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Research Article

Uniform Convergence of the Spectral Expansion for a Differential Operator with Periodic Matrix Coefficients

O. A. Veliev

Departartment of Mathematics, Faculty of Arts and Science, Dogus University, Acibadem, Kadikoy, 34722 Istanbul, Turkey

Correspondence should be addressed to O. A. Veliev, oveliev@dogus.edu.tr

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We obtain asymptotic formulas for eigenvalues and eigenfunctions of the operator generated by a system of ordinary differential equations with summable coefficients and the quasiperiodic boundary conditions. Using these asymptotic formulas, we find conditions on the coefficients for which the root functions of this operator form a Riesz basis. Then, we obtain the uniformly convergent spectral expansion of the differential operators with the periodic matrix coefficients.

1. Introduction

Let $L(P_2, P_3, \dots, P_n) \equiv L$ be the differential operator generated in the space $L_2^m(-\infty, \infty)$ by the differential expression

$$l(y) = y^{(n)}(x) + P_2(x)y^{(n-2)}(x) + P_3(x)y^{(n-3)}(x) + \cdots + P_n(x)y, \tag{1.1}$$

and $L_t(P_2, P_3, \dots, P_n) \equiv L_t$ be the differential operator generated in $L_2^m(0,1)$ by the same differential expression and the boundary conditions

$$U_{\nu,t}(y) \equiv y^{(\nu)}(1) - e^{it}y^{(\nu)}(0) = 0, \quad \nu = 0, 1, \dots, (n-1), \tag{1.2}$$

where $n \ge 2$, $P_\nu = (p_{\nu,i,j})$ is an $m \times m$ matrix with the complex-valued summable entries $p_{\nu,i,j}$, $P_\nu(x+1) = P_\nu(x)$ for $\nu = 2, 3, \dots, n$, the eigenvalues $\mu_1, \mu_2, \dots, \mu_m$ of the matrix,

$$C = \int_0^1 P_2(x)dx, \tag{1.3}$$

are simple, and $y = (y_1, y_2, \dots, y_m)$ is a vector-valued function. Here, $L_2^m(a,b)$ is the space of the vector-valued functions $f = (f_1, f_2, \dots, f_m)$, where $f_k \in L_2(a,b)$ for $k = 1,2,\dots,m$, with the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ defined by

$$\|f\|^2 = \int_a^b |f(x)|^2 dx, \qquad (f,g) = \int_a^b \langle f(x), g(x)\rangle dx, \tag{1.4}$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are the norm and inner product in $\mathbb{C}^m$.

It is well known that (see [1, 2]) the spectrum $\sigma(L)$ of L is the union of the spectra $\sigma(L_t)$ of L_t for $t \in [0, 2\pi)$. First, we derive an asymptotic formula for the eigenvalues and eigenfunctions of L_t which is uniform with respect to t in $Q_\varepsilon(n)$, where

$$Q_\varepsilon(2\mu) = \{t \in Q : |t - \pi k| > \varepsilon,\ \forall k \in \mathbb{Z}\}, \quad Q_\varepsilon(2\mu+1) = Q, \quad \varepsilon \in \left(0, \frac{\pi}{4}\right),\ \mu = 1,2,\dots, \tag{1.5}$$

and Q is a compact subset of $\mathbb{C}$ containing a neighborhood of the interval $[-\pi/2, 2\pi - \pi/2]$. Using these formulas, we prove that the root functions of L_t for $t \in \mathbb{C}(n)$ form a Riesz basis in $L_2^m(0,1)$, where $\mathbb{C}(2\mu) = \mathbb{C} \setminus \{\pi k : k \in \mathbb{Z}\}$, $\mathbb{C}(2\mu+1) = \mathbb{C}$. Then we construct the uniformly convergent spectral expansion for L.

Let us introduce some preliminary results and describe the scheme of the paper. Denote by $L_t(0)$ the operator $L_t(P_2, \dots, P_n)$ when $P_2(x) = 0, \dots, P_n(x) = 0$. Clearly,

$$\varphi_{k,j,t}(x) = e(t) e_j e^{i(2\pi k + t)x} \quad \text{for } k \in \mathbb{Z},\ \ j = 1,2,\dots,m, \ \text{where } (e(t))^{-2} = \int_0^1 |e^{itx}|^2 dx, \tag{1.6}$$

$e_1 = (1,0,0,\dots,0)$, $e_2 = (0,1,0,\dots,0), \dots$, $e_m = (0,0,\dots,0,1)$ are the normalized eigenfunctions of the operator $L_t(0)$ corresponding to the eigenvalue $(2\pi ki + ti)^n$. It easily follows from the classical investigations [3, Chapter 3, Theorem 2] that the boundary conditions (1.2) are regular and the large eigenvalues of L_t consist of m sequences

$$\{\lambda_{k,1}(t) : |k| \ge N\}, \{\lambda_{k,2}(t) : |k| \ge N\}, \dots, \{\lambda_{k,m}(t) : |k| \ge N\}, \tag{1.7}$$

satisfying the following asymptotic formula uniformly with respect to t in Q

$$\lambda_{k,j}(t) = (2\pi ki + ti)^n + O\left(k^{n-1-1/2m}\right) \quad \text{as } k \longrightarrow \pm\infty, \tag{1.8}$$

where $N \gg 1$ and $j = 1,2,\dots,m$. We say that the formula $f(k,t) = O(h(k))$ is uniform with respect to t in Q if there exists positive constants N and c, independent of t, such that $|f(k,t)| < c|h(k)|$ for all $t \in Q$ and $|k| \ge N$.

The method proposed here allows us to obtain the asymptotic formulas of high accuracy for the eigenvalues $\lambda_{k,j}(t)$ and the corresponding normalized eigenfunctions $\Psi_{k,j,t}(x)$ of L_t when $p_{\nu,i,j} \in L_1[0,1]$ for all ν, i, j. Note that to obtain the asymptotic formulas of high accuracy by the classical methods, it is required that $P_2, P_3, \dots, P_n$ be differentiable (see [3]). To obtain the asymptotic formulas for L_t, we take the operator $L_t(C)$, where $L_t(P_2, \dots, P_n)$ is denoted by $L_t(C)$ when $P_2(x) = C$, $P_3(x) = 0, \dots, P_n(x) = 0$ for an unperturbed operator and $L_t - L_t(C)$ for a perturbation. One can easily verify that the eigenvalues and normalized eigenfunctions of $L_t(C)$ are

$$\mu_{k,j}(t) = (2\pi ki + ti)^n + \mu_j (2\pi ki + ti)^{n-2}, \qquad \Phi_{k,j,t}(x) = e(t) v_j e^{i(2\pi k + t)x} \tag{1.9}$$

for $k \in \mathbb{Z}$, $j = 1,2,\dots,m$, where $v_1, v_2, \dots, v_m$ are the normalized eigenvectors of the matrix C corresponding to the eigenvalues $\mu_1, \mu_2, \dots, \mu_m$, respectively.

In Section 2, we investigate the operator L_t and prove the following theorem.

Theorem 1.1. (a) *The large eigenvalues of L_t consist of m sequences* (1.7) *satisfying the following formula uniformly with respect to t in $Q_\varepsilon(n)$:*

$$\lambda_{k,j}(t) = (2\pi ki + ti)^n + \mu_j(2\pi ki + ti)^{n-2} + O(k^{n-3}\ln|k|). \tag{1.10}$$

There exists constant $N(\varepsilon)$ such that if $|k| \geq N(\varepsilon)$ and $t \in Q_\varepsilon(n)$, then $\lambda_{k,j}(t)$ is a simple eigenvalue of L_t and the corresponding normalized eigenfunction $\Psi_{k,j,t}(x)$ satisfies

$$\Psi_{k,j,t}(x) = e(t)v_j e^{i(2\pi k+t)x} + O(k^{-1}\ln|k|). \tag{1.11}$$

This formula is uniform with respect to t and x in $Q_\varepsilon(n)$ and in $[0,1]$, that is, there exists a constant c_1, independent of t, such that the term $O(k^{-1}\ln|k|)$ in (1.11) *satisfies*

$$|O(k^{-1}\ln|k|)| < c_1|k^{-1}\ln|k|| \quad \forall t \in Q_\varepsilon(n),\ x \in [0,1],\ |k| \geq N(\varepsilon). \tag{1.12}$$

(b) *If $t \in \mathbb{C}(n)$, then the root functions of L_t form a Riesz basis in $L_2^m(0,1)$.*

(c) *Let L_t^* be adjoint operator of L_t and $X_{k,j,t}$ be the eigenfunction of L_t^* corresponding to the eigenvalue $\overline{\lambda_{k,j}(t)}$ and satisfying $(X_{k,j,t}, \Psi_{k,j,t}) = 1$, where $|k| \geq N(\varepsilon)$ and $t \in Q_\varepsilon(n)$. Then, $X_{k,j,t}(x)$ satisfies the following formula uniformly with respect to t and x in $Q_\varepsilon(n)$ and in $[0,1]$, respectively,*

$$X_{k,j,t}(x) = u_j(e(t))^{-1}e^{i(2k\pi+\bar{t})x} + O(k^{-1}\ln|k|), \tag{1.13}$$

where u_j is the eigenvector of C^* corresponding to $\overline{\mu_j}$ and satisfying $(u_j, v_j) = 1$.

(d) *If f is absolutely continuous function satisfying* (1.2) *and $f' \in L_2^m[0,1]$, then the expansion series of $f(x)$ by the root functions of L_t converges uniformly, with respect to x in $[0,1]$, where $t \in \mathbb{C}(n)$.*

Shkalikov [4, 5] proved that the root functions of the operators generated by an ordinary differential expression with summable coefficients and regular boundary conditions form a Riesz basis with brackets. Luzhina [6] generalized these results for the matrix case. In [7], we prove that if $n = 2$ and the eigenvalues of the matrix C are simple, then the root functions of L_t for $t \in (0,\pi) \cup (\pi, 2\pi)$ form an ordinary Riesz basis. The case $n > 2$ is more complicated and the most part of the method of [7] does not work here, since in the case $n > 2$ the adjoint operator of the operator generated by expression with arbitrary summable coefficients cannot be defined by the Lagrange's formula.

