{E}\zeta^{2q}(v^q-\tau^{q-1}v)\,d\mu\\
&\quad\le K_1\Sigma E\frac{\omega}{R}\int_{Q(2R)}(\zeta^2v)^{q-1}d\mu+K_2E^2\frac{\omega^2}{R^2}\int_{Q(2R)}(\zeta^2v)^{q-2}d\mu\\
&\qquad+K_3E\omega^2\int_{Q_\tau(2R)}\zeta^{2q}v^{q-2}\left[\left(1-\frac{\tau}{v}\right)\mathscr{C}^2+\mathscr{E}\right]dX+E\omega^2\int_{B(R)\times\{0\}}\zeta^{2q}G\,dx\\
&\qquad+\left(\frac{1}{4}+\omega\varepsilon(\tau)q\right)\int_{Q(2R)}\overline{E}(\zeta^2v)^q d\mu+\frac{1}{2}qE\omega^2\int_{Q(2R)}\mathscr{F}_-\zeta^{2q}v^{q-2}(v-\tau)_+dX
\end{aligned}\tag{6.50}$$

with

$$\begin{aligned}
K_1&=4\beta_5q^2+\frac{1}{2}q^2\beta_1\omega^2+\frac{1}{4}q,\qquad K_2=\beta_5q^3+\beta_2q^3,\\
K_3&=\frac{1}{8}\beta_2q^3+4\beta_2^2q^4+8(\beta_1\omega\varepsilon(\tau)q)^2+\frac{1}{2}q^2.
\end{aligned}\tag{6.51}$$

We now use the remark after Lemma 3.1 with $\chi=v^{q-2}$ and ζ^q in place of ζ. Since $G\le q\Xi$ and $K_3\ge q$, we infer from (6.1), (6.3), and (6.4) that

$$\begin{aligned}
&\omega^2\int_{B(R)\times\{0\}}\zeta^{2q}G\,dx+K_3\omega^2\int_{Q_\tau(2R)}\zeta^{2q}v^{q-2}\left[\left(1-\frac{\tau}{v}\right)\mathscr{C}^2+\mathscr{E}\right]dX\\
&\quad\le K_4\Sigma\frac{\omega}{R}\int_{Q(2R)}(\zeta^2v)^{q-1}d\mu+K_5\frac{\omega^2}{R^2}\int_{Q(2R)}(\zeta^2v)^{q-2}d\mu\\
&\qquad+K_6\omega^2\varepsilon(\tau)^2\int_{Q(2R)}(\zeta^2v)^q d\mu+\frac{q}{2}\omega^2\int_{Q(2R)}\mathscr{F}_-\zeta^{2q}v^{q-2}(v-\tau)_+dX
\end{aligned}\tag{6.52}$$

with

$$\begin{aligned}
K_4&=(2+2\beta_1\omega)K_3,\\
K_5&=(4\beta_5q^2+8\beta_2q^2)K_3,\\
K_6&=20q^2\beta_1^2K_3.
\end{aligned}\tag{6.53}$$

Since $K_3\ge q/2$, it follows that

$$\begin{aligned}
&\frac{q}{4}\int_{Q(2R)}\overline{E}(\zeta^2v)^q d\mu\\
&\quad\le(K_1+K_4)\Sigma E\frac{\omega}{R}\int_{Q(2R)}(\zeta^2v)^{q-1}d\mu+(K_2+K_5)E^2\frac{\omega^2}{R^2}\int_{Q(2R)}(\zeta^2v)^{q-2}d\mu\\
&\qquad+\left(\frac{1}{4}+q\omega\varepsilon(\tau)+K_6E\omega^2\varepsilon(\tau)^2\right)\int_{Q(2R)}\overline{E}(\zeta^2v)^q d\mu+q\tau^{q-1}\int_{Q(2R)}\zeta^{2q}v\,d\mu.
\end{aligned}\tag{6.54}$$

If we now replace τ by τ_1 and write μ_1 for the measure defined by replacing τ by τ_1 in (6.32), we infer that

$$\int_{Q(2R)} (\zeta^2 v)^q d\mu_1 \leq C\tau_1^{q-1} \int_{Q(2R)} (\zeta^2 v)\, d\mu_1 + C\Sigma E \frac{\omega}{R} \int_{Q(2R)} (\zeta^2 v)^{q-1} d\mu_1 + CE^2 \frac{\omega^2}{R^2} \int_{Q(2R)} (\zeta^2 v)^{q-2} d\mu_1. \tag{6.55}$$

