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Research Article

Positive Solutions for Boundary Value Problems of N-Dimension Nonlinear Fractional Differential System

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We study the boundary value problem for a kind N-dimension nonlinear fractional differential system with the nonlinear terms involved in the fractional derivative explicitly. The fractional differential operator here is the standard Riemann-Liouville differentiation. By means of fixed point theorems, the existence and multiplicity results of positive solutions are received. Furthermore, two examples given here illustrate that the results are almost sharp.

1. Introduction

We are interested in the following N-dimension nonlinear fractional differential system:

$$\begin{aligned} D_{0+}^{\alpha_1}x_1(t)+f_1\big(t,x_2(t),D_{0+}^{\mu_1}x_2(t)\big)&=0,\\ &\vdots\\ D_{0+}^{\alpha_{N-1}}x_{N-1}(t)+f_{N-1}\big(t,x_N(t),D_{0+}^{\mu_{N-1}}x_N(t)\big)&=0,\\ D_{0+}^{\alpha_N}x_N(t)+f_N\big(t,x_1(t),D_{0+}^{\mu_N}x_1(t)\big)&=0, \end{aligned}\qquad 0<t<1, \tag{1.1}$$

that is subject to the boundary conditions

$$\begin{aligned} x_1(0)=x_2(0)=\cdots=x_N(0)=0,\\ x_1(1)=x_2(1)=\cdots=x_N(1)=0, \end{aligned} \tag{1.2}$$

where $D_{0^+}^{\alpha_i}$ is the standard Riemann-Liouville fractional derivative of order α_i, $f_i \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $1 < \alpha_i < 2$, $\mu_i > 0$, $i = 1, 2, \ldots, N$, and $\alpha_i - \mu_{i-1} > 1$, $i = 1, 2, \ldots, N$, $\mu_0 = \mu_N$.

Recently, fractional differential equations (in short FDEs) have been studied extensively. The motivation for those works stems from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, and so on. For an extensive collection of such results, we refer the readers to the monographs by Samko et al. [1], Podlubny [2], Miller and Ross [3], and Kilbas et al. [4].

Some basic theory for the initial value problems of FDE involving Riemann-Liouville differential operator has been discussed by Lakshmikantham [5–7], El-Sayed et al. [8, 9], Diethelm and Ford [10], and Bai [11], and so on. Also, there are some papers which deal with the existence and multiplicity of solutions for nonlinear FDE boundary value problems (in short BVPs) by using techniques of topological degree theory. For example, Su [12] considered the BVP of the coupled system

$$\begin{aligned} &D^{\alpha}u(t) = f\left(t, v(t), D^{\mu}v(t)\right), \quad 0 < t < 1, \\ &D^{\beta}v(t) = g\left(t, u(t), D^{\nu}u(t)\right), \quad 0 < t < 1, \\ &u(0) = u(1) = v(0) = v(1) = 0. \end{aligned} \tag{1.3}$$

By using the Schauder fixed point theorem, one existence result was given.

In [13], Bai and Lü obtained positive solutions of the two-point BVP of FDE

$$\begin{gathered} D_{0^+}^{\alpha}u(t) = f\left(t, u(t)\right), \quad 0 < t < 1, \ 1 < \alpha \le 2, \\ u(0) = u(1) = 0 \end{gathered} \tag{1.4}$$

by means of Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem. $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative.

Zhang discussed the existence of solutions of the nonlinear FDE

$${}^{c}D_{0^+}^{\alpha}u(t) = f\left(t, u(t)\right), \quad 0 < t < 1, \ 1 < \alpha \le 2 \tag{1.5}$$

with the boundary conditions

$$u(0) = \nu \neq 0, \qquad u(1) = \rho \neq 0, \tag{1.6}$$

$$u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \tag{1.7}$$

in [14, 15], respectively. Since conditions (1.6) and (1.7) are nonzero boundary values, the Riemann-Liouville fractional derivative $D_{0^+}^{\alpha}$ is not suitable. Therefore, the author investigated the BVPs (1.5)-(1.6) and (1.5)–(1.7) by involving in the Caputo fractional derivative ${}^{c}D_{0^+}^{\alpha}$.

From above works, we can see a fact, although the BVPs of nonlinear FDE have been studied by some authors, to the best of our knowledge, higher-dimension fractional equation systems are seldom considered. Su in [12] studied the two-dimension system, however, the Schauder fixed point theorem cannot ensure the solutions to be positive. Since only positive solutions are useful for many applications, we investigate the existence and multiplicity of positive solutions for BVP (1.1)-(1.2) in this paper. In addition, two examples are given to demonstrate our results.

2. Preliminaries

For the convenience of the reader, we first recall some definitions and fundamental facts of fractional calculus theory, which can be found in the recent literatures [1–4].

Definition 2.1. The fractional integral of order $\tau > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_0^x \frac{f(t)}{(x-t)^{1-\tau}} dt, \quad x > 0, \tag{2.1}$$

provided that the integral exists, where $\Gamma(\tau)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad (z > 0), \tag{2.2}$$

for which, the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(1) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{2.3}$$

the Dirichlet formula

$$\int_0^1 t^{z-1}(1-t)^{\omega-1} dt = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}, \quad \left(z, \omega \notin \mathbb{Z}_0^-\right) \tag{2.4}$$

hold.

Definition 2.2. The fractional derivative of order $\tau > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ can be written as

$$D_{0^+}^{\tau} f(x) = \frac{1}{\Gamma(n-\tau)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\tau+1-n}} dt, \quad n = [\tau] + 1, \tag{2.5}$$

where $[\tau]$ denotes the integer part of τ, provided that the right side is pointwise defined on $(0,\infty)$.