In Section 3 using Theorem 1.1, we obtain spectral expansion for the nonself-adjoint differential operator L with the periodic matrix coefficients. The spectral expansion for the self-adjoint differential operators with the periodic coefficients was constructed by Gelfand [8], Titchmarsh [9], and Tkachenko [10]. In [11], it was proved that the nonself-adjoint Hill operator H can be reduced to the triangular form if all eigenvalues of the operators H_t for $t \in [0,2\pi)$ are simple, where H and H_t denote the operators L and L_t in the case $m = 1$, $n = 2$. McGarvey [2, 12] proved that L, in the case $m = 1$, is a spectral operator if the projections of the operator L are uniformly bounded. Gesztesy and Tkachenko [13] proved that the Hill operator H is a spectral operator of scalar type (see [14] for the definition of the spectral operator) if and only if for all $t \in [0,2\pi)$ the operators H_t have not associated function, the multiple point of either the periodic or antiperiodic spectrum is a point of its Dirichlet spectrum, and some other conditions hold. (Recall that a function Ψ is called an associated function of H_t corresponding to the eigenvalue λ if $(H_t - \lambda I)\Psi \neq 0$ and there exists an integer $k > 1$ such that $(H_t - \lambda I)^k \Psi = 0$ (see [3]).) However, in general, the eigenvalues are not simple, projections are not uniformly bounded, and L_t has associated function, since the Hill operator with simple potential $q(x) = e^{i2\pi x}$ has infinitely many spectral singularities (see [15], where Gasymov investigated the Hill operator with special potential). Note that the spectral singularity of L is the point of $\sigma(L)$ in neighborhood on which the projections of L are not uniformly bounded. In [16], we proved that a number $\lambda \in \sigma(L_t) \subset \sigma(L)$ is a spectral singularity if and only if L_t has an associated function corresponding to the eigenvalue λ. The existence of the spectral singularities and the absence of the Parseval's equality for the nonself-adjoint operator L_t do not allow us to apply the elegant method of Gelfand (see [8]) for construction of the spectral expansion for the nonself-adjoin operator L. These situations essentially complicate the construction of the spectral expansion for the nonself-adjoint case. In [17, 18], we constructed the spectral expansion for the Hill operator with continuous complex-valued potential q and with locally summable complex-valued potential q, respectively. Then, in [19, 20], we constructed the spectral expansion for the nonself-adjoint operator L in the case $m = 1$, with coefficients $p_k \in C^{(k-1)}[0,1]$ and with $p_k \in L_1[0,1]$ for $k = 2, 3, \ldots, n$, respectively. In the paper [21], we constructed the spectral expansion of L when $p_{k,i,j} \in C^{(k-1)}[0,1]$. In this paper, we do it when $p_{k,i,j}$ is arbitrary Lebesgue integrable on $(0,1)$ function. Besides, in [21], the expansion is obtained for compactly supported continuous vector functions, while in this paper, we obtain the spectral expansion for each function $f \in L_2^m(-\infty,\infty)$ satisfying

$$\sum_{k=-\infty}^{\infty} |f(x+k)| < \infty \tag{1.14}$$

if $n = 2\mu + 1$ and for each function from Ω, where $f(x) \in \Omega \subset L_2^m(-\infty,\infty)$ if and only if there exist positive constants M and α such that

$$|f(x)| < Me^{-\alpha|x|} \quad \forall x \in (-\infty,\infty) \tag{1.15}$$

if $n = 2\mu$. Moreover, using Theorem 1.1, we prove that the spectral expansion of L converges uniformly in every bounded subset of $(-\infty,\infty)$ if f is absolutely continuous compactly supported function and $f' \in L_2^m(-\infty,\infty)$. Note that the spectral expansion obtained in [21], when $p_{k,i,j} \in C^{(k-1)}[0,1]$, converges in the norm of $L_2^m(a,b)$, where a and b are arbitrary real numbers. Some parts of the proofs of the spectral expansions for $L$are just writing in the vector form of the corresponding proofs obtained in [19] for the case $m = 1$. These parts are given in appendices in order to give a possibility to read this paper independently.

Thus, in this paper, we obtain the spectral expansion for the nonself-adjoint differential operators L_t and L with the periodic matrix coefficients. There exist many important papers about spectral theory of the self-adjoint differential operators with the periodic matrix coefficients (see [22, 23] and references therein). We do not discuss the results of those papers, since those results have no any relation with the spectral expansion for the nonself-adjoint differential operators L_t and L.

2. On the eigenvalues and root functions of L_t

The formula (1.8) shows that the eigenvalue $\lambda_{k,j}(t)$ of L_t is close to the eigenvalue $(2k\pi i + ti)^n$ of $L_t(0)$. By (1.5), if $t \in Q_\varepsilon(n)$, $|k| \gg 1$, then the eigenvalue $(2\pi ki + ti)^n$ of $L_t(0)$ lies far from the other eigenvalues $L_t(0)$. Thus, (1.5) and (1.8) imply that

$$|\lambda_{k,j}(t) - (2\pi pi + ti)^n| > (||k| - |p|| + 1)(|k| + |p|)^{n-1} \tag{2.1}$$

for $p \neq k$, $t \in Q_\varepsilon(n)$, where $|k| \gg 1$. Using this, one can easily verify that

$$\sum_{p:p>d} \frac{|p|^{n-\nu}}{|\lambda_{k,j}(t) - (2\pi pi + ti)^n|} = O\left(\frac{1}{d^{\nu-1}}\right) \quad \forall d > 2|k|, \tag{2.2}$$

$$\sum_{p:p\neq k} \frac{|p|^{n-\nu}}{|\lambda_{k,j}(t) - (2\pi pi + ti)^n|} = O\left(\frac{\ln |k|}{k^{\nu-1}}\right), \tag{2.3}$$

where $|k| \gg 1$, $\nu \geq 2$, and (2.2), (2.3) are uniform with respect to t in $Q_\varepsilon(n)$.

The boundary conditions adjoint to (1.2) is $U_{\nu,\bar{t}}(y) = 0$. Therefore, the eigenfunctions $\varphi^*_{k,s,t}$ and $\Phi^*_{k,s,t}$ of the operators $L^*_t(0)$ and $L^*_t(C)$ corresponding to the eigenvalues $\overline{(2\pi pi + ti)^n}$ and $\overline{\mu_{k,s}(t)}$, respectively, and satisfying $(\varphi_{k,s,t}, \varphi^*_{k,s,t}) = 1$, $(\Phi_{k,s,t}, \Phi^*_{k,s,t}) = 1$ are

$$\varphi^*_{k,s,t}(x) = e_s(e(t))^{-1} e^{i(2\pi k + \bar{t})x}, \qquad \Phi^*_{k,s,t}(x) = u_s(e(t))^{-1} e^{i(2\pi k + \bar{t})x}, \tag{2.4}$$

where $\mu_{k,s}(t)$ and u_s are defined in (1.9) and (1.13).

To prove the asymptotic formulas for the eigenvalue $\lambda_{k,j}(t)$ and the corresponding normalized eigenfunction $\Psi_{k,j,t}(x)$ of L_t, we use the formula

$$(\lambda_{k,j} - \mu_{k,s})(\Psi_{k,j,t}, \Phi^*_{k,s,t}) = \left((P_2 - C)\Psi^{(n-2)}_{k,j,t}, \Phi^*_{k,s,t}\right) + \sum_{\nu=3}^{n} \left(P_\nu \Psi^{(n-\nu)}_{k,j,t}, \Phi^*_{k,s,t}\right), \tag{2.5}$$

which can be obtained from

$$L_t \Psi_{k,j,t}(x) = \lambda_{k,j}(t) \Psi_{k,j,t}(x) \tag{2.6}$$

by multiplying scalarly by $\Phi^*_{k,s,t}(x)$. To estimate the right-hand side of (2.5), we use (2.2), (2.3), the following lemma, and the formula

$$\left(\lambda_{k,j}(t) - (2\pi pi + ti)^n\right)(\Psi_{k,j,t}, \varphi^*_{p,s,t}) = \sum_{\nu=2}^{n} \left(P_\nu \Psi^{(n-\nu)}_{k,j,t}, \varphi^*_{p,s,t}\right), \tag{2.7}$$

which can be obtained from (2.6) by multiplying scalarly by $\varphi^*_{p,s,t}(x)$.

Lemma 2.1. *If $|k| \gg 1$ and $t \in Q_\varepsilon(n)$, then*

$$\left(P_\nu \Psi^{(n-\nu)}_{k,j,t}, \varphi^*_{p,s,t}\right) = \sum_{q=1}^{m} \left(\sum_{l=-\infty}^{\infty} p_{\nu,s,q,p-l} (2\pi li + it)^{n-\nu} \left(\Psi_{k,t}, \varphi^*_{l,q,t}\right)\right), \tag{2.8}$$

where $p_{\nu,s,q,k} = \int_0^1 p_{\nu,s,q}(x) e^{-i2\pi kx}\, dx$. Moreover, there exists a constant c_2, independent of t, such that

$$\max_{p\in\mathbb{Z},\ s=1,2,\ldots,m} \left| \sum_{\nu=2}^{n} \left(P_\nu \Psi^{(n-\nu)}_{k,j,t}, \varphi^*_{p,s,t}\right)\right| < c_2 |k|^{n-2} \quad \forall t \in Q_\varepsilon(n),\ j = 1, 2, \ldots, m. \tag{2.9}$$

Proof. Since $P_2 \Psi^{(n-2)}_{k,j,t} + P_3 \Psi^{(n-3)}_{k,j,t} + \cdots + P_n \Psi_{k,j,t} \in L^m_1[0,1]$, we have

$$\lim_{p\to\infty} \left| \sum_{\nu=2}^{n} \left(P_\nu \Psi^{(n-\nu)}_{k,j,t}, \varphi^*_{p,s,t}\right)\right| = 0. \tag{2.10}$$

Therefore, there exists a positive constant $M(k,j)$ and indices p_0, s_0 satisfying

$$\max_{\substack{p\in\mathbb{Z},\\ s=1,2,\ldots,m}}\left|\sum_{\nu=2}^{n}\left(P_\nu\Psi_{k,j,t}^{(n-\nu)},\varphi_{p,s,t}^{*}\right)\right|=\left|\sum_{\nu=2}^{n}\left(P_\nu\Psi_{k,j,t}^{(n-\nu)},\varphi_{p_0,s_0,t}^{*}\right)\right|=M(k,j). \tag{2.11}$$

Then using (2.7) and (2.2), we get

$$\begin{gathered}\left|\left(\Psi_{k,j,t},\varphi_{p,s,t}^{*}\right)\right|\le\frac{M(k,j)}{\left|\lambda_{k,j}(t)-(2\pi pi+it)^{n}\right|},\\ \sum_{p:|p|>d}\left|\left(\Psi_{k,j,t},\varphi_{p,s,t}^{*}\right)\right|=M(k,j)O\left(\frac{1}{d^{n-1}}\right),\end{gathered} \tag{2.12}$$

where $d>2|k|$. This implies that the decomposition of $\Psi_{k,j,t}(x)$ by the basis $\{\varphi_{p,s,t}(x):p\in\mathbb{Z},$ $s=1,2,\ldots,m\}$ has the form

$$\Psi_{k,j,t}(x)=\sum_{p:|p|\le d}\left(\Psi_{k,j,t},\varphi_{p,s,t}^{*}\right)\varphi_{p,s,t}(x)+g_{0,d}(x), \tag{2.13}$$

where

$$\sup_{x\in[0,1]}\left|g_{0,d}(x)\right|=M(k,j)O\left(\frac{1}{d^{n-1}}\right). \tag{2.14}$$

Now using the integration by parts, (1.2), and the inequality (2.12), we obtain

$$\begin{gathered}\left(\Psi_{k,j,t}^{(n-\nu)},\varphi_{p,s,t}^{*}\right)=(2\pi ip+it)^{n-\nu}\left(\Psi_{k,j,t},\varphi_{p,s,t}^{*}\right),\\ \left|\left(\Psi_{k,j,t}^{(n-\nu)},\varphi_{p,s,t}^{*}\right)\right|\le\frac{|2\pi ip+it|^{n-\nu}M(k,j)}{\left|\lambda_{k}(t)-(2\pi pi+it)^{n}\right|}.\end{gathered} \tag{2.15}$$

Therefore, arguing as in the proof of (2.13) and using (2.2), we get

$$\Psi_{k,j,t}^{(n-\nu)}(x)=\sum_{p:|p|\le d}\left(\Psi_{k,j,t}^{(n-\nu)},\varphi_{p,s,t}^{*}\right)\varphi_{p,s,t}(x)+g_{\nu,d}(x), \tag{2.16}$$

$$\sup_{x\in[0,1]}\left|g_{\nu,d}(x)\right|=M(k,j)O\left(\frac{1}{d^{\nu-1}}\right), \tag{2.17}$$

where $\nu=2,3,\ldots,n$. Now using (2.16) in $(P_\nu\Psi_{k,j,t}^{(n-\nu)},\varphi_{p,s,t}^{*})$ and letting $q\to\infty$, we get (2.8).