Applying Young's inequality yields

$$\int_{Q(2R)} (\zeta^2 v)^q d\mu_1 \leq C\left(\tau_1 + \Sigma E \frac{\omega}{R}\right)^q \int_{Q(2R)} d\mu_1, \tag{6.56}$$

so

$$\int_{Q_{\tau_1}(2R)} (\zeta^2 v)^q Du \cdot A\, dX \leq C\left(\tau_1 + \Sigma E \frac{\omega}{R}\right)^q \int_{Q_{\tau_1}(2R)} Du \cdot A\, dX, \tag{6.57}$$

and it is clear that

$$\int_{Q_\tau(2R)\setminus Q_{\tau_1}(2R)} (\zeta^2 v)^q Du \cdot A\, dX \leq (\tau_1)^q \int_{Q_\tau(2R)\setminus Q_{\tau_1}(2R)} Du \cdot A\, dX. \tag{6.58}$$

Adding these last two inequalities gives the desired result.

The case $q < 2$ follows from this one via Hölder's inequality. □

Note that we can take ε to be a constant provided that a modulus of continuity is known for u; all we need is to take R small enough that (6.5) holds.

The estimate of $\int Du \cdot A\, dX$ is given in [5, Lemma 11.13], so we give the estimate without proof.

Lemma 6.2. *Suppose conditions (6.1) hold and set* $\omega = \operatorname{osc}_{Q(R)} u$. *Also set* $\tau_2 = \max\{\tau_0, 8\beta_5\omega/R\}$ *and*

$$\Delta = \sup_{v<\tau_2} \left\{ (B - \beta_6 Du \cdot A)_+ + (Du \cdot A)_+ + \frac{\omega}{R}|A| \right\}. \tag{6.59}$$

Then

$$\int_{Q_\tau(R/2)} Du \cdot A\, dX \leq C(n) \exp(\beta_6 \omega) R^n [\omega^2 + \Delta R^2]. \tag{6.60}$$

We can combine all of these results into a single estimate although we will see in the next section that sometimes a different combination is more useful.

Theorem 6.3. *Suppose there are functions* w, Λ_0, Λ_1, Λ_2, Λ, λ, *and* ε *such that conditions (3.3), (5.1), (5.2), (5.3), (5.4), (5.15), (6.1), (6.2), (6.3), and (6.4) hold for some nonnegative constants* β, $\beta_1, \ldots, \beta_7$, *and* $\tau_1 \geq \max\{2, \tau_0\}$ *with* $\omega = \operatorname{osc}_{Q(R)} u$ *and* $q_* = \max\{\beta_4, 2\}$. *Set* $\tau_2 = \max\{\tau_0, 8\beta_5\omega/R\}$, $E = \exp(\beta_6\omega)$, *and* $\Sigma = 1 + \beta_7\omega/R$, *and define* Δ *by (6.59). Then* *there is a constant C, determined only by n, β, $\beta_1 R$, $\beta_1\omega\varepsilon(\tau_1)$, β_2, β_3, β_4, β_5 such that*

$$\sup_{Q(R/8)} w(v) \le \max\left\{w(2\tau_0), C\left(\tau_1+\Sigma E\frac{\omega}{R}\right)^{\beta_4} E\left[\frac{\omega^2}{R^2}+\Delta\right]\right\}. \tag{6.61}$$

7. Examples

We start by assuming that the functions A and B satisfy the conditions

$$\begin{gathered} Du\cdot A \ge \gamma_0 v\Psi(v)-\gamma_1, \qquad |A| \le \gamma_2\Psi(v), \\ \gamma_0\frac{\Psi(v)}{v}|\xi|^2 \le a^{ij}\xi_i\xi_j, \\ v|A_z|+|A_x|+|B| \le \varepsilon_1(v)v\Psi(v) \end{gathered} \tag{7.1}$$

for a positive constant γ_0, nonnegative constants γ_1 and γ_2, an increasing function $\Psi \in C^1([1,\infty))$ such that $\Psi(1)=1$ and

$$\Psi'(v) \le \psi_0 v^{\alpha-1}\Psi(v) \tag{7.2}$$

for some nonnegative constants ψ_0 and α and a decreasing, positive function ε_1 such that $\lim_{\tau\to\infty}\varepsilon_1(\tau)=0$. Then conditions (3.3) are satisfied with $D^i_k=0$,