Remark 2.3. The following properties are useful for our discussion:

$$I_{0+}^{\tau} I_{0+}^{\sigma} f(t) = I_{0+}^{\tau+\sigma} f(t), \quad D_{0+}^{\tau} I_{0+}^{\tau} f(t) = f(t), \quad \tau > 0,\ \sigma > 0,\ f \in L[0,1],$$
$$I_{0+}^{\tau} D_{0^+}^{\tau} f(t) = f(t) + c_1 t^{\tau-1} + c_2 t^{\tau-2} + \cdots + c_n t^{\tau-n}, \quad c_i \in \mathbb{R},\ i = 1,2,\ldots,n, \tag{2.6}$$
$$I_{0+}^{\tau} : C[0,1] \longrightarrow C[0,1], \quad D_{0+}^{\tau} f \in C(0,1) \cap L[0,1], \quad \tau > 0,\ f \in C[0,1].$$

In the following, we present the useful lemmas which are fundamental in the proof of our main results.

Lemma 2.4 (see [16]). *Let C be a convex subset of a normed linear space E and U be an open subset of C with $p^* \in U$. Then every compact continuous map $N : \overline{U} \to C$ has at least one of the following two properties:*

(A1) *N has a fixed point;*

(A2) *there is an $x \in \partial U$ with $x = (1-\overline{\lambda})p^* + \overline{\lambda} N x$, for some $0 < \overline{\lambda} < 1$.*

Definition 2.5. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geq t\alpha(x) + (1-t)\alpha(y), \tag{2.7}$$

for all $x, y \in P$, and $t \in [0,1]$.

Let α and β be nonnegative continuous convex functionals on the cone P, ψ be a nonnegative continuous concave functional on P. Then for positive real numbers $r > a$ and L, one defines the following convex sets:

$$\begin{aligned} P(\alpha, r; \beta, L) &= \{x \in P : \alpha(x) < r,\ \beta(x) < L\}, \\ \overline{P}(\alpha, r; \beta, L) &= \{x \in P : \alpha(x) \leq r,\ \beta(x) \leq L\}, \\ P(\alpha, r; \beta, L; \psi, a) &= \{x \in P : \alpha(x) < r,\ \beta(x) < L,\ \psi(x) > a\}, \\ \overline{P}(\alpha, r; \beta, L; \psi, a) &= \{x \in P : \alpha(x) \leq r,\ \beta(x) \leq L,\ \psi(x) \geq a\}. \end{aligned} \tag{2.8}$$

The assumptions below about the nonnegative continuous convex functionals α, β will be used as follows:

(B1) there exists $M > 0$ such that $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$, for all $x \in P$;

(B2) $P(\alpha, r; \beta, L) \neq \emptyset$, for all $r > 0$, $L > 0$.

Lemma 2.6 (see [17]). *Let P be a cone in a real Banach space E, $r_2 \geq d > b > r_1 > 0$, and $L_2 \geq L_1 > 0$. Assume that α and β are nonnegative continuous convex functionals satisfying (B1) and (B2), ψ is a nonnegative continuous concave functional on P such that $\psi(y) \leq \alpha(y)$, for all*

$y \in \overline{P}(\alpha, r_1; \beta, L_1)$ and $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$, is a completely continuous operator. Suppose

(C1) $\{y \in P(\alpha, d; \beta, L_2; \psi, b) : \psi(y) > b\} \neq \emptyset$, $\psi(Ty) > b$, *for* $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$;

(C2) $\alpha(Ty) < r_1$, $\beta(Ty) < L_1$, *for all* $y \in \overline{P}(\alpha, r_1; \beta, L_1)$;

(C3) $\psi(Ty) > b$, *for all* $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$ *with* $\alpha(Ty) > d$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P}(\alpha, r_2; \beta, L_2)$ with

$$\begin{aligned} &y_1 \in P(\alpha, r_1; \beta, L_1), \\ &y_2 \in \{y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b) : \psi(y) > b\}, \\ &y_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)). \end{aligned} \tag{2.9}$$

3. Related lemmas

Let $X = X_1 \times X_2 \times \cdots \times X_N$ with the norm

$$\|x\| = \max\{\|x_i\|_{X_i} : i = 1, 2, \ldots, N\}, \quad \text{for } x = (x_1, x_2, \ldots, x_N) \in X, \tag{3.1}$$

where $X_i = \{x_i \in C[0,1] : D_{0^+}^{\mu_{i-1}} x_i \in C[0,1]\}$, $i = 1, 2, \ldots, N$ with

$$\|x_i\|_{X_i} = \|x_i\|_\infty + \|D^{\mu_{i-1}} x_i\|_\infty, \tag{3.2}$$

where $\|\cdot\|_\infty$ is the standard sup norm of the space $C[0,1]$. Throughout, we denote $\mu_0 = \mu_N$ and $x_{N+1} = x_1$. Then X is a Banach space (see [12]).

Define the cone $P \subset X$ by

$$P = \{x = (x_1, x_2, \ldots, x_N) \in X : x_i(t) \geq 0,\ x_i(0) = 0,\ t \in [0,1],\ i = 1, 2, \ldots, N\}. \tag{3.3}$$

Lemma 3.1. *If $x \in P$, then $\|x_i\|_\infty \leq (1/\Gamma(1 + \mu_{i-1}))\|D^{\mu_{i-1}} x_i\|_\infty$, $i = 1, 2, \ldots, N$.*

Proof. For $x = (x_1, x_2, \ldots, x_N) \in P$, we have

$$\begin{aligned} x_i(t) &= I_{0+}^{\mu_{i-1}} D_{0+}^{\mu_{i-1}} x_i(t) \\ &\leq \frac{1}{\Gamma(\mu_{i-1})} \int_0^t \frac{|D^{\mu_{i-1}} x_i(s)|}{(t-s)^{1-\mu_{i-1}}} ds \\ &\leq \frac{1}{\Gamma(1 + \mu_{i-1})} \|D^{\mu_{i-1}} x_i\|_\infty, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{3.4}$$

That is, $\|x_i\|_\infty \leq (1/\Gamma(1 + \mu_{i-1}))\|D^{\mu_{i-1}} x_i\|_\infty$, $i = 1, 2, \ldots, N$. □