Let us prove (2.9). It follows from (2.11) and (2.8) that

$$\begin{aligned}M(k,j)&=\left|\sum_{\nu=2}^{n}\left(P_\nu\Psi_{k,j,t}^{(n-\nu)},\varphi_{p_0,s_0,t}^{*}\right)\right|\\ &=\left|\sum_{\nu=2}^{n}\sum_{q=1}^{m}\left(\sum_{l=-\infty}^{\infty}p_{\nu,s_0,q,p_0-l}(2\pi im+it)^{n-\nu}\left(\Psi_{k,j,t},\varphi_{l,q,t}^{*}\right)\right)\right|.\end{aligned} \tag{2.18}$$

By (2.12) and (2.3), we have

$$\begin{gathered}\left|\sum_{\nu=2}^{n}\sum_{q=1}^{m}\left(\sum_{l\neq k}p_{\nu,s_0,q,p_0-l}(2\pi im+it)^{n-\nu}\left(\Psi_{k,j,t},\varphi_{l,q,t}^{*}\right)\right)\right|=M(k,j)O\left(\frac{\ln|k|}{|k|}\right),\\ \left|\sum_{\nu=2}^{n}\sum_{q=1}^{m}\left(p_{\nu,s_0,q,p_0-k}(2\pi im+it)^{n-\nu}\left(\Psi_{k,j,t},\varphi_{k,q,t}^{*}\right)\right)\right|=O\left(k^{n-2}\right).\end{gathered} \tag{2.19}$$

Therefore, using (2.18), we get $M(k,j)=M(k,j)O(\ln|k|/k)+O(|k|^{n-2})$, $M(k,j)=O(|k|^{n-2})$ which means that (2.9) holds. □

It follows from (2.9)–(2.12) that

$$\left|\left(\Psi_{k,j,t},\varphi^*_{p,q,t}\right)\right| \le \frac{c_2|k|^{n-2}}{|\lambda_{k,j}(t)-(2\pi pi+it)^n|} \quad \forall p\neq k. \tag{2.20}$$

Lemma 2.2. *The equalities,*

$$\begin{gathered}\left((P_2-C)\Psi^{(n-2)}_{k,j,t},\Phi^*_{k,s,t}\right)=O\left(k^{n-3}\ln|k|\right),\\ \left(P_\nu\Psi^{(n-\nu)}_{k,j,t},\Phi^*_{k,s,t}\right)=O\left(k^{n-3}\right),\end{gathered} \tag{2.21}$$

hold uniformly with respect to t in $Q_\varepsilon(n)$, where $\nu\ge 3$.

Proof. Using (2.8) for $\nu=2$, $p=k$ and the obvious relation

$$\left(C\Psi^{(n-2)}_{k,j,t},\varphi^*_{k,s,t}\right)=\sum_{q=1}^{m}\left(p_{2,s,q,0}(2\pi ki+it)^{n-2}\left(\Psi_{k,j,t},\varphi^*_{k,q,t}\right)\right), \tag{2.22}$$

we see that

$$\left((P_2-C)\Psi^{(n-2)}_{k,j,t},\varphi^*_{k,s,t}\right)=\sum_{q=1}^{m}\left(\sum_{l\neq k}p_{2,s,q,k-l}(2\pi li+it)^{n-2}\left(\Psi_{k,j,t},\varphi^*_{l,q,t}\right)\right). \tag{2.23}$$

This with (2.20) and (2.3) for $\nu=2$ implies that

$$\left((P_2-C)\Psi^{(n-2)}_{k,j,t},\varphi^*_{k,s,t}\right)=O\left(k^{n-3}\ln|k|\right). \tag{2.24}$$

Similarly, using (2.8), (2.20), (2.3), we obtain

$$\left((P_\nu\Psi^{(n-\nu)}_{k,j,t},\varphi^*_{k,s,t}\right)=O\left(k^{n-3}\right)\quad \forall\nu\ge 3. \tag{2.25}$$

Since (2.3) is uniform with respect to t in $Q_\varepsilon(n)$ and the constant c_2 in (2.20) does not depend on t (see Lemma 2.1), these formulas are uniform with respect to t in $Q_\varepsilon(n)$. Hence, using the definitions of $\Phi^*_{k,s,t}$ and $\varphi^*_{k,q,t}$ (see (2.4)), we get the proof of (2.21). □

Lemma 2.3. *There exist positive numbers $N_1(\varepsilon)$ and c_3, independent of t, such that*

$$\max_{s=1,2,\ldots,m}\left|\left(\Psi_{k,j,t},\Phi^*_{k,s,t}\right)\right|>c_3 \tag{2.26}$$

for all $|k|\ge N_1(\varepsilon)$, $t\in Q_\varepsilon(n)$, and $j=1,2,\ldots,m$.

Proof. It follows from (2.20) and (2.3) that

$$\sum_{s=1,2,\ldots,m}\left(\sum_{p:p\neq k}\left|\left(\Psi_{k,j,t},\varphi^*_{p,s,t}\right)\right|\right)=O\left(\frac{\ln|k|}{k}\right) \tag{2.27}$$

and this formula is uniform with respect to t in $Q_\varepsilon(n)$. Then, the decomposition of $\Psi_{k,j,t}(x)$ by the basis $\{\varphi_{p,s,t}(x): s=1,2,\ldots,m,\ p\in\mathbb{Z}\}$ has the form

$$\Psi_{k,j,t}(x)=\sum_{s=1,2,\ldots,m}\left(\Psi_{k,j,t},\varphi^*_{k,s,t}\right)\varphi_{k,s,t}(x)+O\left(\frac{\ln|k|}{k}\right). \tag{2.28}$$

Since $\|\Psi_{k,j,t}\| = \|\varphi_{k,j,t}\| = 1$ and (2.28) is uniform with respect to t in $Q_\varepsilon(n)$, there exists a positive constant $N_1(\varepsilon)$, independent of t, such that

$$\max_{s=1,2,\ldots,m} |(\Psi_{k,j,t}, \varphi^*_{k,s,t})| > \frac{1}{m+1} \tag{2.29}$$

for all $|k| \geq N_1(\varepsilon)$, $t \in Q_\varepsilon(n)$, and $j = 1,2,\ldots,m$. Therefore, using (2.4) and taking into account that the vectors $u_1, u_2, \ldots, u_m$ form a basis in $\mathbb{C}^m$, that is, e_s is a linear combination of these vectors, we get the proof of (2.26). □

Proof of Theorem 1.1(a). It follows from Lemma 2.2 that there exist positive constants $N_2(\varepsilon)$ and c_4, independent of t, such that if $|k| \geq N_2(\varepsilon)$, $t \in Q_\varepsilon(n)$, then the right-hand side of (2.5) is less than $c_4|k|^{n-3}\ln|k|$. Therefore, (2.5) and Lemma 2.3 imply that there exist positive constants c_5, $N(\varepsilon)$, independent of t, such that if $t \in Q_\varepsilon(n)$ and $|k| \geq N(\varepsilon)$, then

$$\{\lambda_{k,1}(t), \lambda_{k,2}(t), \ldots, \lambda_{k,m}(t)\} \subset (D(k,1,t) \cup D(k,2,t) \cup \cdots \cup D(k,m,t)), \tag{2.30}$$

where $D(k,s,t) = U(\mu_{k,s}(t), c_5|k|^{n-3}\ln|k|)$, $U(\mu, c) = \{\lambda \in \mathbb{C} : |\lambda - \mu| < c\}$. Now let us prove that in each of the disks $D(k,s,t)$ for $s = 1,2,\ldots,m$ and $|k| \geq N(\varepsilon)$, there exists a unique eigenvalue of L_t. For this purpose, we consider the following family of operators:

$$L_{t,z} = L_t(C) + z(L_t - L_t(C)), \quad 0 \leq z \leq 1. \tag{2.31}$$

It is clear that (2.30) holds for $L_{t,z}$, that is, the eigenvalues $\lambda_{k,1,z}(t), \lambda_{k,2,z}(t), \ldots, \lambda_{k,m,z}(t)$, where $|k| \geq N(\varepsilon)$, of $L_{t,z}$ lie in the union of the pairwise disjoint m disks $D(k,1,t), D(k,2,t), \ldots, D(k,m,t)$. Besides, in each of these disks, there exists a unique eigenvalue of $L_{t,0}$. Therefore, taking into account that the family $L_{t,z}$ is holomorphic with respect to z, and the boundaries of these disks lie in the resolvent set of the operators $L_{t,z}$ for all $z \in [0,1]$, we obtain the following proposition.

Proposition 2.4. *There exists a positive constant $N(\varepsilon)$, independent of t, such that if $t \in Q_\varepsilon(n)$ and $|k| \geq N(\varepsilon)$, then the disk $D(k,j,t)$ contains unique eigenvalue, denoted by $\lambda_{k,j}$, of L_t and this eigenvalue is a simple eigenvalue of L_t, where $j = 1,2,\ldots,m$ and the sets $Q_\varepsilon(n)$, $D(k,j,t)$ are defined in* (1.5), (2.30).