$$\begin{gathered} \Lambda_0=\frac{\gamma_0}{2}\frac{\varepsilon_1(v)}{\varepsilon_1(1)}v^3\Psi(v), \qquad \Lambda_1=\frac{\gamma_0}{2}\frac{\varepsilon_1(v)}{\varepsilon_1(1)}v^2\Psi(v), \qquad \Lambda_2=\frac{\gamma_0}{2}v\Psi(v), \\ \beta_1=(2n)^{1/2}\frac{\varepsilon_1(1)}{\gamma_0}, \qquad \beta_2=n^{1/2}\frac{\gamma_2}{\gamma_0}. \end{gathered} \tag{7.3}$$

In addition, we can take $w=v$, $\Lambda=(1/2)\gamma_0 v^2\Psi(v)$, and

$$\lambda=\frac{\gamma_0}{1+4\alpha^2+\psi_0^2}v^{-2\alpha}\Psi(v) \tag{7.4}$$

to satisfy (5.1)–(5.4) with $\beta=(1+\alpha)N$. Condition (5.15) holds with

$$\beta_3=\left(2\left[1+4\alpha^2+\psi_0^2\right]\right)^{N/2}\frac{\gamma_0}{2} \tag{7.5}$$

and $\beta_4=(1+\alpha)N+1$ if τ_0 is sufficiently large. Finally, conditions (6.1)–(6.4) hold with

$$\begin{gathered} \beta_5=2\frac{\gamma_2}{\gamma_0}, \qquad \beta_6=2\frac{\varepsilon_1(1)}{\gamma_0}, \qquad \beta_7=\frac{1}{\gamma_0}, \\ \varepsilon(v)=\frac{\varepsilon_1(v)}{\varepsilon_1(1)}. \end{gathered} \tag{7.6}$$

Since $\varepsilon(v)\to 0$ as $v\to\infty$, we have a gradient estimate under these hypotheses.

In particular, the equation

$$u_t=\operatorname{div}\left(\exp\left(\frac{1}{2}v^2\right)Du\right)+B(X,u,Du) \tag{7.7}$$

is included under these hypotheses if $|B| = o(v^2 \exp((1/2)v^2))$ as $|p| \to \infty$: we take $\Psi(v) = v\exp((1/2)[v^2-1])$, and note that (7.2) is satisfied with $\psi_0 = 2$ and $\alpha = 2$. It would be of interest to know if a gradient estimate can be obtained for $|B| = O(v^2 \exp((1/2)v^2))$.

The difficulty with [7, Lemma 5.4] is easy to explain in terms of the notation here. We write $\operatorname{div} A = a^{ij}D_{ij}u$ since, in this case, A is independent of z and x. Moreover, under the hypotheses of that lemma, one needs to estimate the integral

$$I = \int_{Q(R)} w^q \left(1 - \frac{\tau}{v}\right)_+ \zeta^{2q} a^{ij} D_{ij} u \, dX \tag{7.8}$$

for some function w, which was claimed to equal v^2 in [7]. The structure of the function A shows that

$$a^{ij} D_{ij} u \le \left(\mathscr{C}^2 + \mathscr{E}\right)^{1/2} (Du \cdot A)^{1/2}, \tag{7.9}$$

so

$$I \le \left(\int_{Q_\tau(R)} (\zeta w)^q \left[\left(1 - \frac{\tau}{v}\right) \mathscr{C}^2 + \mathscr{E} \right] dX \right)^{1/2} \left(\int_{Q(R)} (\zeta w)^q d\mu \right)^{1/2}, \tag{7.10}$$

and the integral

$$\int_{Q_\tau(R)} (\zeta w)^q \left[\left(1 - \frac{\tau}{v}\right) \mathscr{C}^2 + \mathscr{E} \right] dX \tag{7.11}$$

cannot be estimated by a small multiple of

$$\int_{Q(R)} (\zeta w)^q d\mu. \tag{7.12}$$

(Note that this estimation does not arise in the proof of [13, Lemma 2.3], so the latter result is correct.)