It is well known that the solution for the system BVP (1.1)-(1.2) is equivalent to the fixed point of the following integral system:

$$\begin{aligned}
T_1x_2(t) &= \int_0^1 G_1(t,s)f_1\big(s,x_2(s),D_{0+}^{\mu_1}x_2(s)\big)ds,\\
&\;\;\vdots\\
T_{N-1}x_N(t) &= \int_0^1 G_{N-1}(t,s)f_{N-1}\big(s,x_N(s),D_{0+}^{\mu_{N-1}}x_N(s)\big)ds, \qquad 0<t<1,\\
T_Nx_1(t) &= \int_0^1 G_N(t,s)f_N\big(s,x_1(s),D_{0+}^{\mu_N}x_1(s)\big)ds,
\end{aligned} \tag{3.5}$$

for $x \in X$, where

$$G_i(t,s) = \frac{1}{\Gamma(\alpha_i)} \begin{cases} \big(t(1-s)\big)^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \le s \le t \le 1,\\ \big(t(1-s)\big)^{\alpha_i-1}, & 0 \le t \le s \le 1. \end{cases} \tag{3.6}$$

Denote $Tx := (T_1x_2,\dots,T_{N-1}x_N,T_Nx_1)^{\top}$, we can see

$$T_ix_{i+1}(t) = t^{\alpha_i-1}I_{0+}^{\alpha_i}f_i\big(1,x_{i+1}(1),D^{\mu_i}x_{i+1}(1)\big) - I_{0+}^{\alpha_i}f_i\big(t,x_{i+1}(t),D^{\mu_i}x_{i+1}(t)\big), \tag{3.7}$$

$i = 1,2,\dots,N$. For the Green functions $G_i(t,s)$, $i = 1,2,\dots,N$, we can obtain

(i) $G_i(t,s) \ge 0$, for $t,s \in [0,1]$, $\gamma_i(s)G_i(s,s) \le G_i(t,s) \le G_i(s,s)$, for $(t,s) \in [\theta,1-\theta] \times [0,1]$, $\theta \in (0,1/2)$, where

$$\gamma_i(s) = \begin{cases} \dfrac{\big((1-\theta)(1-s)\big)^{\alpha_i-1} - (1-\theta-s)^{\alpha_i-1}}{\big(s(1-s)\big)^{\alpha_i-1}}, & 0 < s \le r_i,\\ \dfrac{\theta^{\alpha_i-1}}{s^{\alpha_i-1}}, & r_i \le s < 1, \end{cases} \tag{3.8}$$

here, $r_i \in (\theta,1-\theta)$ is the unique solution of the equation

$$\big((1-\theta)(1-s)\big)^{\alpha_i-1} - (1-\theta-s)^{\alpha_i-1} = \big(\theta(1-s)\big)^{\alpha_i-1}; \tag{3.9}$$

(ii) $\max_{t\in[0,1]}\int_0^1 G_i(t,s)ds = (\alpha_i-1)^{\alpha_i-1}/\alpha_i^{\alpha_i}\Gamma(\alpha_i+1) =: \rho_{i1}$ and $\min_{t\in[\theta,1-\theta]}\int_0^1 G_i(t,s)ds = \theta(1-\theta)^{\alpha_i-1}/\Gamma(\alpha_i+1) =: \rho_{i2}$.

Lemma 3.2. *$T : P \to P$ is completely continuous.*

Proof. We divide the proof into three steps.

Step 1. $T : P \longrightarrow P$. In fact, for any $x \in P$, since $f_i(t, x_{i+1}(t), D_{0+}^{\mu_i} x_{i+1}(t)) \geq 0$ for $t \in [0,1]$ and $G_i(t,s) \geq 0$, for $t,s \in [0,1]$, $T_i x_{i+1}(t) \geq 0$, for $t \in [0,1]$. Moreover, $G(0,s) = 0$ implies that $T_i x_{i+1}(0) = 0$.

Step 2. T is continuous on P, which is valid due to the continuity of the function f.

Step 3. We will show that T is relatively compact. For any given bounded set $U \subset P$, there exists $M > 0$ such that $\|x\| \leq M$, for all $x \in U$. We take $\kappa_i = \max\{|f_i(t,u,v)| : t \in [0,1],\ |u| \leq M,\ |v| \leq M\}$. For $x \in U$, let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$, we have

$$\begin{aligned}
\left|T_i x_{i+1}(t_1) - T_i x_{i+1}(t_2)\right| &= \Big|\left(t_1^{\alpha_i - 1} - t_2^{\alpha_i - 1}\right) I_{0+}^{\alpha_i} f_i\left(1, x_{i+1}(1), D_{0+}^{\mu_i} x_{i+1}(1)\right) \\
&\quad - \left[I_{0+}^{\alpha_i} f_i\left(t_1, x_{i+1}(t_1), D_{0+}^{\mu_i} x_{i+1}(t_1)\right) - I_{0+}^{\alpha_i} f_i\left(t_2, x_{i+1}(t_2), D_{0+}^{\mu_i} x_{i+1}(t_2)\right)\right]\Big| \\
&\leq \left|t_1^{\alpha_i - 1} - t_2^{\alpha_i - 1}\right| \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\
&\quad + \Big|\frac{1}{\Gamma(\alpha_i)} \int_0^{t_2} (t_2 - s)^{\alpha_i - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\
&\quad - \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} (t_1 - s)^{\alpha_i - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds\Big| \\
&\leq \frac{\kappa_i}{\Gamma(\alpha_i + 1)} \left|t_1^{\alpha_i - 1} - t_2^{\alpha_i - 1}\right| \\
&\quad + \frac{\kappa_i}{\Gamma(\alpha_i)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\alpha_i - 1} ds + \int_0^{t_1} \left|(t_2 - s)^{\alpha_i - 1} - (t_1 - s)^{\alpha_i - 1}\right| ds\right] \\
&= \frac{\kappa_i}{\Gamma(\alpha_i + 1)} \left(t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1} + t_2^{\alpha_i} - t_1^{\alpha_i}\right) \longrightarrow 0, \quad \text{as } t_2 - t_1 \longrightarrow 0.
\end{aligned} \tag{3.10}$$