Using this proposition and the definition of $\mu_{k,s}$ (see (1.9)) and taking into account that the eigenvalues of C are simple, we get

$$|\lambda_{k,j} - \mu_{k,s}| > a_j|k|^{n-2} \quad \forall s \neq j,\ |k| \geq N(\varepsilon), \tag{2.32}$$

where $a_j = \min_{s\neq j}|\mu_j - \mu_s|$. This together with (2.5), (2.21) gives

$$(\Psi_{k,j,t}, \Phi^*_{k,s,t}) = O(k^{-1}\ln|k|) \quad \forall s \neq j. \tag{2.33}$$

On the other hand, by (2.4) and (2.27), we have

$$\sum_{s=1,2,\ldots,m} \left(\sum_{p:p\neq k} |(\Psi_{k,j,t}, \Phi^*_{p,s,t})| \right) = O(k^{-1}\ln|k|). \tag{2.34}$$

Since (2.21), (2.27) are uniform with respect to t in $Q_\varepsilon(n)$, the formulas (2.33) and (2.34) are also uniform. Therefore, decomposing $\Psi_{k,j,t}$ by basis $\{\Phi_{p,s,t} : s = 1,2,\ldots,m,\ p \in \mathbb{Z}\}$, we see that (1.11) and (1.12) hold. Theorem 1.1(a) is proved. □

Proof of Theorem 1.1(b). It follows from (1.11) that the root functions of L_t quadratically close to the system,

$$\{v_j e(t) e^{i(2\pi k+t)x} : k \in \mathbb{Z},\ l = 1, 2, \ldots, m\}, \tag{2.35}$$

which form a Riesz basis in $L_2^m(0,1)$. On the other hand, the system of the root functions of L_t is complete and minimal in $L_2^m(0,1)$ (see [6]). Therefore, by Bari theorem (see [24]), the system of the root functions of L_t forms a Riesz basis in $L_2^m(0,1)$. □

Proof of Theorem 1.1(c). To prove the asymptotic formulas for normalized eigenfunction $\Psi^*_{k,j,t}$ of L_t^* corresponding to the eigenvalue $\overline{\lambda_{k,j}(t)}$, we use the formula

$$\left(\overline{\lambda_{k,j}(t)} - \overline{(2\pi pi + ti)^n}\right)\left(\Psi^*_{k,j,t}, \varphi_{p,s,t}\right) = \sum_{\nu=2}^{n}\left(\Psi^*_{k,j,t}, \overline{(2\pi pi + ti)^{n-\nu}} P_\nu \varphi_{p,s,t}\right) \tag{2.36}$$

obtained from $L_t^* \Psi^*_{k,j,t} = \overline{\lambda_{k,j}(t)} \Psi^*_{k,j,t}$ by multiplying by $\varphi_{p,s,t}$ and using

$$\left(L_t^* \Psi^*_{k,j,t}, \varphi_{p,s,t}\right) = \left(\Psi^*_{k,j,t}, L_t \varphi_{p,s,t}\right). \tag{2.37}$$

Instead of (2.7) using this formula and arguing as in the proof of (2.20), we obtain

$$\left|\left(\Psi^*_{k,j,t}, \varphi_{p,q,t}\right)\right| = \frac{1}{\left|\lambda_{k,j}(t) - (2\pi pi + it)^n\right|} O\left(k^{n-2}\right) \quad \forall p \neq k. \tag{2.38}$$

This together with (1.9) and (2.3) implies the following relations:

$$\left|\left(\Psi^*_{k,j,t}, \Phi_{p,q,t}\right)\right| = \frac{1}{\left|\lambda_{k,j}(t) - (2\pi pi + it)^n\right|} O\left(k^{n-2}\right) \quad \forall p \neq k, \tag{2.39}$$

$$\sum_{s=1,2,\ldots,m} \sum_{p : p \neq k} \left|\left(\Psi^*_{k,j,t}, \Phi_{p,s,t}\right)\right| \Big) = O\left(k^{-1} \ln |k|\right). \tag{2.40}$$

On the other hand (1.11) and the equality $(\Psi^*_{k,j,t}, \Psi_{k,s,t}) = 0$ for $j \neq s$ give

$$\left(\Psi^*_{k,j,t}, \Phi_{k,s,t}\right) = O\left(k^{-1} \ln |k|\right) \quad \forall s \neq j. \tag{2.41}$$

Clearly, the formulas (2.39)–(2.41) are uniform with respect to t in $Q_\varepsilon(n)$ and they yield

$$\Psi^*_{k,j,t}(x) = u_j (e(t))^{-1} e^{(2k\pi i + i\bar{t})x} + O\left(k^{-1} \ln |k|\right), \tag{2.42}$$

where u_j is defined in (1.13). Now, (1.11) and (2.42) imply (1.13), since

$$X_{k,j,t} = \frac{\Psi^*_{k,j,t}}{\left(\Psi^*_{k,j,t}, \Psi_{k,j,t}\right)} = \left(1 + O\left(k^{-1} \ln |k|\right)\right) \Psi^*_{k,j,t}. \tag{2.43}$$

□

Proof of Theorem 1.1(d). To investigate the convergence of the expansion series of L_t, we consider the series

$$\sum_{k:|k|\geq N,\ j=1,2,\ldots,m} (f, X_{k,j,t})\Psi_{k,j,t}(x), \tag{2.44}$$

where $N = N(\varepsilon)$ and $N(\varepsilon)$ is defined in Theorem 1.1(a), $f(x)$ is absolutely continuous function satisfying (1.2) and $f'(x) \in L_2^m(0,1)$. Without loss of generality, instead of the series (2.44), we consider the series

$$\sum_{k:|k|\geq N,\ j=1,2,\ldots,m} (f_t, X_{k,j,t})\Psi_{k,j,t}(x), \tag{2.45}$$

since (2.45) will be used in the next section for spectral expansion of L, where $f_t(x)$ is defined by Gelfand transform (see [8, 9])

$$f_t(x) = \sum_{k=-\infty}^{\infty} f(x+k)e^{-ikt}, \tag{2.46}$$

f is an absolutely continuous compactly supported function and $f' \in L_2^m(-\infty,\infty)$. It follows from (2.46) that

$$f_t(x+1) = e^{it} f_t(x), \qquad f_t' \in L_2^m[0,1]. \tag{2.47}$$

To prove the uniform convergence of (2.45), we consider the series

$$\sum_{|k|\geq N,\ j=1,2,\ldots,m} |(f_t, X_{k,j,t})|. \tag{2.48}$$

To estimate the terms of this series, we decompose $X_{k,j,t}$ by basis $\{\Phi_{p,s,t}^* : p \in \mathbb{Z},\ s = 1,2,\ldots,m\}$ and then use the inequality

$$\begin{aligned} |(f_t, X_{k,j,t})| \leq &\sum_{s=1,2,\ldots,m} |(f_t, \Phi_{k,s,t}^*)||(X_{k,j,t}, \Phi_{k,s,t})| \\ &+ \sum_{p\neq k,\, s=1,2,\ldots,m} |(f_t, \Phi_{p,j,t}^*)||(X_{k,j,t}, \Phi_{p,s,t})|. \end{aligned} \tag{2.49}$$

Using the integration by parts and then Schwarz inequality, we get

$$\sum_{\substack{|k|\geq N,\\ s=1,2,\ldots,m}} |(f_t, \Phi_{k,s,t}^*)| = \sum_{\substack{|k|\geq N,\\ s=1,2,\ldots,m}} \left| \frac{1}{2\pi ki + it} (f_t', \Phi_{k,s,t}^*) \right| < \infty. \tag{2.50}$$

Again using the integration by parts, Schwarz inequality, and (2.39), (2.43), we obtain that there exists a constant c_6, independent of t, such that the expression in the second row of (2.49) is less than

$$c_6 \|f_t'\| \left(\sum_{p\neq k,\, s=1,2,\ldots,m} \left| \frac{1}{p} \frac{|k|^{n-2}}{|\lambda_{k,s}(t) - (2\pi pi + it)^n|} \right|^2 \right)^{1/2}, \tag{2.51}$$

which is $O(k^{-2})$. Therefore, the relations (2.49), (2.50) imply that the expression in (2.48) tends to zero, uniformly with respect to t in $Q_\varepsilon(n)$, as $N \to \infty$, and the expression in (2.45) tends to zero, uniformly with respect to t and x in $Q_\varepsilon(n)$ and in $[0,1]$, respectively, as $N \to \infty$. Since in the proof of the uniform convergence of (2.45) we used only the properties (2.47) of f_t, the series (2.44) converges uniformly with respect to x in $[0,1]$, that is, Theorem 1.1(d) is proved. Moreover, we proved the following theorem, which will be used in the next section. □

Theorem 2.5. *If f is absolutely continuous compactly supported function and $f' \in L_2^m(-\infty,\infty)$, then the series* (2.45) *converges uniformly with respect to t and x in $Q_\varepsilon(n)$ and in any bounded subset of $(-\infty,\infty)$.*

Indeed, we proved that (2.45) converges uniformly with respect to t and x in $Q_\varepsilon(n)$ and in $[0,1]$. Therefore, taking into account that (1.2) implies the equality

$$\Psi_{k,j,t}(x+1) = e^{it}\Psi_{k,j,t}(x), \tag{2.52}$$

we get the proof of Theorem 2.5.

3. Spectral expansion for L

Let $Y_1(x,\lambda), Y_2(x,\lambda), \dots, Y_n(x,\lambda)$ be the solutions of the matrix equation

$$Y^{(n)}(x) + P_2(x)Y^{(n-2)}(x) + P_3(x)Y^{(n-3)}(x) + \cdots + P_n(x)Y = \lambda Y(x), \tag{3.1}$$

satisfying $Y_k^{(j)}(0,\lambda) = 0_m$ for $j \neq k-1$ and $Y_k^{(k-1)}(0,\lambda) = I_m$, where 0_m and I_m are $m\times m$ zero and identity matrices, respectively. The eigenvalues of the operator L_t are the roots of the characteristic determinant

$$\begin{aligned}\Delta(\lambda,t) &= \det\left(Y_j^{(\nu-1)}(1,\lambda) - e^{it}Y_j^{(\nu-1)}(0,\lambda)\right)_{j,\nu=1}^{n} \\ &= e^{inmt} + f_1(\lambda)e^{i(nm-1)t} + f_2(\lambda)e^{i(nm-2)t} + \cdots + f_{nm-1}(\lambda)e^{it} + 1\end{aligned} \tag{3.2}$$

which is a polynomial of e^{it} with entire coefficients $f_1(\lambda), f_2(\lambda), \dots$. Therefore, the multiple eigenvalues of the operators L_t are the zeros of the resultant $R(\lambda) \equiv R(\Delta,\Delta')$ of the polynomials $\Delta(\lambda,t)$ and $\Delta'(\lambda,t) \equiv (\partial/\partial\lambda)\Delta(\lambda,t)$. Since $R(\lambda)$ is entire function and the large eigenvalues of L_t for $t \neq 0,\pi$ are simple (see Theorem 1.1(a)):

$$\ker R = \{\lambda : R(\lambda) = 0\} = \{a_1, a_2, \dots\}, \qquad \lim_{k\to\infty}|a_k| = \infty. \tag{3.3}$$

For each a_k, there are nm values $t_{k,1}, t_{k,2}, \dots, t_{k,nm}$ of t satisfying $\Delta(a_k,t) = 0$. Hence, the set

$$A = \bigcup_{k=1}^{\infty}\{t : \Delta(a_k,t) = 0\} = \{t_{k,i} : i = 1,2,\dots,nm;\ k = 1,2,\dots\} \tag{3.4}$$

is countable and for $t \notin A$, all eigenvalues of L_t are simple eigenvalues. By Theorem 1.1(a), the possible accumulation points of the set A are πk, where $k \in \mathbb{Z}$.

Lemma 3.1. *The eigenvalues of L_t can be numbered as $\lambda_1(t), \lambda_2(t), \dots$, such that for each p, the function $\lambda_p(t)$ is continuous in Q and is analytic in $Q \setminus A(p)$, where Q is defined in* (1.5)*, $A(p)$ is a subset of A consisting of finite numbers $t_1^p, t_2^p, \dots, t_{s_p}^p$, and $|\lambda_p(t)| \to \infty$ as $p \to \infty$. Moreover, there exists a number N_0 such that if $|k| \geq N_0$, $t \in Q_\varepsilon(n)$, then*

$$\lambda_{p(k,j)}(t) = \lambda_{k,j}(t), \tag{3.5}$$

where $N_0 \geq N(\varepsilon)$, $p(k,j) = 2|k|m + j$ if $k > 0$, $p(k,j) = (2|k|-1)m + j$ if $k < 0$, and the set $Q_\varepsilon(n)$ and the number $N(\varepsilon)$ are defined in (1.5) *and in Theorem 1.1(a), respectively.*

Proof. Let $t \in Q$. It easily follows from the classical investigations [3, Chapter 3, Theorem 2] (see (1.7), (1.8)) that there exist numbers r, c, independent of t, and an integer $N_0 \geq N(\varepsilon)$ such that all eigenvalues of the operators $L_{t,z}$ for $z \in [0,1]$, where $L_{t,z}$ is defined by (2.31), lie in the set

$$U(0,r) \cup \left(\bigcup_{k:|k| \geq N_0} U\left((2\pi k i + ti)^n, ck^{n-1-1/2m}\right) \right), \tag{3.6}$$

where $U(\mu, c) = \{\lambda \in \mathbb{C} : |\lambda - \mu| < c\}$. Clearly, there exists a closed curve Γ such that the following hold.