Note also that when Ψ satisfies (7.2) with $\alpha = 0$, we have the uniformly parabolic equations described in [16, Example 4] but without any assumptions on the maximum eigenvalue of the matrix $[a^{ij}]$. In particular, we reproduce the usual gradient estimate for parabolic p-Laplacian equations once we observe that the condition $\Psi(1) = 1$ can be replaced by $\Psi(\tau^*) = 1$ for some $\tau^* \ge 1$. If we further assume that $\varepsilon_1(v) = \gamma_3/v$ for some positive constant γ_3 and that $\Psi(v) \ge v$ (which is the case if $v\Psi'(v) \ge \Psi(v)$), then we can take as structure functions

$$\Lambda_0 = \Lambda_1 = \Lambda_2 = \frac{\gamma_0}{2} v \Psi(v), \tag{7.13}$$

and hence $\Lambda = \lambda = (1/2)\gamma_0 \Psi(v)$. With $w = v^2$, (5.5) reads

$$\sup_{Q_\tau(R/2)} v^2 \le C\left(\tau^2 + R^{-n-2} \int_{Q_\tau(R)} v\Psi(v)\, dX\right). \tag{7.14}$$

The integral here can be estimated directly via Lemma 6.2 and our estimate has the same form as [6, Equation VIII.5.1] although we have used the choices $\sigma = 1/2$, $\theta = \rho^2 = R^2$ for the parameters in [6]. Moreover, if $\Psi(v) = (v^2 - 1)^{(m-1)/2}$ with $m \in (1,2)$, then we choose $r \geq 2$ so that $n[m-2] + 2r > 2$, and we take $\Lambda = v$, $\lambda = v^{m-1}$, and $w = v^{r+N(2-m)/2}$. In this way, we also reproduce [6, Equation VIII.5.3] (with the same choice of parameters).

On the other hand, when $A = v$ and $B \equiv 0$, our method does not apply. To see why, we examine (3.3g) and (6.3c) with $\xi = v$. First, $|A|v \cdot \xi \geq 1/8$ for v sufficiently large, while $a^{ij}\xi_i\xi_j \leq v^{-3}$, so the structure function Λ_2 needs to be at least (some multiple of) v^3 and this choice of Λ_2 clearly does not satisfy (6.3c). This example is important because it is the motivating case for the structure described in [11]. Moreover, the hypotheses for gradient estimates in [11] and [5] are clearly satisfied for this choice of A and B.

8. Gradient estimates without boundary data

In [8], Ecker showed that the gradient of a solution to a prescribed mean curvature equation can be estimated, locally in space, just in terms of its initial data. Here, we show how that result follows from a simple modification of our estimates. In fact, we obtain a corresponding estimate for a larger class of equations.

To this end, we need to adjust our notation slightly. First, for any $R > 0$ and $T > 0$, we set

$$Q(R,T) = \{X \in \mathbb{R}^{n+1} : |x| < R, 0 < t < T\}, \tag{8.1}$$

and we write $Q_\tau(R,T)$ for the subset of $Q(R,T)$ on which $v > \tau$. We then have the following form of the energy inequality.

Lemma 8.1. *Let χ be a nonnegative Lipschitz function defined on $[\tau,\infty)$ for some $\tau \geq \tau_0$ and let ζ be a nonnegative $C^2(\overline{B(R)})$ function which vanishes on $\partial B(R)$. Suppose conditions (3.3) hold, and define Ξ by (3.4). If $v(x,0) \leq \tau$ for all $x \in B(R)$, then*

$$\begin{aligned}
&\int_{q_\tau(R,s)} \Xi(v)\zeta^2\,dx + \int_{Q_\tau(R,T)} \left[\left(1-\frac{\tau}{v}\right)\mathscr{C}^2 + \mathscr{E}\right]\chi\zeta^2\,dX \\
&\quad \leq 20\beta_1^2 \int_{Q_\tau(R,T)} \Lambda_0((v-\tau)\chi' + \chi)\zeta^2\,dX + 4\beta_1 \int_{Q_\tau(R,T)} \Lambda_1\chi\zeta|D\zeta|\,dX \\
&\qquad + 4\int_{Q_\tau(R,T)} |A|\chi[\,|D^2\zeta|\,\zeta + |D\zeta|^2]v\,dX + 32\beta_2 \int_{Q_\tau(R,T)} \Lambda_2((v-\tau)\chi' + \chi)|D\zeta|^2\,dX
\end{aligned} \tag{8.2}$$

for any $s \in (0,T)$.