Notice that

$$D_{0+}^{\mu_{i-1}} T_i x_{i+1}(t) = I_{0+}^{\alpha_i} f_i\left(1, x_{i+1}(1), D^{\mu_i} x_{i+1}(1)\right) \cdot D_{0+}^{\mu_{i-1}} t^{\alpha_i - 1} - I_{0+}^{\alpha_i - \mu_{i-1}} f_i\left(t, x_{i+1}(t), D^{\mu_{i-1}} x_{i+1}(t)\right), \tag{3.11}$$

one gets

$$\begin{aligned}
&\left|D_{0+}^{\mu_{i-1}} T_i x_{i+1}(t_1) - D_{0+}^{\mu_{i-1}} T_i x_{i+1}(t_2)\right| \\
&\quad = \Big|I_{0+}^{\alpha_i} f_i\left(1, x_{i+1}(1), D^{\mu_i} x_{i+1}(1)\right)\left(D_{0+}^{\mu_{i-1}} t_1^{\alpha_i - 1} - D_{0+}^{\mu_{i-1}} t_2^{\alpha_i - 1}\right) \\
&\qquad - \left[I_{0+}^{\alpha_i - \mu_{i-1}} f_i\left(t_1, x_{i+1}(t_1), D^{\mu_i} x_{i+1}(t_1)\right) - I_{0+}^{\alpha_i - \mu_{i-1}} f_i\left(t_2, x_{i+1}(t_2), D^{\mu_i} x_{i+1}(t_2)\right)\right]\Big| \\
&\quad \leq \frac{\kappa_i}{\alpha_i \Gamma(\alpha_i - \mu_{i-1})} \left|t_1^{\alpha_i - \mu_{i-1} - 1} - t_2^{\alpha_i - \mu_{i-1} - 1}\right| \\
&\qquad + \frac{\kappa_i}{\Gamma(\alpha_i - \mu_{i-1})} \left[\int_{t_1}^{t_2} (t_2 - s)^{\alpha_i - \mu_{i-1} - 1} ds + \int_0^{t_1} \left|(t_2 - s)^{\alpha_i - \mu_{i-1} - 1} - (t_1 - s)^{\alpha_i - \mu_{i-1} - 1}\right| ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\kappa_i}{\alpha_i \Gamma(\alpha_i - \mu_{i-1})} \left(t_2^{\alpha_i - \mu_{i-1} - 1} - t_1^{\alpha_i - \mu_{i-1} - 1} \right) \\
&\quad + \frac{\kappa_i}{\Gamma(\alpha_i - \mu_{i-1} + 1)} \left(t_2^{\alpha_i - \mu_{i-1}} - t_1^{\alpha_i - \mu_{i-1}} \right) \longrightarrow 0, \quad \text{as } t_2 - t_1 \longrightarrow 0,
\end{aligned}
\tag{3.12}
$$

where $i = 1, 2, \ldots, N$, we can see that TU is an equicontinuous set. Now, we proof that T is uniformly bounded. For any $x \in U$,

$$
\begin{aligned}
\left| T_i x_{i+1}(t) \right| &= \left| t^{\alpha_i - 1} I_{0+}^{\alpha_i} f_i\left(1, x_{i+1}(1), D_{0+}^{\mu_i} x_{i+1}(1)\right) - I_{0+}^{\alpha_i} f_i\left(t, x_{i+1}(t), D_{0+}^{\mu_i} x_{i+1}(t)\right) \right| \\
&\leq \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1 - s)^{\alpha_i - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\
&\leq \frac{2\kappa_i}{\Gamma(\alpha_i + 1)} < \infty, \\
\left| D_{0+}^{\mu_{i-1}} T_i x_{i+1}(t) \right| &= \left| I_{0+}^{\alpha_i} f_i\left(1, x_{i+1}(1), D^{\mu_i} x_{i+1}(1)\right) D_{0+}^{\mu_{i-1}} t^{\alpha_i - 1} - I_{0+}^{\alpha_i - \mu_{i-1}} f_i\left(t, x_{i+1}(t), D^{\mu_i} x_{i+1}(t)\right) \right| \\
&\leq \frac{\kappa_i}{\alpha_i \Gamma(\alpha_i)} \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i - \mu_{i-1})} + \frac{\kappa_i}{\Gamma(\alpha_i - \mu_{i-1})} \int_0^t (t - s)^{\alpha_i - \mu_{i-1} - 1} ds \\
&\leq \frac{\kappa_i (2\alpha_i - \mu_{i-1})}{\alpha_i \Gamma(\alpha_i - \mu_{i-1} + 1)} < \infty,
\end{aligned}
\tag{3.13}
$$

where $i = 1, 2, \ldots, N$. That is, TU is uniformly bounded. Thus, T is relatively compact. By means of the Arzela-Ascoli theorem, $T : P \rightarrow P$ is completely continuous. □

4. The existence of one positive solution

Theorem 4.1. *If there exist* $a_i, b_i, c_i \in C([0,1], \mathbb{R}_+)$, $i = 1, 2, \ldots, N$ *satisfying*

$$
\left\| b_i \right\|_\infty + \left\| c_i \right\|_\infty < \min \left\{ \frac{\alpha_i^{\alpha_i} \Gamma(\alpha_i + 1)}{(\alpha_i - 1)^{\alpha_i - 1}}, \frac{\alpha_i \Gamma(\alpha_i - \mu_{i-1} + 1)}{2\alpha_i - \mu_{i-1}} \right\},
\tag{4.1}
$$

such that

$$
f_i(t, x, y) \leq a_i(t) + b_i(t) x + c_i(t) y.
\tag{4.2}
$$

Then the BVP (1.1)-(1.2) *has at least one positive solution.*

Proof. Lemma 3.2 indicates that $T : P \to P$ is completely continuous.