(a) The curve Γ lies in the resolvent set of the operator $L_{t,z}$ for all $z \in [0,1]$.

(b) All eigenvalues of $L_{t,z}$, for all $z \in [0,1]$ that do not lie in $U((2\pi k i + ti)^n, ck^{n-1-1/2m})$ for $|k| \geq N_0$, belong to the set enclosed by Γ.

Therefore, taking into account that the family $L_{t,z}$ is holomorphic with respect to z, we obtain that the number of eigenvalues of the operators $L_{t,0} = L_t(C)$ and $L_{t,1} = L_t$ lying inside of Γ are the same. It means that apart from the eigenvalues $\lambda_{k,j}(t)$, where $|k| \geq N_0$, $j = 1,2,\ldots,m$, there exist $(2N_0 - 1)m$ eigenvalues of the operator L_t. We define $\lambda_p(t)$ for $p > (2N_0 - 1)m$ and $t \in Q_\varepsilon(n)$ by (3.5). Let us first prove that these eigenvalues, that is, the eigenvalues $\lambda_{k,j}(t)$ for $|k| \geq N_0$ are the analytic functions on $Q_\varepsilon(n)$. By Theorem 1.1(a) if $t_0 \in Q_\varepsilon(n)$ and $|k| \geq N_0$, where $N_0 \geq N(\varepsilon)$, then $\lambda_{k,j}(t_0)$ is a simple zero of (3.2), that is, $\Delta(\lambda, t_0) = 0$, and $\Delta'(\lambda, t_0) \neq 0$ for $\lambda = \lambda_{k,j}(t_0)$. By implicit function theorem, there exist a neighborhood $U(t_0)$ of t_0 and an analytic function $\lambda(t)$ on $U(t_0)$ such that $\Delta(\lambda(t), t) = 0$ for $t \in U(t_0)$ and $\lambda(t_0) = \lambda_{k,j}(t_0)$. By Proposition 2.4, $\lambda_{k,j}(t_0) \in D(k,j,t_0)$. Since $\mu_{k,j}(t)$ and $\lambda(t)$ are continuous functions, the neighborhood $U(t_0)$ of t_0 can be chosen so that $\lambda(t) \in D(k,j,t)$ for all $t \in U(t_0)$. On the other hand, by Proposition 2.4, there exists a unique eigenvalue of L_t lying in $D(k,j,t)$ and this eigenvalue is denoted by $\lambda_{k,j}(t)$. Therefore, $\lambda(t) = \lambda_{k,j}(t)$ for all $t \in U(t_0)$, that is, $\lambda_{k,j}(t)$ is an analytic function in $U(t_0)$ for any $t_0 \in Q_\varepsilon(n)$.

Now let us construct the analytic continuation of $\lambda_{p(k,j)}(t)$ from $Q_\varepsilon(n)$ to the sets $U(0,\varepsilon)$, $U(\pi,\varepsilon)$ by using (3.2) and the implicit function theorem. Consider (3.2) for $t \in U(0,\varepsilon)$, $\lambda \in U_0 = U\left((2\pi k i)^n, 2n(2\pi k)^{n-1}\varepsilon\right)$. Since U_0 is a bounded region, $(\ker R) \cap U_0$ is a finite set (see (3.3)). Therefore, the subset $A(U_0)$ of A corresponding to $(\ker R) \cap U_0$, that is, the values of t corresponding to the multiple zeros of (3.2) lying in U_0 is finite. It follows from (1.7) and (1.8) that for any $t \in U(0,\varepsilon) \setminus A(U_0)$, the equation $\Delta(\lambda, t) = 0$ has $2m$ different solutions $d_1(t), d_2(t), \ldots, d_{2m}(t)$ in U_0 and $\Delta'(\lambda, t) \neq 0$ for $\lambda = d_1(t), d_2(t), \ldots, d_{2m}(t)$. Using the implicit function theorem and taking into account (1.8), we see that there exists a neighborhood $U(t,\delta)$ of t such that the following hold.

(i) There exist analytic functions $d_{1,t}(z), d_{2,t}(z), \ldots, d_{2m,t}(z)$ in $U(t,\delta)$ coinciding with $d_1(t), d_2(t), \ldots, d_{2m}(t)$ for $z = t$, respectively, and satisfying

$$\Delta(d_{s,t}(z), z) = 0, \quad d_{s,t}(z) \neq d_{j,t}(z) \quad \forall z \in U(t,\delta),\ s = 1,2,\ldots,2m,\ j \neq s. \tag{3.7}$$

(ii) $U(t,\delta) \cap A(U_0) = \varnothing$ and $d_{s,t}(z) \in U_0$ for $z \in U(t,\delta)$, $s = 1,2,\ldots,2m$.

Now, take any point t_0 from $U(0,\varepsilon) \setminus A(U_0)$. Let γ be a line segment in $U(0,\varepsilon) \setminus A(U_0)$ joining t_0 and a point of the circle $S(0,\varepsilon) = \{t : |t| = \varepsilon\}$. For any t from γ, there exists $U(t,\delta)$

satisfying (i) and (ii). Since γ is a compact set, the cover $\{U(t,\delta) : t \in \gamma\}$ of γ contains a finite cover $U(t_0,\delta), U(t_1,\delta), \ldots, U(t_v,\delta)$, where $t_v \in S(0,\varepsilon)$. For any $z \in U(t_v,\delta) \cap Q_\varepsilon(n)$, the eigenvalue $\lambda_{p(k,j)}(z)$ coincides with one of the eigenvalues $d_{1,t_v}(z), d_{2,t_v}(z), \ldots, d_{2m,t_v}(z)$ since there exists $2m$ eigenvalue of L_z lying in U_0. Denote by B_s the subset of the set $U(t_v,\delta) \cap Q_\varepsilon(n)$ for which the function $\lambda_{p(k,j)}(z)$ coincides with $d_{s,t_v}(z)$. Since $d_{s,t}(z) \neq d_{i,t}(z)$ for $s \neq i$, the sets $B_1, B_2, \ldots, B_{2m}$ are pairwise disjoint and the union of these sets is $U(t_v,\delta) \cap Q_\varepsilon(n)$. Therefore, there exists index s for which the set B_s contains an accumulation point and hence $\lambda_{p(k,j)}(z) = d_{s,t_v}(z)$ for all $z \in U(t_v,\delta) \cap Q_\varepsilon(n)$. Thus, $d_{s,t_v}(z)$ is an analytic continuation of $\lambda_{p(k,j)}(z)$ to $U(t_v,\delta)$. In the same way, we get the analytic continuation of $\lambda_{p(k,j)}(z)$ to $U(t_{v-1},\delta), U(t_{v-2},\delta), \ldots, U(t_0,\delta)$. Since t_0 is arbitrary point of $U(0,\varepsilon) \setminus A(U_0)$, we obtain the analytic continuation of $\lambda_{p(k,j)}(z)$ to $U(0,\varepsilon) \setminus A(U_0)$. The analytic continuation of $\lambda_{p(k,j)}(z)$ to $U(\pi,\varepsilon) \setminus A(U_\pi)$ can be obtained in the same way, where $A(U_\pi)$ can be defined as $A(U_0)$. Thus, the function $\lambda_{p(k,j)}(t)$ is analytic in $Q \setminus A(p)$, where $A(p)$consists of finite numbers $t_1^p, t_2^p, \ldots, t_{s_p}^p$. Since $\Delta(\lambda,t)$ is continuous with respect to (λ,t), the function $\lambda_{p(k,j)}(t)$ can be extended continuously to the set Q.

Now let us define the eigenvalues $\lambda_p(t)$ for $p \leq (2N_0-1)m$, $t \in Q$, which are apart from the eigenvalues defined by (3.5). These eigenvalues lie in a bounded set B, and by (3.3), the set $B \cap \ker R$ and the subset $A(B)$ of A corresponding to B are finite. Take a point a from the set $Q \setminus A$. Denote the eigenvalues of L_a in an increasing (of absolute value) order $|\lambda_1(a)| \leq |\lambda_2(a)| \leq \cdots \leq |\lambda_{(2N_0-1)m}(a)|$. If $|\lambda_p(a)| = |\lambda_{p+1}(a)|$, then by $\lambda_p(a)$, we denote the eigenvalue that has a smaller argument, where argument is taken in $[0,2\pi)$. Since $a \notin A$, the eigenvalues $\lambda_1(a), \lambda_2(a), \ldots, \lambda_{(2N_0-1)m}(a)$ are simple zeros of $\Delta(\lambda,a) = 0$. Therefore, using the implicit function theorem, we obtain the analytic functions $\lambda_1(t), \lambda_2(t), \ldots, \lambda_{(2N_0-1)m}(t)$ on a neighborhood $U(a,\delta)$ of a which are eigenvalues of L_t for $t \in U(a,\delta)$. These functions can be continued analytically to $Q_\varepsilon(n) \setminus A$, being the eigenvalues of L_t, where, as we noted above, $A \cap Q_\varepsilon(n)$consists of a finite number of points. Taking into account that $A(B)$ is finite, arguing as we have done in the proof of analytic continuation and continuous extension of $\lambda_p(t)$ for $p > (2N_0-1)m$, we obtain the analytic continuations of these functions to the set Q except finite points and the continuous extension to Q. □

By Gelfand's lemma (see [8, 9]), every compactly supported vector function $f(x)$ can be represented in the form

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f_t(x)\,dt, \tag{3.8}$$

where $f_t(x)$ is defined by (2.46). This representation can be extended to all functions of $L_2^m(-\infty,\infty)$, and

$$\int_0^1 \langle f_t(x), X_{k,t}(x)\rangle dx = \int_{-\infty}^{\infty} \langle f(x), X_{k,t}(x)\rangle dx, \tag{3.9}$$

where $\{X_{k,t} : k = 1,2,\ldots\}$ is a biorthogonal system of $\{\Psi_{k,t} : k = 1,2,\ldots\}$, $\Psi_{k,t}(x)$ is the normalized eigenfunction corresponding to $\lambda_k(t)$, the eigenvalue $\lambda_k(t)$ is defined in Lemma 3.1, $\Psi_{k,t}(x)$, and $X_{k,t}(x)$ are extended to $(-\infty,\infty)$ by (2.52) and by $X_{k,t}(x+1) = e^{i\bar{t}}X_{k,t}(x)$.