Proof. We proceed exactly as in Lemma 3.1 except that the integral involving ζ_t is not present. □

Next, we note (see, e.g., [5, Corollary 6.9]) that our Sobolev inequality (4.1) holds if we replace $Q(R)$ by $Q(R,T)$ and $(-R^2,0)$ by $(0,T)$. Then the proof of Lemma 5.1 gives the following gradient bound.

Lemma 8.2. *Suppose that all the hypotheses of Lemma 5.1 hold except for (5.2d), which is replaced by the assumption that $v(x,0) \leq \tau$ for all $x \in B(R)$. Then there is a constant*

$c_3(n,\beta,\beta_1 R,\beta_2)$ such that

$$\sup_{Q_\tau(R/2,T)} \left(1-\frac{\tau}{v}\right)^{N+2} w \le c_3 R^{-n-2} \int_{Q_\tau(R,T)} w\left(\frac{\Lambda}{\lambda}\right)^{N/2} \frac{\Lambda}{v}\, dX. \tag{8.3}$$

Note that if $\Lambda = \lambda = v^\theta$ for some constant $\theta < 1$, then we can take $w = v^{1-\theta}$ to infer that the integrand in (8.3) is identically one, and hence we obtain a gradient bound directly which depends only on a gradient bound for the initial function and on data of the equation. In particular, we have the following result for p-Laplacian equations.

Corollary 8.3. *Let $m \in (1,2)$, and suppose u is a solution of the equation*

$$-u_t + \operatorname{div}\left(|Du|^{m-2} Du\right) = 0 \tag{8.4}$$

in some cylinder $Q(R,T)$ with $|Du|$ bounded on $B(R)\times\{0\}$. Then

$$\sup_{B(R/2)\times(0,T)} |Du| \le C(m,n)\left(1 + \sup_{B(R)\times\{0\}} |Du| + \left(\frac{T}{R^2}\right)^{1/(2-m)}\right). \tag{8.5}$$

To include the mean curvature equations, we must modify our structure conditions to include a condition on the maximum eigenvalue of the matrix $\partial A/\partial p$. Following [11], we assume that there is a positive function $\overline{\mu}$ such that

$$a^{ij}\psi_i\xi_j \le \left(\overline{\mu}|\psi|^2\right)^{1/2}\left(a^{ij}\xi_i\xi_j\right)^{1/2} \tag{8.6}$$

for all vectors ξ and ψ. Of course if $[a^{ij}]$ is symmetric, then we can take $\overline{\mu}$ to be the maximum eigenvalue of this matrix. With this hypothesis in hand, we have the following version of the energy inequality.

Lemma 8.4. *Let χ be a nonnegative Lipschitz function defined on $[\tau,\infty)$ for some $\tau \ge \tau_0$ and let ζ be a nonnegative $C^1(\overline{B(R)})$ function which vanishes on $\partial B(R)$. Suppose conditions (3.3a)–(3.3d) and (8.6) hold, and define Ξ by (3.4). If $v(x,0) \le \tau$ for all $x \in B(R)$, then*

$$\begin{aligned} &\int_{q_\tau(R,s)} \Xi(v)\zeta^2\, dx + \int_{Q_\tau(R,T)} \left[\left(1-\frac{\tau}{v}\right)\mathscr{C}^2 + \mathscr{E}\right]\chi\zeta^2\, dX \\ &\quad \le 12\beta_1^2 \int_{Q_\tau(R,T)} \Lambda_0\left((v-\tau)\chi' + \chi\right)\zeta^2\, dX + 4\int_{Q_\tau(R,T)} v^2\overline{\mu}|D\zeta|^2\, dX \end{aligned} \tag{8.7}$$

for any $s \in (0,T)$.

Proof. This is an easy modification of the proof of Lemma 5.1. See [5, Lemma 11.10] for details but note the differences in notation between that reference and the current paper. □

From this energy inequality, we obtain the following gradient estimate.

Lemma 8.5. *Suppose that conditions (3.3a)–(3.3d), (5.1), (5.2a), (5.2d), (5.3), (8.6) and*

$$v\overline{\mu} \le \Lambda \tag{8.8}$$

are satisfied and that $v(x,0) \le \tau$ for all $x \in B(R)$, then there is a constant $c_3(n,\beta,\beta_1 R,\beta_2)$ such that (8.3) holds.