For $i = 1, 2, \ldots, N$, let

$$Q_i > \max\left\{ \frac{(\alpha_i - 1)^{\alpha_i - 1}\|a_i\|_\infty}{\alpha_i^{\alpha_i}\Gamma(\alpha_i + 1) - (\alpha_i - 1)^{\alpha_i - 1}(\|b_i\|_\infty + \|c_i\|_\infty)}, \frac{(2\alpha_i - \mu_{i-1})\|a_i\|_\infty}{\alpha_i\Gamma(\alpha_i - \mu_{i-1} + 1) - (2\alpha_i - \mu_{i-1})(\|b_i\|_\infty + \|c_i\|_\infty)} \right\},$$
$$Q = \max\{Q_i : i = 1, 2, \ldots, N\}. \tag{4.3}$$

Define $\Omega = \{x = (x_1, x_2, \ldots, x_N) \in P : \|x_i\|_{X_i} < Q_i,\ i = 1, 2, \ldots, N\}$, then $\|x\| < Q$. For $\forall x \in \partial\Omega$, $\|x_i\|_{X_i} = Q_i$. Thus, $\|x_i\|_\infty \le Q_i$ and $\|D_{0+}^{\mu_{i-1}} x_i\|_\infty \le Q_i$:

$$\begin{aligned}
\left|T_i x_{i+1}(t)\right| &= \int_0^1 G_i(t,s) f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\
&\le \int_0^1 G_i(t,s)\left[a_i(s) + b_i(s)x_{i+1}(s) + c_i(s)D_{0+}^{\mu_i} x_{i+1}(s)\right] ds \\
&\le \left[\|a_i\|_\infty + (\|b_i\|_\infty + \|c_i\|_\infty)Q_i\right] \frac{(\alpha_i - 1)^{\alpha_i - 1}}{\alpha_i^{\alpha_i}\Gamma(\alpha_i + 1)} < Q_i, \\
\left|D_{0+}^{\mu_{i-1}} T_i x_{i+1}(t)\right| &= \left|I_{0+}^{\alpha_i} f_i(1, x_{i+1}(1), D^{\mu_i} x_{i+1}(1)) D_{0+}^{\mu_{i-1}} t^{\alpha_i - 1} - I_{0+}^{\alpha_i - \mu_{i-1}} f_i(t, x_{i+1}(t), D^{\mu_i} x_{i+1}(t))\right| \\
&\le \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \cdot \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i - \mu_{i-1})} t^{\alpha_i - \mu_{i-1} - 1} \\
&\quad + \frac{1}{\Gamma(\alpha_i - \mu_{i-1})} \int_0^t (t-s)^{\alpha_i - \mu_{i-1} - 1} f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\
&\le \frac{\|a_i\|_\infty + (\|b_i\|_\infty + \|c_i\|_\infty)Q_i}{\alpha_i\Gamma(\alpha_i - \mu_{i-1})} + \frac{\|a_i\|_\infty + (\|b_i\|_\infty + \|c_i\|_\infty)Q_i}{\Gamma(\alpha_i - \mu_{i-1} + 1)} \\
&= \left[\|a_i\|_\infty + (\|b_i\|_\infty + \|c_i\|_\infty)Q_i\right] \frac{(2\alpha_i - \mu_{i-1})}{\alpha_i\Gamma(\alpha_i - \mu_{i-1} + 1)} < Q_i
\end{aligned} \tag{4.4}$$

indicate that $\|T_i x_{i+1}\|_{X_i} < Q_i$, and then $\|Tx\| = \max\{\|T_i x_{i+1}\|_{X_i} : i = 1, 2, \ldots, N\} < Q$. Take $p^* = 0$ in Lemma 2.4, for any $x \in \partial\Omega$, $x = \overline{\lambda} Tx (0 < \overline{\lambda} < 1)$ does not hold. Hence, the operator T has at least a fixed point, then the BVP (1.1)-(1.2) has at least one positive solution. □

Example 4.2. Consider the problem

$$\begin{gathered}
D_{0+}^{5/3} x_1(t) + f_1\left(t, x_2(t), D_{0+}^{1/4} x_2(t)\right) = 0, \quad 0 < t < 1, \\
D_{0+}^{3/2} x_2(t) + f_2\left(t, x_1(t), D_{0+}^{1/3} x_1(t)\right) = 0, \quad 0 < t < 1, \\
x_1(0) = x_1(1) = x_2(0) = x_2(1) = 0,
\end{gathered} \tag{4.5}$$

where

$$f_1(t,u,v) = \frac{10}{9}\Gamma\left(\frac{2}{3}\right) - \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\left(1 + \frac{2\sqrt{\pi}}{\Gamma(1/4)}\right)t + \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\left(1 + \frac{12\sqrt{\pi}}{5\Gamma(1/4)}\right)t^2 + \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\sqrt{t}u + \frac{1}{9}\Gamma\left(\frac{1}{3}\right)t^{3/4}v,$$
$$f_2(t,u,v) = \frac{3}{4}\sqrt{\pi} - \frac{1}{2}\left(1 + \frac{\Gamma(2/3)}{\Gamma(1/3)}\right)t + \frac{1}{4}\left(2 + \frac{5\Gamma(2/3)}{2\Gamma(1/3)}\right)t^2 + \frac{1}{2}t^{1/3}u + \frac{1}{4}t^{2/3}v,$$
$$\alpha_1 = \frac{5}{3}, \qquad \alpha_2 = \frac{3}{2}, \qquad \mu_1 = \frac{1}{4}, \qquad \mu_2 = \frac{1}{3}. \tag{4.6}$$

Choose

$$a_1(t) = \frac{10}{9}\Gamma\left(\frac{2}{3}\right) + \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\left(1 + \frac{12\sqrt{\pi}}{5\Gamma(1/4)}\right)t^2, \quad b_1(t) = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\sqrt{t}, \quad c_1(t) = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)t^{3/4},$$
$$a_2(t) = \frac{3}{4}\sqrt{\pi} + \frac{1}{4}\left(2 + \frac{5\Gamma(2/3)}{2\Gamma(1/3)}\right)t^2, \quad b_2(t) = \frac{1}{2}t^{1/3}, \quad c_2(t) = \frac{1}{4}t^{2/3}. \tag{4.7}$$

It is easy to check that (4.1) holds. Thus, by Theorem 4.1, the BVP (4.5) has at least one positive solution. In fact, $x(t) = (t^{3/2}(1-t), t^{1/2}(1-t))^{\top}$ is such a solution.