Let $a \in (0,\pi/2) \setminus A$, $\varepsilon \in (0,a/2)$ and let $l(\varepsilon)$ be a smooth curve joining the points $-a$ and $2\pi - a$ and satisfying

$$l(\varepsilon) \subset (Q_\varepsilon(n) \cap \Pi(a,\varepsilon)) \setminus A, \qquad l(-\varepsilon) \cap A = \varnothing, \qquad D(\varepsilon) \cup \overline{D(-\varepsilon)} \subset Q, \tag{3.10}$$

where $\Pi(a,\varepsilon) = \{x + iy : x \in [-a, 2\pi - a], y \in [0, 2\varepsilon)\}$, $l(-\varepsilon) = \{t : \bar{t} \in l(\varepsilon)\}$, the sets Q, $Q_\varepsilon(n)$, and A are defined in (1.5) and (3.4), $D(\varepsilon)$ and $D(-\varepsilon)$ are the domains enclosed by $l(\varepsilon) \cup [-a, 2\pi - a]$ and $l(-\varepsilon) \cup [-a, 2\pi - a]$, respectively, and $\overline{D(-\varepsilon)}$ is the closure of $D(-\varepsilon)$. Clearly, the domain $D(\varepsilon) \cup \overline{D(-\varepsilon)}$ is enclosed by the closed curve $l(\varepsilon) \cup l^-(-\varepsilon)$, where $l^-(-\varepsilon)$ is the opposite arc of $l(-\varepsilon)$. Suppose $f \in \Omega$, that is, (1.15) holds. If $2\varepsilon < \alpha$, then $f_t(x)$ is an analytic function of t in a neighborhood of $D(\varepsilon)$. Hence, the Cauchy's theorem and (3.8) give

$$f(x) = \frac{1}{2\pi} \int_{l(\varepsilon)} f_t(x) dt. \tag{3.11}$$

Since $l(\varepsilon) \in \mathbb{C}(n)$ (see (3.10) and the definition of $\mathbb{C}(n)$ in Section 1, it follows from Theorem 1.1(b) and Lemma 3.1 that for each $t \in l(\varepsilon)$, we have a decomposition

$$f_t(x) = \sum_{k=1}^{\infty} a_k(t) \Psi_{k,t}(x), \tag{3.12}$$

where $a_k(t) = (f_t, X_{k,t})$. Using (3.12) in (3.11), we get

$$f(x) = \frac{1}{2\pi} \int_{l(\varepsilon)} f_t(x) dt = \frac{1}{2\pi} \int_{l(\varepsilon)} \sum_{k=1}^{\infty} a_k(t) \Psi_{k,t}(x) dt. \tag{3.13}$$

Remark 3.2. If $\lambda \in \sigma(L)$, then there exist points $t_1, t_2, \ldots, t_k$ of $[0, 2\pi)$ such that λ is an eigenvalue $\lambda(t_j)$ of L_{t_j} of multiplicity s_j for $j = 1, 2, \ldots, k$. Let $S(\lambda, b) = \{z : |z - \lambda| = b\}$ be a circle containing only the eigenvalue $\lambda(t_j)$ of L_{t_j} for $j = 1, 2, \ldots, k$. Using Lemma 3.1, we see that there exists a neighborhood $U(t_j, \delta) = \{t : |t - t_j| \leq \delta\}$ of t_j such that the following hold.

(a) The circle $S(\lambda, b)$ lies in the resolvent set of L_t for all $t \in U(t_j, \delta)$ and $j = 1, 2, \ldots, k$.

(b) If $t \in (U(t_j, \delta) \setminus \{t_j\})$, then the operator L_t has only s_j eigenvalues, denoted by $\Lambda_{j,1}(t), \Lambda_{j,2}(t), \ldots, \Lambda_{j,s_j}(t)$, lying in $S(\lambda, b)$ and these eigenvalues are simple.

Thus, the spectrum of L_t for $t \in U(t_j, \delta)$, $j = 1, 2, \ldots, k$is separated by $S(\lambda, b)$ into two parts in the sense of [25] (see [25, Chapter 3, Section 6.4]). Since $\{L_t : t \in U(t_j, \delta)\}$ is a holomorphic family of operators in the sense of [25] (see [25, Chapter 7, Section 1]), the theory of holomorphic family of the finite dimensional operators can be applied to the part of L_t for $t \in U(t_j, \delta)$ corresponding to the inside of $S(\lambda, b)$. Therefore, (see [13, Chapter 2, Section 1]) the eigenvalues $\Lambda_{j,1}(t), \Lambda_{j,2}(t), \ldots, \Lambda_{j,s_j}(t)$ and corresponding eigenprojections $P(\Lambda_{j,1}(t)), P(\Lambda_{j,2}(t)), \ldots, P(\Lambda_{j,s_j}(t))$ are branches of an analytic function. These eigenprojections are represented by a Laurent series in $t^{1/\nu}$, where $\nu \leq s_j$, with finite principal parts. One can easily see that if $\lambda_p(t)$ is a simple eigenvalue of L_t, then

$$P(\lambda_p(t)) f = (f, X_{p,t}) \Psi_{p,t}, \qquad \|P(\lambda_p(t))\| = \frac{1}{\|X_{p,t}\|} = \left| \frac{1}{\alpha_p(t)} \right|, \tag{3.14}$$

and $P(\lambda_p(t))$ is analytic function in a neighborhood of t, where $\alpha_p(t) = (\Psi_{p,t}, \Psi^*_{p,t})$. This and Lemma 3.1 show that $a_p(t)\Psi_{p,t}$ is analytic function of t on $D(\varepsilon) \cup \overline{D(-\varepsilon)}$ except finite points.

Theorem 3.3. (a) *If f is absolutely continuous, compactly supported function and $f' \in L_2^m(-\infty,\infty)$, then*

$$f(x) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\int_{l(\varepsilon)} a_k(t)\Psi_{k,t}(x)dt, \tag{3.15}$$

$$f(x) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\int_{[0,2\pi)^+} a_k(t)\Psi_{k,t}(x)dt, \tag{3.16}$$

where

$$\int_{[0,2\pi)^+} a_k(t)\Psi_{k,t}(x)dt = \lim_{\varepsilon\to 0}\int_{l(\varepsilon)} a_k(t)\Psi_{k,t}(x)dt, \tag{3.17}$$

and the series (3.15), (3.16) *converge uniformly in any bounded subset of* $(-\infty,\infty)$.

(b) *Every function $f \in \Omega$, where Ω is defined in* (1.15), *has decompositions* (3.15) *and* (3.16), *where the series converges in the norm of $L_2^m(a,b)$ for every $a,b \in \mathbb{R}$.*

Proof. The proof of (3.15) in the case (a) follows from (3.13), Theorem 2.5, and Lemma 3.1. In Appendix A, by writing the proof of Theorem 2 of [19] in the vector form, we obtain the proof of (3.15) in the case (b). In Appendix B, the formula (3.16) is obtained from (3.15) by writing the proof of Theorem 3 of [19] in the vector form. □

Definition 3.4. Let λ be a point of the spectrum $\sigma(L)$ of L and $t_1, t_2, \ldots, t_k$ be the points of $[0,2\pi)$ such that λ is an eigenvalue of L_{t_j} of multiplicity s_j for $j = 1,2,\ldots,k$. The point λ is called a spectral singularity of L if

$$\sup \|P(\Lambda_{j,i}(t))\| = \infty, \tag{3.18}$$

where supremum is taken over all $t \in (U(t_j,\delta) \setminus \{t_j\})$, $j = 1,2,\ldots,k$; $i = 1,2,\ldots,s_j$, the set $U(t_j,\delta)$, and the eigenvalues $\Lambda_{j,1}(t), \Lambda_{j,2}(t), \ldots, \Lambda_{j,s_j}(t)$ are defined in Remark 3.2. In other words, λ is called a spectral singularity of L if there exist indices j, i such that the point t_j is a pole of $P(\Lambda_{j,i}(t))$. Briefly speaking, a point $\lambda \in \sigma(L)$ is called a spectral singularity of L if the projections of L_t corresponding to the simple eigenvalues lying in the small neighborhood of λ are not uniformly bounded. We denote the set of the spectral singularities by $S(L)$.

Remark 3.5. Note that if $\gamma = \{\lambda_p(t) : t \in (\alpha,\beta)\}$ is a curve lying in $\sigma(L)$ and containing no multiple eigenvalues of L_t, where $t \in [0,2\pi)$, then arguing as in [16, 21], one can prove that the projection $P(\gamma)$ of L corresponding to γ satisfies the following relations:

$$P(\gamma)f = \int_{(\alpha,\beta)} (f, X_{p,t})\Psi_{p,t}\,dt, \qquad \|P(\gamma)\| = \sup_{t\in(\alpha,\beta)} \|P(\lambda_p(t))\|. \tag{3.19}$$

These relations show that Definition 3.4 is equivalent to the definition of the spectral singularity given in [16, 21], where the spectral singularity is defined as a point in the neighborhood of which the projections $P(\gamma)$ are not uniformly bounded. The proof of (3.19) is long and technical. In order to avoid eclipsing, the essence by the technical details and taking into account that in the spectral expansion of L, the eigenfunctions and eigenprojections of L_t for $t \in [0,2\pi)$ are used (see (3.16)), in this paper, in the definition of the spectral singularity, without loss of naturalness, instead of the boundlessness of the projections $P(\gamma)$ of L, we use the boundlessness of the projections $P(\lambda_p(t))$ of L_t, that is, we use Definition 3.4. In any case, the spectral singularity is a point of $\sigma(L)$ that requires the regularization in order to get the spectral expansion.

Theorem 3.6. (a) *All spectral singularities of L are contained in the set of the multiply eigenvalues of L_t for $t \in 0, 2\pi$, that is, $S(L) = \{\Lambda_1, \Lambda_2, \dots\} \subset \ker R \cap \sigma(L)$, where $S(L)$ and* $\ker R$ *are defined in Definition 3.4 and in* (3.3)*, respectively.*

(b) *Let $\lambda = \lambda_p(t_0) \in \sigma(L) \setminus S(L)$, where $t_0 \in (a, 2\pi - a)$. If $\gamma_1, \gamma_2, \dots,$ is a sequence of smooth curves lying in a neighborhood $U = \{t \in \mathbb{C} : |t - t_0| \le \delta_0\}$ of t_0 and approximating the interval $[t_0 - \delta_0, t_0 + \delta_0]$, then*

$$\lim_{k\to\infty} \int_{\gamma_k} a_p(t)\Psi_{p,t}(x)dt = \int_{t_0-\delta_0}^{t_0-\delta_0} a_p(t)\Psi_{p,t}(x)dt, \tag{3.20}$$

where U is a neighborhood of t_0 such that if $t \in U$, then $\lambda_p(t)$ is not a spectral singularity.

(c) *If the operator L has no spectral singularities, then we have the following spectral expansion in term of the parameter t:*

$$f(x) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\int_0^{2\pi} a_k(t)\Psi_{k,t}(x)dt. \tag{3.21}$$

If $f(x)$ is an absolutely continuous, compactly supported function and $f' \in L_2^m(-\infty, \infty)$, then the series in (3.21) *converges uniformly in any bounded subset of $(-\infty, \infty)$. If $f(x) \in \Omega$, where Ω is defined in* (1.15)*, then the series converges in the norm of $L_2^m(a, b)$ for every $a, b \in \mathbb{R}$.*

Proof. (a) If $\lambda_p(t_0)$ is a simple eigenvalue of L_{t_0}, then due to Remark 3.2(see (3.14) and the end of Remark 3.2, the projection $P(\lambda_p(t))$ and $|\alpha_p(t)|$ continuously depend on t in some neighborhood of t_0. On the other hand, $\alpha_p(t_0) \neq 0$, since the system of the root functions of L_{t_0} is complete. Thus, it follows from Definition 3.4 that λ is not a spectral singularity of L.