Note that Corollary 8.3 also follows from this lemma. Furthermore, in case $A(X,z,p) = v$ and B depends only on X and z, we suppose that B is nonincreasing as a function of z and Lipschitz with respect to x, all the conditions of this lemma are satisfied (cf. [5, pages 279-280]) with $C_k^i = 0$,

$$\begin{gathered}\Lambda_0 = v, \qquad \overline{\mu} = \frac{1}{v}, \qquad w = v, \qquad \lambda = \Lambda = 1, \\ \beta = 2, \qquad \beta_1 = \left(\sup |B_x|\right)^{1/2}, \qquad \tau_0 = \sup_{B(R)\times\{0\}} v,\end{gathered} \tag{8.9}$$

and hence we infer that

$$\sup_{Q(R/2,T)} v \le C(n, R^2 \sup |B_x|)\left(\sup_{B(R)\times\{0\}} v + TR^{-2}\right), \tag{8.10}$$

which is a sharper form of [8, Theorem 3.1] in case the constant $\overline{\kappa}$ there is zero. To infer the estimate for general $\overline{\kappa}$, we perform a simple transformation. In our notation, the assumption involving $\overline{\kappa}$ is that $B_z \le \overline{\kappa}$, so let us note that $\overline{u} = \exp(\overline{\kappa}t)u$ is a solution of the equation

$$-\overline{u}_t + \operatorname{div} \overline{A}(X,\overline{u},D\overline{u}) + \overline{B}(X,\overline{u},D\overline{u}) = 0 \tag{8.11}$$

with

$$\begin{aligned}\overline{A}(X,z,p) &= \exp(-\overline{\kappa}t)A(X, \exp(\overline{\kappa}t)z, \exp(\overline{\kappa}t)p), \\ \overline{B}(X,z,p) &= \exp(-\overline{\kappa}t)B(x, \exp(\overline{\kappa}t)z) - \overline{\kappa}z.\end{aligned} \tag{8.12}$$

Now the hypotheses of Lemma 8.5 are satisfied for $\overline{u}$, $\overline{A}$, and $\overline{B}$ with $C_k^i = 0$,

$$\begin{gathered}\Lambda_0 = v, \qquad \overline{\mu} = \frac{1}{v}, \qquad w = v, \qquad \lambda = \exp(-2\overline{\kappa}T), \qquad \Lambda = 1, \\ \beta = 2, \qquad \beta_1 = \exp(\overline{\kappa}T)\left(\sup |B_x|\right)^{1/2}, \qquad \tau_0 = \sup_{B(R)\times\{0\}} v.\end{gathered} \tag{8.13}$$

The corresponding estimate for $v(D\overline{u})$ then implies that

$$\sup_{Q(R/2,T)} v \le C(n, \exp(\overline{\kappa}T), R^2 \sup |B_x|)\left(\sup_{B(R)\times\{0\}} v + TR^{-2}\right), \tag{8.14}$$

which is a sharper version of the full force of [8, Theorem 3.1].

9. Equations with faster than exponential growth

An important element in the theory of *a priori* estimates is the question of what classes of operators are encompassed. As we have already seen, if $A(p) = \Psi(v)v$ for some increasing scalar function Ψ, then our method provides a gradient estimate for some choices of Ψ

but not others. In particular, if Ψ grows too slowly (e.g., if Ψ is a constant), then our method does not supply a gradient estimate. In this section, we examine this structure when Ψ grows more rapidly than any exponential function.

Our first step is a positive one, reducing the pointwise gradient estimate to an integral estimate. We only need a slight variant of the argument in Section 5.

We assume that there is a positive increasing function y with $y(\tau_0) \geq 1$ such that

$$w'(v) \leq \frac{y(v)}{v} w(v),$$
$$\frac{d}{dv}\left(\frac{\Lambda(v)}{\lambda(v)}\right)^{N/2} \leq \frac{y(v)}{v}\left(\frac{\Lambda(v)}{\lambda(v)}\right)^{N/2}. \tag{9.1}$$

$$\left[\left(\frac{vy'}{y}\right)^2 + \left(\frac{v\lambda'}{\lambda}\right)^2 + y^2\right] y\lambda g^{ij}\xi_i\xi_j \leq v a^{ij}\xi_i\xi_j \tag{9.2}$$

for all $\xi \in \mathbb{R}^n$. In addition, we assume that

$$\Lambda_0 y \leq v\Lambda,$$
$$\Lambda_2 y \leq v\Lambda. \tag{9.3}$$

Theorem 9.1. *Suppose u is a solution of (1.1), and suppose that w, Λ, and λ satisfy conditions (3.3), (5.1a), (5.1c), (5.2), (5.3), (9.1), (9.2), and (9.3). Then (5.5) holds with c_1 determined by n, β, $\beta_1 R$, and β_2.*