5. The existence of triple positive solutions

Let the nonnegative continuous convex functionals α, β and the nonnegative continuous concave functional ψ be defined on the cone P by

$$\alpha(x) = \max\left\{\left\|x_i\right\|_\infty : i = 1,2,\dots,N\right\},$$
$$\beta(x) = \max\left\{\left\|D_{0+}^{\mu_{i-1}}x_i\right\|_\infty : i = 1,2,\dots,N\right\},$$
$$\psi(x) = \min\left\{\min_{\theta\le t\le 1-\theta}\left|x_i(t)\right| : i = 1,2,\dots,N\right\}. \tag{5.1}$$

Obviously, α and β satisfy (B1) and (B2), $\psi(x) \le \alpha(x)$, for all $x \in P$.

For simplicity, we denote

$$\rho_{i3} := \int_\theta^{1-\theta}\gamma_i(s)G_i(s,s)\,ds, \qquad \rho_{i4} := \frac{2\alpha_i - \mu_{i-1}}{\alpha_i\Gamma(\alpha_i - \mu_{i-1} + 1)},$$
$$\sigma := \max\left\{\frac{1}{\Gamma(1+\mu_i)} : i = 1,2,\dots,N\right\}. \tag{5.2}$$

Theorem 5.1. *Assume that there exist constants $\sigma L \geq b/\theta > b > \sigma l > 0$ such that $b\Gamma(\mu_i + 1) \leq \theta L$, for $i = 1, 2, \ldots, N$. Suppose*

(H1) $f_i(t, u, v) \leq \min\{\sigma L/\rho_{i1}, L/\rho_{i4}\}$, $(t, u, v) \in [0, 1] \times [0, \sigma L] \times [-L, L]$;

(H2) $f_i(t, u, v) > b/\rho_{i2}$, $(t, u, v) \in [0, 1] \times [b, b/\theta] \times [-L, L]$;

(H3) $f_i(t, u, v) < \min\{\sigma l/\rho_{i1}, l/\rho_{i4}\}$, $(t, u, v) \in [0, 1] \times [0, \sigma l] \times [-l, l]$;

(H4) $f_i(t, u, v) > b/\rho_{i3}$, $(t, u, v) \in [\theta, 1-\theta] \times [b, \sigma L] \times [-L, L]$.

Then the BVP (1.1)-(1.2) *has at least three positive solutions $x = (x_1, x_2, \ldots, x_N)$, $y = (y_1, y_2, \ldots, y_N)$, and $z = (z_1, z_2, \ldots, z_N)$ such that*

$$\begin{gathered} 0 \leq x_i(t) \leq \sigma l, \quad 0 \leq y_i(t) \leq \sigma L, \quad \sigma l \leq z_i(t) \leq \sigma L, \quad t \in [0,1], \\ \left\| D_{0+}^{\mu_{i-1}} x_i \right\|_\infty \leq l, \quad \left\| D_{0+}^{\mu_{i-1}} y_i \right\|_\infty \leq L, \quad -l \leq D_{0+}^{\mu_{i-1}} z_i(t) \leq L, \quad t \in [0,1], \\ y_i(t) > b, \quad z_i(t) \leq b, \quad t \in [\theta, 1-\theta], \text{ for } i = 1, 2, \ldots, N. \end{gathered} \tag{5.3}$$

Proof. Lemma 3.2 has showed that $T : P \to P$ is completely continuous. Now, we will verify that all the conditions of Lemma 2.6 are satisfied. The proof is based on the following steps.

Step 1. We will show that (H1) implies $T : \overline{P}(\alpha, \sigma L; \beta, L) \to \overline{P}(\alpha, \sigma L; \beta, L)$.

In fact, for $x \in \overline{P}(\alpha, \sigma L; \beta, L)$, $\alpha(x) \leq \sigma L$, $\beta(x) \leq L$, and then $\|x_i\|_\infty \leq \sigma L$, $\|D_{0+}^{\mu_{i-1}} x_i\|_\infty \leq L$, $i = 1, 2, \ldots, N$. In view of (H1), we have

$$\begin{aligned} \left\| (T_i x_{i+1}) \right\|_\infty &= \max_{0 \leq t \leq 1} \int_0^1 G_i(t,s) f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i} x_{i+1}(s)\right) ds \\ &\leq \max_{(t,u,v) \in [0,1]\times[0,\sigma L]\times[-L,L]} f_i(t,u,v) \cdot \max_{0 \leq t \leq 1} \int_0^1 G_i(t,s)\, ds \\ &\leq \frac{\sigma L}{\rho_{i1}} \cdot \rho_{i1} = \sigma L, \\ \left\| (D^{\mu_{i-1}} T_i x_{i+1}) \right\|_\infty &= \max_{0 \leq t \leq 1} \Big| I_{0+}^{\alpha_i} f_i\left(1, x_{i+1}(1), D^{\mu_i} x_{i+1}(1)\right) \\ &\qquad \cdot D_{0+}^{\mu_{i-1}} t^{\alpha_i - 1} - I_{0+}^{\alpha_i - \mu_{i-1}} f_i\left(t, x_{i+1}(t), D^{\mu_i} x_{i+1}(t)\right) \Big| \\ &\leq \max_{(t,u,v) \in [0,1]\times[0,\sigma L]\times[-L,L]} f(t,u,v) \\ &\quad \cdot \max_{0 \leq t \leq 1} \left[\frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i - 1} ds \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i - \mu_{i-1})} t^{\alpha_i - \mu_{i-1} - 1} \right. \\ &\qquad \left. + \frac{1}{\Gamma(\alpha_i - \mu_{i-1})} \int_0^t (t-s)^{\alpha_i - \mu_{i-1} - 1} ds \right] \\ &\leq \frac{L}{\rho_{i4}} \cdot \rho_{i4} = L. \end{aligned} \tag{5.4}$$

Then $\alpha(Tx) \leq \sigma L$ and $\beta(Tx) \leq L$, that is, $Tx \in \overline{P}(\alpha, \sigma L; \beta, L)$.