(b) It follows from (3.3) and Theorem 3.6(a) that there exists a neighborhood U of t_0 such that if $t \in U$, then $\lambda_p(t)$ is not a spectral singularity of L. If $\lambda_p(t_0) \in \sigma(L) \setminus S(L)$, then by Definition 3.4, t_0 is not a pole of $P(\lambda_p(t))$, that is, by Remark 3.2, the Laurent series in $t^{1/\nu}$, where $\nu \le s$, of $P(\lambda_p(t))$ at t_0 has no principal part. Therefore, (3.14) implies that

$$\frac{1}{|\alpha_p(t)|}(f_t, \Psi_{p,t}^*)\Psi_{p,t} \tag{3.22}$$

is a bounded continuous functions in a neighborhood of t_0, which implies the proof of (b).

(c) By Theorem 3.6(b) if the operator L has not spectral singularities, then

$$\int_{[0,2\pi)^+} a_k(t)\Psi_{k,t}(x)dt = \int_0^{2\pi} a_k(t)\Psi_{k,t}(x)dt, \tag{3.23}$$

where the left-hand side of this equality is defined by (3.17). Thus, (3.21) follows from (3.23), (3.16) and Theorem 3.6(c) follows from Theorem 3.3. □

Now, we change the variables to λ by using the characteristic equation $\Delta(\lambda, t) = 0$ and the implicit-function theorem. By (3.2), $\Delta(\lambda, t)$ and $\partial\Delta(\lambda, t)/\partial t$ are polynomials of e^{it} and their resultant $T(\lambda)$ is entire function. It is clear that $T(\lambda)$ is not zero function. Let $b_1, b_2, \dots,$ be zeros of $T(\lambda)$. Then, $|b_k| \to \infty$ as $k \to \infty$ and the equation $\Delta(\lambda, t) = 0$ defines a function $t(\lambda)$ such that

$$\Delta(\lambda, t(\lambda)) = 0, \qquad \frac{dt}{d\lambda} = -\frac{\partial\Delta/\partial\lambda}{\partial\Delta/\partial t}, \qquad \frac{\partial\Delta(\lambda, t)}{\partial t}\Big/_{t=t(\lambda)} \neq 0 \tag{3.24}$$

for all $\lambda \in \mathbb{C} \setminus \{b_1, b_2, \dots\}$. Consider the functions

$$F_{p,t}(x) = \sum_{k=1,2,\dots,n} Y_k(x, \lambda_p(t)) A_k(t, \lambda_p(t)) = \left(\sum_{k=1,2,\dots,n} Y_k(x, \lambda) A_k(t(\lambda), \lambda) \right)_{\lambda=\lambda_p(t)}, \tag{3.25}$$

where $Y_1(x,\lambda), Y_2(x,\lambda), \dots, Y_n(x,\lambda)$ are linearly independent solutions of (3.1), $A_k = (A_{k,1}, A_{k,2}, \dots, A_{k,m})$, $A_{k,i} = A_{k,i}(t,\lambda)$ is the cofactor of the entry in mn row and $(k-1)m+i$ column of the determinant (3.2). One can readily see that

$$A_{k,i}(t,\lambda) = g_s(\lambda) e^{ist} + g_{s-1}(\lambda) e^{i(s-1)t} + \cdots + g_1(\lambda) e^{it} + g_0(\lambda), \tag{3.26}$$

where $g_0(\lambda), g_1(\lambda), \dots,$ are entire functions. By (3.24), $A_{k,i}(t(\lambda),\lambda)$ is an analytic function of λ in $\mathbb{C} \setminus \{b_1, b_2, \dots\}$. Since the operator L_t for $t \neq 0, \pi$ has a simple eigenvalue, there exists a nonzero cofactor of the determinant (3.2). Without loss of generality, it can be assumed that $A_{k,1}(t(\lambda),\lambda)$ is nonzero function. Then, $A_{k,1}(t(\lambda),\lambda)$ has a finite number zeros in each compact subset of $\mathbb{C} \setminus \{b_1, b_2, \dots\}$. Thus, there exists a countable set E_1 such that

$$\{b_1, b_2, \dots\} \subset E_1, \qquad A_{k,1}(t(\lambda),\lambda) \neq 0 \quad \forall \lambda \notin E_1. \tag{3.27}$$

Let A_1 be the set of all t satisfying $\Delta(\lambda,t) = 0$ for some $\lambda \in E_1$. Clearly, A_1 is a countable set. Now, using Lemma 3.1, (3.25), (3.27) and taking into account that the functions $Y_1(x,\lambda), Y_2(x,\lambda), \dots, Y_n(x,\lambda)$ are linearly independent, we obtain

$$\Psi_{p,t}(x) = \frac{F_{p,t}(x)}{\|F_{p,t}\|}, \quad \|F_{p,t}\| \neq 0 \ \forall t \in \left(D(\varepsilon) \cup \overline{D(-\varepsilon)}\right) \setminus (A \cup A_1), \tag{3.28}$$

where $\Psi_{p,t}(x)$ is a normalized eigenfunction corresponding to $\lambda_p(t)$. Since the set $A \cup A_1$ is countable, there exist the curves $l(\varepsilon_1), l(\varepsilon_2), \dots,$ such that

$$\lim_{s \to \infty} l(\varepsilon_s) = [-a, 2\pi - a], \quad l(\varepsilon_s) \in \left(D(\varepsilon) \cup \overline{D(-\varepsilon)}\right) \setminus (A \cup A_1) \ \forall s. \tag{3.29}$$

Now let us do the change of variables in (3.15). Using (3.24), (3.25), (3.28), we get

$$a_p(t(\lambda)) \Psi_{p,t(\lambda)}(x) = \frac{h(\lambda)}{\alpha(\lambda)} F(x,\lambda), \quad \text{where } F(x,\lambda) = \sum_{j=1,2,\dots,n} Y_j(x,\lambda) A_j(\lambda), \tag{3.30}$$

$A_j(\lambda) = A_j(t(\lambda),\lambda)$, $A_j(t,\lambda)$ is defined in (3.25), and $(F(x,\lambda))_{\lambda=\lambda_p(t)} = F_{p,t}(x)$, $h(\lambda) = (f(\cdot), \Phi(\cdot,\lambda))$, $\Phi(x, \lambda_p(t))$ is eigenfunction of L_t^* corresponding to $\overline{\lambda_p(t)}$ and $\alpha(\lambda) \equiv (F(\cdot,\lambda), \Phi(\cdot,\lambda))$. Using these notations and (3.24), we obtain

$$\int_{l(\varepsilon_s)} a_p(t) \Psi_{p,t}(x) dt = \int_{\Gamma_p(\varepsilon_s)} \frac{-h(\lambda)\varphi(\lambda)}{\alpha(\lambda)\phi(\lambda)} \left(\sum_{j=1}^{n} Y_j(x,\lambda) A_j(\lambda) \right) d\lambda, \tag{3.31}$$

where $\Gamma_p(\varepsilon_s) = \{\lambda = \lambda_p(t) : t \in l(\varepsilon_s)\}$, $\varphi = \partial\Delta/\partial\lambda$, $\phi = \partial\Delta/\partial t$. Note that it follows from (3.24) and (3.29) that $\phi(\lambda) \neq 0$ for $\lambda \in \Gamma_p(\varepsilon_s)$. If $t \in l(\varepsilon_s)$, then by the definition of A and by (3.29) $\lambda_p(t)$ is a simple eigenvalue. Hence, $\alpha_p(t) \neq 0$, since the root functions of L_t is complete in $L_2^m(0,1)$. Therefore, $\alpha(\lambda) \neq 0$ for $\lambda \in \Gamma_p(\varepsilon_s)$.

To do the regularization about the spectral singularities $\Lambda_1, \Lambda_2, \dots,$ we take into account that there exist numbers i_l, δ, and c_7 such that if $|\lambda - \Lambda_l| < \delta$, then the equality

$$\left| \frac{(\lambda - \Lambda_l)^{i_l} h(\lambda)\varphi(\lambda) A_j(\lambda)}{\alpha(\lambda)\phi(\lambda)} \right| < c_7 \tag{3.32}$$

holds for $j = 1, 2, \dots, n$ and $U(\Lambda_1, \delta), U(\Lambda_2, \delta), \dots$ are pairwise disjoint disks, where $U(\Lambda, \delta) = \{\lambda : |\lambda - \Lambda| < \delta\}$. Introduce the mapping B as follows:

$$Bf(x,\lambda) = f(x,\lambda) - \sum_{l} \sum_{\nu=0}^{i_l - 1} B_{l,\nu}(\lambda) \frac{\partial^{\nu}(f(x,\Lambda_l))}{\partial \lambda^{\nu}}, \tag{3.33}$$

where $B_{l,\nu}(\lambda) = (\lambda - \Lambda_l)^{\nu}/\nu!$ for $\lambda \in U(\Lambda_l, \delta)$ and $B_{l,\nu}(\lambda) = 0$ for $\lambda \notin U(\Lambda_l, \delta)$. We set

$$\Gamma_k = \{\lambda = \lambda_k(t) : t \in [0, 2\pi)\}, \qquad S_k = \{l : \Lambda_l \in \Gamma_k \cap S(L)\}. \tag{3.34}$$

Now, using these notations and formulas (3.16), (3.17), (3.31), we get

$$f(x) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \left(\int_{\Gamma_k} \frac{-h(\lambda)\varphi(\lambda)}{\alpha(\lambda))\phi(\lambda)} \left(\sum_{j=1}^{n} B(Y_j(x,\lambda)) A_j(\lambda) \right) d\lambda + \sum_{l \in S_k} M_{k,l}(x) \right), \tag{3.35}$$

where

$$M_{k,l}(x) = \lim_{s \to \infty} \frac{1}{2\pi} \int_{\Gamma_k(\varepsilon_s)} \frac{-h(\lambda)\varphi(\lambda)}{\alpha(\lambda))\phi(\lambda)} \left(\sum_{j=1}^{n} \left(\sum_{\nu=0}^{i_l - 1} B_{l,\nu}(\lambda) \frac{\partial^{\nu}(Y_j(x,\Lambda_l))}{\partial \lambda^{\nu}} A_j(\lambda) \right) d\lambda. \right. \tag{3.36}$$

Thus, Theorem 3.3 implies the following spectral expansion of L.

Theorem 3.7. *Every function $f(x) \in \Omega$ has decomposition* (3.35)*, where the series in* (3.35) *converges in the norm of $L_2^m(a,b)$ for every $a, b \in \mathbb{R}$. If $f(x)$ is absolutely continuous, compactly supported function and $f' \in L_2^m(-\infty,\infty)$, then the series in* (3.35) *converges uniformly in any bounded subset of* $(-\infty,\infty)$.