Proof. Take χ as in Lemma 5.1 so that χ is increasing. It is not hard to see that

$$(v-\tau)\chi'(v) \leq Y(v)\chi(v) \tag{9.4}$$

for $Y(v) = (n+2)qy(v)$, and hence

$$\Xi(v) = \int_\tau^v (\sigma-\tau)\chi(\sigma)d\sigma \geq \int_\tau^v \frac{(\sigma-\tau)^2}{Y(\sigma)}\chi'(\sigma)d\sigma$$
$$\geq \frac{1}{Y(v)}\int_\tau^v (\sigma-\tau)^2\chi'(\sigma)d\sigma = \frac{1}{Y(v)}\left[(v-\tau)^2\chi(v) - 2\Xi(v)\right]. \tag{9.5}$$

Simple rearrangement gives

$$\Xi(v) \geq \frac{1}{Y(v)+2}(v-\tau)^2\chi(v) \tag{9.6}$$

and a simple calculation (cf. [5, Lemma 6.15]) gives

$$\Xi(v) \leq \frac{1}{2}(v-\tau)^2\chi(v). \tag{9.7}$$

We now define h by

$$h^2 = \chi y\lambda\left(1-\frac{\tau}{v}\right)^2 v\zeta^{(N+2)q-N} \tag{9.8}$$

to infer (5.12). From this inequality, the proof is exactly the same as for Lemma 5.1. □

As a specific example, we suppose that

$$A(X,z,p) = \exp\left(\exp\left(\frac{1}{2}v^2\right)\right)p, \qquad B(X,z,p) \equiv 0. \tag{9.9}$$

Then (3.3) is easily checked with

$$\Lambda_0 = \Lambda_1 = 0, \qquad \Lambda_2 = \exp\left(\exp\left(\frac{1}{2}v^2\right)\right), \qquad \beta_1 = 0, \tag{9.10}$$

and suitable β_2. The remainder of the hypotheses are satisfied with $w = v^K$ for any $K > 0$ and

$$\Lambda = v^3 \exp\left(\exp\left(\frac{1}{2}v^2\right)\right), \qquad \lambda = v^{-5}\exp(-v^2)\exp\left(\exp\left(\frac{1}{2}v^2\right)\right), \qquad y = (N+1)v^2. \tag{9.11}$$

In particular, for $K = 1$, we thus obtain

$$\sup v \le C\left(1 + R^{-n-2}\int_Q v^{4N+3}\exp\left(\frac{N}{2}v^2\right)\exp\left(\exp\left(\frac{1}{2}v^2\right)\right)dX\right). \tag{9.12}$$

To see why we cannot infer a complete gradient estimate for this example, we note that (9.2) immediately implies that

$$\lambda \le \exp\left(\exp\left(\frac{1}{2}v^2\right)\right)\exp(-v^2)v^{-2} \tag{9.13}$$

while (9.3) implies that $\Lambda \ge \exp(\exp((1/2)v^2))$, so we must take y no less than some constant times v^2. Hence, the integral in (5.5) is at least

$$\int \frac{w}{v}\exp\left(\exp\left(\frac{1}{2}v^2\right)\right)\exp\left(\frac{N}{2}v^2\right)dX, \tag{9.14}$$

and this integrand cannot be estimated by an expression of the form $v^q Du \cdot A$ for any power q, so Lemma 6.1 does not apply to this example. If we note that the integrand can be estimated by an expression of the form $w_1^q Du \cdot A$ with $w_1 = \exp(v^2)$, then it would seem that the proof of that lemma could be modified. If we try to imitate the proof of Lemma 6.1 but with $G(w_1)$ in place of $G(v)$ (as was done in [11]), the integral I_3 causes problems since the integrand has the form

$$\zeta^{2q-1}w_1^q|A|, \tag{9.15}$$

and this cannot be estimated by $\zeta^{2q-1}w_1^{q-\theta}Du \cdot A$ for any positive constant θ. Hence it is not possible to adapt the proof of Lemma 6.1 to this situation.

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