Step 2. To check the condition (C1) in Lemma 2.6, we choose $x^*(t) = ((b/\theta)t^{\mu_N}, (b/\theta)t^{\mu_1}, \ldots, (b/\theta)t^{\mu_{N-1}})$, $t \in [0,1]$. It is easy to see that

$$\alpha(x^*) = \max\left\{\max_{t\in[0,1]}\left|\frac{b}{\theta}t^{\mu_i}\right| : i = 1,2,\ldots,N\right\} = \frac{b}{\theta},$$
$$\beta(x^*) = \max\left\{\max_{t\in[0,1]}\left|\frac{b}{\theta}D_{0+}^{\mu_i}t^{\mu_i}\right| : i = 1,2,\ldots,N\right\} = \max\left\{\frac{b}{\theta}\Gamma(1+\mu_i) : i = 1,2,\ldots,N\right\} \le L,$$
$$\psi(x^*) = \min\left\{\min_{t\in[\theta,1-\theta]}\left|\frac{b}{\theta}t^{\mu_i}\right| : i = 1,2,\ldots,N\right\} = \min\left\{\frac{b}{\theta}\theta^{\mu_i} : i = 1,2,\ldots,N\right\} > b. \tag{5.5}$$

Consequently, $\{x \in \overline{P}(\alpha, b/\theta; \beta, L; \psi, b) : \psi(x) > b\} \ne \varnothing$. For any $x \in \overline{P}(\alpha, b/\theta; \beta, L; \psi, b)$, from (H2), one gets

$$\begin{aligned}
\min_{t\in[\theta,1-\theta]}\left|T_i x_{i+1}(t)\right| &= \min_{t\in[\theta,1-\theta]}\int_0^1 G_i(t,s)f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i}x(s)\right)ds \\
&\ge \min_{(t,u,v)\in[0,1]\times[b,b/\theta]\times[-L,L]} f_i(t,u,v)\cdot\min_{t\in[\theta,1-\theta]}\int_0^1 G_i(t,s)ds \\
&> \frac{b}{\rho_{i2}}\cdot\rho_{i2} = b,
\end{aligned} \tag{5.6}$$

then we can obtain $\psi(Tx) > b$.

Step 3. It is similar to Step 1 that we can prove $T : \overline{P}(\alpha, \sigma l; \beta, l) \to \overline{P}(\alpha, \sigma l; \beta, l)$ by condition (H3), that is, (C2) in Lemma 2.6 holds.

Step 4. We verify that (C3) in Lemma 2.6 is satisfied. For $x \in \overline{P}(\alpha, \sigma L; \beta, L; \psi, b)$ with $\alpha(Tx) > b/\theta$, we have

$$\begin{aligned}
\min_{t\in[\theta,1-\theta]}\left|T_i x_{i+1}(t)\right| &\ge \int_0^1 \gamma_i(s)G_i(s,s)f_i\left(s, x_{i+1}(s), D_{0+}^{\mu_i}x_{i+1}(s)\right)ds \\
&\ge \int_\theta^{1-\theta}\gamma_i(s)G_i(s,s)ds\cdot\min_{(t,u,v)\in[\theta,1-\theta]\times[b,\sigma L]\times[-L,L]} f_i(t,u,v) \\
&> \rho_{i3}\cdot\frac{b}{\rho_{i3}} = b.
\end{aligned} \tag{5.7}$$

Thus, $\psi(Tx) > b$, (C3) in Lemma 2.6 is satisfied.

Therefore, the operator T has three points $x, y, z \in \overline{P}(\alpha, \sigma L; \beta, L)$ with

$$\begin{gathered}
x \in P(\alpha, \sigma l; \beta, l), \qquad y \in P(\alpha, \sigma L; \beta, L; \psi, b), \\
z \in \overline{P}(\alpha, \sigma L; \beta, L) \setminus \left(P(\alpha, \sigma L; \beta, L; \psi, b) \cup P(\alpha, \sigma l; \beta, l)\right).
\end{gathered} \tag{5.8}$$

Then the BVP (1.1)-(1.2) has three positive solutions $x, y, z \in \overline{P}(\alpha, \sigma L; \beta, L)$ such that

$$
\begin{gathered}
0 \le x_i(t) \le \sigma l, \quad 0 \le y_i(t) \le \sigma L, \quad \sigma l \le z_i(t) \le \sigma L, \quad t \in [0,1], \\
\left\| D_{0+}^{\mu_{i-1}} x_i \right\|_\infty \le l, \quad \left\| D_{0+}^{\mu_{i-1}} y_i \right\|_\infty \le L, \quad -l \le D_{0+}^{\mu_{i-1}} z_i(t) \le L, \quad t \in [0,1], \\
y_i(t) > b, \quad z_i(t) \le b, \quad t \in [\theta, 1-\theta], \text{ for } i = 1, 2, \dots, N.
\end{gathered} \tag{5.9}
$$