Remark 3.8. Let $n = 2\mu + 1$. Then by Theorem 1.1 all large eigenvalue of L_t for $t \in Q$ are simple and hence the set $A \cap Q$, where A is defined in (3.4), is finite. The number of spectral singularities is finite and (3.23) holds for $k \gg 1$. If $\varepsilon \ll 1$, then $D(\varepsilon) \cap A = \varnothing$ and $D(-\varepsilon) \cap A = \varnothing$, where $D(\varepsilon)$ and $D(-\varepsilon)$ are defined in (3.10). Therefore, the spectral expansion (3.35) has a simpler form. Moreover, repeating the proof of Corollary 1(a) of [20], we obtain that every function $f \in L_2^m(-\infty,\infty)$, satisfying (1.14), has decomposition (3.35).

Appendices

A. Proof of (3.15)

Here, we justify the term by term integration of the series in (3.13). Let $H_{N,t}$ be the linear span of $\Psi_{1,t}(x), \Psi_{2,t}(x), \dots, \Psi_{N,t}(x)$ and $f_{N,t}$ be the projection of $f_t(x)$ onto $H_{N,t}$. Since $\{\Psi_{k,t}(x)\}$ and $\{X_{k,t}(x)\}$ are biorthogonal system, we have

$$f_{N,t}(x) = \sum_{k=1,2,\dots,N} a_k^N(t) \Psi_{k,t}(x), \tag{A.1}$$

where $a_k^N(t) = (f_{N,t}, X_{k,t})$. Using the notations $g_{N,t} = f_t - f_{N,t}, b_k^N(t) = (g_{N,t}, X_{k,t})$, the equality (A.1), and then (3.11), we obtain $a_k^N(t) = a_k(t) - b_k^N(t)$,

$$f_t = \sum_{k=1,2,\ldots,N} (a_k(t) - b_k^N(t))\Psi_{k,t} + g_{N,t},$$
$$f(x) = \frac{1}{2\pi}\left(\sum_{k=1}^{N}\int_{l(\varepsilon)}\left(a_k(t)\Psi_{k,t}(x)dt + \int_{l(\varepsilon)}\left(g_{N,t}(x) - \sum_{k=1}^{n} b_k^N(t)\right)\Psi_{k,t}(x)\right)dt\right). \tag{A.2}$$

To obtain (3.15), we need to prove that the last integral in (A.2) tends to zero as $N \to \infty$. For this purpose, we prove the following.

Lemma A.1. *The functions*

$$\|g_{N,t}\|, \qquad \left\|\sum_{k=1,2,\ldots,N} b_k^N(t))\Psi_{k,t}\right\| \tag{A.3}$$

tend to zero as $N \to \infty$ *uniformly with respect to* t *in* $l(\varepsilon)$.

Proof. First, we prove that $\|g_{N,t}\|$ tends to zero uniformly. Let $P_{N,t}$ and $P_{\infty,t}$ be projections of $L_2^m[0,1]$ onto $H_{N,t}$ and $H_{\infty,t}$, respectively, where $H_{\infty,t} = \cup_{n=1}^{\infty} H_{N,t}$. If follows from (3.12) that $f_t \in H_{\infty,t}$. On the other hand, one can readily see that

$$H_{N,t} \subset H_{N+1,t} \subset H_{\infty,t}, \qquad P_{N,t} \subset P_{\infty,t}, \quad P_{N,t} \longrightarrow P_{\infty,t}. \tag{A.4}$$

Therefore, $P_{N,t}f_t \to f_t$, that is, $\|g_{N,t}\| \to 0$. Since $\|g_{N,t}\|$ is a distance from f_t to $H_{N,t}$, for each sequence $\{t_1, t_1, \ldots\} \subset l(\varepsilon)$ converging to t_0, we have

$$\begin{aligned}\|g_{N,t_s}\| &\le \left\|f_{t_s} - \sum_{k=1,2,\ldots,N} a_k^N(t_0)\Psi_{k,t_s}(x)\right\| \\ &\le \|g_{N,t_0}\| + \|f_{t_s} - f_{t_0}\| + \left\|\sum_{k=1,2,\ldots,N} a_k^N(t_0)(\Psi_{k,t_0} - \Psi_{k,t_s})\right\| \\ &\le \|g_{N,t_0}\| + \alpha_s,\end{aligned} \tag{A.5}$$

where $\alpha_s \to 0$ as $s \to \infty$ by continuity of f_t and $\Psi_{k,t}$ on $l(\varepsilon)$. Similarly (interchanging t_0 and t_s), we get $\|g_{N,t_0}\| \le \|g_{N,t_s}\| + \beta_s$, where $\beta_s \to 0$ as $s \to \infty$. Hence, $\|g_{N,t}\|$ is a continuous function on the compact $l(\varepsilon)$. On the other hand, the first inclusion of (A.4) implies that $\|g_{N,t}\| \ge \|g_{N+1,t}\|$. Now, it follows from the proved three properties of $\|g_{N,t}\|$ that $\|g_{N,t}\|$ tends to zero as $N \to \infty$ uniformly on the compact $l(\varepsilon)$.

Now, to prove that the second function in (A.3) tends to zero uniformly, we consider the family of operators $\Gamma_{p,t}$ for $t \in l(\varepsilon)$, $p = 1, 2, \ldots$, defined by formula

$$\Gamma_{p,t}(f) = \sum_{k=1,2,\ldots,p} (f, X_{k,t})\Psi_{k,t}(x). \tag{A.6}$$

First, let us prove that the set $\Gamma(f) = \{\Gamma_{p,t}(f) : t \in l(\varepsilon), p = 1, 2, \ldots\}$ is a bounded subset of $L_2^m[0,1]$. Since in the Hilbert space every weakly bounded subset is a strongly bounded subset, it is enough to show that for each $g \in L_2^m[0,1]$, there exists a constant M such that

$$|(g, \varphi)| < M, \quad \forall \varphi \in \Gamma(f). \tag{A.7}$$

Decomposing g by the basis $\{X_{k,t} : k = 1,2,\ldots\}$, using definition of φ and then the uniform asymptotic formulas (1.11), (1.13), we obtain

$$\begin{aligned}|(g,\varphi)| &\le \sum_{k=1,2,\ldots,p} |(\varphi, X_{k,t})(g,\Psi_{k,t})| \\ &\le \sum_{k=1,2,\ldots,p} |(\varphi, X_{k,t})|^2 + \sum_{k=1,2,\ldots,p} |(g,\Psi_{k,t})|^2 \\ &= \|\varphi\|^2 + \|g\|^2 + O(1),\end{aligned} \tag{A.8}$$

which implies (A.7). Thus, $\Gamma(f)$ is a bounded set. On the other hand, one can readily see that $\Gamma_{p,t}$, for $t \in l(\varepsilon)$, $p = 1,2,\ldots$, is a linear continuous operator. Therefore, by Banach- Steinhaus theorem, the family of operators $\Gamma_{p,t}$ is equicontinuous. Now, using the equality

$$\Gamma_{N,t} g_{N,t} = \sum_{k=1,2,\ldots,N} b_{k,j}^N(t)\Psi_{k,j,t}, \tag{A.9}$$

and taking into account that the first function in (A.3) tends to zero uniformly, we obtain that the second function in (A.3) also tends to zero uniformly. □

Using Lemma A.1 and Schwarz inequality, we get

$$\begin{aligned}&\left\| \int_{l(\varepsilon)} \left(g_{N,t}(x) - \sum_{k=1,2,\ldots,N} b_k^N(t) \right) \Psi_{k,t}(x) \right\| dt \\ &\quad \le C_\varepsilon \int_a^b \int_{l(\varepsilon)} \left| g_{N,t}(x) - \sum_{k=1,2,\ldots,N} b_{k,}^N(t)\Psi_{k,t}(x) \right| |dt| dx \\ &\quad = C_\varepsilon \int_{l(\varepsilon)} \left\| \left(g_{N,t}(x) - \sum_{k=1,2,\ldots,N} b_k^N(t) \right) \Psi_{k,t}(x) \right\| |dt| \longrightarrow 0 \quad \text{as } N \longrightarrow \infty,\end{aligned} \tag{A.10}$$

where C_ε is the length of $l(\varepsilon)$, the norm used here is the norm of $L_2^m(a,b)$, a and b are the real numbers. This and (A.2) justify the term by term integration of the series in (3.13).

B. Proof of (3.16)

Here, we use the notation introduced in (3.10) and prove (3.16). Since for fixed k the function $a_k(t)\Psi_{k,t}$ is analytic on $D(\varepsilon)$ except finite number points $t_1^k, t_2^k, \ldots, t_{p_k}^k$ (see the end of Remark 3.2), we have

$$\int_{l(\varepsilon)} a_k(t)\Psi_{k,t}\, dt = \int_{[0,2\pi]^+} a_k(t)\Psi_{k,t}\, dt + \sum_{s: t_s^k \in D(\varepsilon)} \operatorname{Res}_{t=t_s^k} a_k(t)\Psi_{k,t}. \tag{B.1}$$

Similarly,

$$\int_{l(-\varepsilon)} a_k(t)\Psi_{k,t}\, dt = \int_{[0,2\pi]^+} a_k(t)\Psi_{k,t}\, dt + \sum_{s: t_s^k \in \overline{D(-\varepsilon)}} \operatorname{Res}_{t=t_s^k} a_k(t)\Psi_{k,t}. \tag{B.2}$$

Since $l(\varepsilon) \cup l^-(-\varepsilon)$ is a closed curve enclosing $D(-\varepsilon) \cup \overline{D(-\varepsilon)}$, we have

$$\int_{l(\varepsilon)\cup l^-(-\varepsilon)} a_{k,j}(t)\Psi_{k,t}(x)dt = \sum_{s: t_s^k \in D(-\varepsilon)\cup\overline{D(-\varepsilon)}} \operatorname{Res}_{t=t_s^k} a_k(t)\Psi_{k,t}. \tag{B.3}$$

Now applying (3.15) to the curves $l(\varepsilon), l(-\varepsilon), l(\varepsilon) \cup l^{-}(-\varepsilon)$, using (B.1), (B.2), (B.3) and taking into account that $l(\varepsilon) \cup l^{-}(-\varepsilon)$ is a closed curve, we obtain

$$f(x) = \frac{1}{2\pi} \sum_{k=1,2,\ldots} \left(\int_{[0,2\pi]^{+}} a_k(t) \Psi_{k,t}(x)\,dt + \sum_{s:t_s^k \in D(\varepsilon)} \operatorname{Res}_{t=t_s^k} a_k(t) \Psi_{k,t} \right), \tag{B.4}$$

$$f(x) = \frac{1}{2\pi} \sum_{k=1,2,\ldots} \left(\int_{[0,2\pi]^{+}} a_k(t) \Psi_{k,t}(x)\,dt + \sum_{s:t_s^k \in \overline{D(-\varepsilon)}} \operatorname{Res}_{t=t_s^k} a_k(t) \Psi_{k,t} \right), \tag{B.5}$$

$$0 = \frac{1}{2\pi} \int_{l(\varepsilon) \cup l^{-}(-\varepsilon)} f_t(x)\,dt = \frac{1}{2\pi} \sum_{k=1,2,\ldots} \left(\sum_{s:t_s^k \in (D(-\varepsilon) \cup \overline{D(-\varepsilon)})} \operatorname{Res}_{t=t_s^k} a_k(t) \Psi_{k,t} \right). \tag{B.6}$$

Adding (B.4) and (B.5) and then using (B.6), we get the proof of (3.16).

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