□

Example 5.2. Consider the problem

$$
\begin{gathered}
D_{0+}^{3/2} x_1(t) + f_1\left(t, x_2(t), D_{0+}^{1/2} x_2(t)\right) = 0, \quad 0 < t < 1, \\
D_{0+}^{7/4} x_2(t) + f_2\left(t, x_1(t), D_{0+}^{1/4} x_1(t)\right) = 0, \quad 0 < t < 1, \\
x_1(0) = x_1(1) = x_2(0) = x_2(1) = 0,
\end{gathered} \tag{5.10}
$$

where

$$
f_1(t,u,v) = \begin{cases}
\left(\dfrac{1}{2}\right)^t + \dfrac{u^2}{10^2} + \dfrac{|v|}{10^6}, & u \in \left[0, \dfrac{56}{25}\right], \\
\left(\dfrac{1}{2}\right)^t + \dfrac{213749}{24890} u^2 + \dfrac{|v|}{10^6} + \dfrac{3136}{62500} - \dfrac{3136}{625} \cdot \dfrac{213749}{24890}, & u \in \left[\dfrac{56}{25}, 3\right], \\
\left(\dfrac{1}{2}\right)^t + \dfrac{1070313}{31250} + \dfrac{|v|}{10^6}, & u \in [3, +\infty],
\end{cases}
$$

$$
f_2(t,u,v) = \begin{cases}
\left(\dfrac{1}{5}\right)^t + \dfrac{u^2}{10^3} + \dfrac{v^2}{10^{10}}, & u \in \left[0, \dfrac{56}{25}\right], \\
\left(\dfrac{1}{5}\right)^t + \dfrac{14740614}{2489000} u^2 + \dfrac{v^2}{10^{10}} + \dfrac{3136}{625000} - \dfrac{3136}{625} \cdot \dfrac{14740614}{2489000}, & u \in \left[\dfrac{56}{25}, 3\right], \\
\left(\dfrac{1}{5}\right)^t + \dfrac{2359}{100} + \dfrac{v^2}{10^{10}}, & u \in [3, +\infty].
\end{cases} \tag{5.11}
$$

Here, we have $\alpha_1 = 3/2$, $\alpha_2 = 7/4$, $\mu_1 = 1/2$, $\mu_2 = 1/4$. By choosing $\theta = 1/4$ and the definition of σ and ρ_{ij}, $i = 1, 2$, $j = 1, 2, 3, 4$, one gets

$$
\begin{aligned}
\sigma &= \max\left\{ \frac{1}{\Gamma(1+1/2)}, \frac{1}{\Gamma(1+1/4)} \right\} = \frac{1}{\Gamma(1+1/2)} = \frac{1}{(1/2)\sqrt{\pi}} \approx 1.12, \\
\rho_{11} &= \frac{(3/2-1)^{3/2-1}}{(3/2)^{3/2}\Gamma(3/2+1)} = \frac{8}{9\sqrt{3\pi}} \approx 0.28,
\end{aligned}
$$

$$\rho_{12} = \frac{(1/4)(1-1/4)^{3/2-1}}{\Gamma(3/2+1)} = \frac{1}{2\sqrt{3\pi}} \approx 0.16,$$

$$\rho_{13} = \int_{1/4}^{3/4} \gamma_1(s) G_1(s,s)ds = \frac{2}{\sqrt{\pi}} \int_{1/4}^{3/4} \frac{1}{2}\sqrt{1-s}ds + \frac{2}{\sqrt{\pi}} \int_{1/4}^{1-\sqrt{3}/6} \sqrt{\frac{3}{4}-s}ds \approx 0.12,$$

$$\rho_{14} = \frac{3-1/4}{(3/2)\Gamma(3/2-1/4+1)} = \frac{88}{15\Gamma(1/4)} \approx 1.61,$$

$$\rho_{21} \approx 0.18, \qquad \rho_{22} \approx 0.12, \qquad \rho_{23} \approx 0.06, \qquad \rho_{24} \approx 1.53. \tag{5.12}$$

Taking $l = 2$, $b = 3$, and $L = 1000$, we have

$$f_1(t,u,v) \leq \min\left\{\frac{\sigma L}{\rho_{11}}, \frac{L}{\rho_{14}}\right\} \approx 621.11, \quad \text{for } (t,u,v) \in [0,1] \times [0,1120] \times [-1000,1000],$$

$$f_1(t,u,v) > \frac{b}{\rho_{13}} \approx 16.67, \quad \text{for } (t,u,v) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [3,1120] \times [-1000,1000],$$

$$f_1(t,u,v) < \min\left\{\frac{\sigma l}{\rho_{11}}, \frac{l}{\rho_{14}}\right\} \approx 1.24, \quad \text{for } (t,u,v) \in [0,1] \times \left[0, \frac{56}{25}\right] \times [-3,3],$$

$$f_1(t,u,v) > \frac{b}{\rho_{12}} \approx 18.75, \quad \text{for } (t,u,v) \in [0,1] \times [3,12] \times [-1000,1000], \tag{5.13}$$

that is, f_1 satisfies the conditions (H1)–(H4) of Theorem 5.1. Similarly, we can show that f_2 satisfies (H1)–(H4). Thus, by Theorem 5.1, the BVP (5.10) has at least three positive solutions $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$ such that

$$\begin{gathered}
0 \leq x_i(t) \leq 2.24, \quad 0 \leq y_i(t) \leq 1120, \quad 2.24 \leq z_i(t) \leq 1120, \quad t \in [0,1], \ i = 1,2, \\
\left\|D_{0+}^{1/4} x_1\right\|_\infty \leq 2, \quad \left\|D_{0+}^{1/2} x_2\right\|_\infty \leq 2, \quad \left\|D_{0+}^{1/4} y_1\right\|_\infty \leq 1000, \quad \left\|D_{0+}^{1/2} y_2\right\|_\infty \leq 1000, \\
-2 \leq D_{0+}^{1/4} z_1(t) \leq 1000, \quad -2 \leq D_{0+}^{1/2} z_2(t) \leq 1000, \quad t \in [0,1], \\
y_i(t) > 3, \quad z_i(t) \leq 3, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \ i = 1,2.
\end{gathered} \tag{5.14}$$

Remark 5.3. The particular case $N = 2$ has been studied by [12] for the existence of one solution, our paper generalizes [12] for the obtaining of one and three positive solutions. For $N = 1$, we develop [13–15] by the nonlinear terms f_i involved in the μ_i-order Riemann-Liouville derivative explicitly.

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