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Research Article

The Existence and Uniqueness of Solution of Duffing Equations with Non-C^2 Perturbation Functional at Nonresonance

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This paper deals with a boundary value problem for Duffing equation. The existence of unique solution for the problem is studied by using the minimax theorem due to Huang Wenhua. The existence and uniqueness result was presented under a generalized nonresonance condition.

1. Introduction

In recent years, many authors are greatly attached to investigation for the existence and uniqueness of solution of Duffing equations, for example, [1–11], and so forth. Some authors ([8, 11, 12], etc.) proved the existence and uniqueness of solution of Duffing equations under C^2 perturbation functions and other conditions at nonresonance by employing minimax theorems. In 1986, Tersian investigated the equation $u'' + f(t, u(t)) = -p(t)$ using a minimax theorem proved by himself and reaped a result of generalized solution [13]. In 2005, Huang and Shen generalized the minimax theorem of Tersian in [13]. Using the generalized minimax theorem, Huang and Shen proved a theorem of existence and uniqueness of solution for the equation $u'' + f(t, u(t)) + e(t) = \mathbf{0}$ [14] under the weaker conditions than those in [13].

Stimulated by the works in [13, 14], in the present paper, we investigate the solutions of the boundary value problems of Duffing equations with non-C^2 perturbation functions at nonresonance using the minimax theorem proved by Huang in [15].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, X and Y be two orthogonal closed subspaces of H such that $H = X \oplus Y$. Let $Q : H \to X$, $P : H \to Y$

denote the projections from H to X and from H to Y, respectively. The following theorem will be employed to prove our main theorem.

Theorem 2.1 (see [15]). *Let H be a real Hilbert space, let X and Y be orthogonal closed vector subspace of H such that $H = X \oplus Y$, let $f : H \to \mathbf{R}$ be an everywhere defined functional with Gâteaux derivative, $\nabla f : H \to H$ everywhere defined and hemicontinuous. Suppose that there exist two continuous functions $\alpha : [0, +\infty) \to (0, +\infty)$, $\beta : [0, +\infty) \to (0, +\infty)$ satisfying*

$$\begin{gathered}\alpha(s) \longrightarrow +\infty, \quad \beta(s) \longrightarrow +\infty, \quad \text{as } s \longrightarrow \infty, \\ \langle \nabla f(u) - \nabla f(v), x_1 - x_2 \rangle \le -\alpha(\|u - v\|)\|x_1 - x_2\|, \\ \langle \nabla f(u) - \nabla f(v), y_1 - y_2 \rangle \ge \beta(\|u - v\|)\|y_1 - y_2\|,\end{gathered} \tag{2.1}$$

for $u \in H$, $v \in H$, $x_1 = Qu \in X$, $x_2 = Qv \in X$, $y_1 = Pu \in Y$, $y_2 = Pv \in Y$. Then, the following hold:

(a) *f has a unique critical point $v_0 \in H$ such that $\nabla f(v_0) = 0$;*

(b) *$f(v_0) = \max_{x\in X} \min_{y\in Y} f(x + y) = \min_{y\in Y} \max_{x\in X} f(x + y)$.*

It is easy to prove the following corollary of the above theorem.

Corollary 2.2. *Let H be a real Hilbert space, let X and Y be orthogonal closed vector subspace of H such that $H = X \oplus Y$, and let $f : H \to \mathbf{R}$ be an everywhere defined functional with second Gâteaux differential. Suppose that there exist two continuous functions $\alpha : [0, +\infty) \to (0, +\infty)$, $\beta : [0, +\infty) \to (0, +\infty)$ satisfying*

$$\begin{gathered}\alpha(s) \longrightarrow +\infty, \quad \beta(s) \longrightarrow +\infty, \quad \text{as } s \longrightarrow \infty, \\ \langle \nabla^2 f(v + t(u - v))(u - v), x_1 - x_2 \rangle \le -\alpha(\|u - v\|)\|x_1 - x_2\|, \\ \langle \nabla^2 f(v + t(u - v))(u - v), y_1 - y_2 \rangle \ge \beta(\|u - v\|)\|y_1 - y_2\|,\end{gathered} \tag{2.2}$$

for $u \in H$, $v \in H$, $x_1 = Qu \in X$, $x_2 = Qv \in X$, $y_1 = Pu \in Y$, $y_2 = Pv \in Y$, $0 < t < 1$. Then, the following hold:

(a) *f has a unique critical point $v_0 \in H$ such that $\nabla f(v_0) = 0$;*

(b) *$f(v_0) = \max_{x\in X} \min_{y\in Y} f(x + y) = \min_{y\in Y} \max_{x\in X} f(x + y)$.*

Proof. We note that f is a second Gâteaux differentiable functional, the mean-value theorem ensures that there exists $\theta \in (0, 1)$ such that $\nabla f(u) - \nabla f(v) = \nabla^2 f(v + \theta(u - v))(u - v)$. Therefore, for $u \in H$, $v \in H$, $x_1 = Qu \in X$, $x_2 = Qv \in X$, $y_1 = Pu \in Y$, $y_2 = Pv \in Y$, we have

$$\begin{gathered}\langle \nabla f(u) - \nabla f(v), x_1 - x_2 \rangle = \langle \nabla^2 f(v + \theta(u - v))(u - v), x_1 - x_2 \rangle \le -\alpha(\|u - v\|)\|x_1 - x_2\|, \\ \langle \nabla f(u) - \nabla f(v), y_1 - y_2 \rangle = \langle \nabla^2 f(v + \theta(u - v))(u - v), y_1 - y_2 \rangle \ge \beta(\|u - v\|)\|y_1 - y_2\|.\end{gathered} \tag{2.3}$$

The conclusion of the corollary follows immediately from Theorem 2.1. □

3. The main theorems

Consider the boundary value problem

$$u'' + g(t,u) = e(t), \qquad u(\mathbf{0}) = a, \qquad u(\mathbf{2\pi}) = b, \tag{3.1}$$

where $u : [\mathbf{0},\mathbf{2\pi}] \to \mathbf{R}$, $g : [\mathbf{0},\mathbf{2\pi}]\times\mathbf{R} \to \mathbf{R}$ is a potential Carathéodory function, $e : [\mathbf{0},\mathbf{2\pi}] \to \mathbf{R}$ is a given function in $L^2[\mathbf{0},\mathbf{2\pi}]$.

Let $u(t) = v(t) + \boldsymbol{\omega}(t)$, $\boldsymbol{\omega}(t) = [a(\mathbf{2\pi} - t) + bt]/\mathbf{2\pi}$, $t \in [\mathbf{0},\mathbf{2\pi}]$, then (3.1) may be written in the form of

$$v'' + g^*(t,v) = e(t), \qquad v(\mathbf{0}) = v(\mathbf{2\pi}) = \mathbf{0}, \tag{3.2}$$

where $g^*(t,v) = g(t,v+\boldsymbol{\omega})$. Clearly, $g^*(t,v)$ is a potential Carathéodory function, and if v_0 is a solution of (3.2), then $u_0 = v_0 + \boldsymbol{\omega}$ will be a solution of (3.1).

It is well known that $L^2[\mathbf{0},\mathbf{2\pi}]$ is a Hilbert space with inner product:

$$(u,v) = \int_0^{2\pi} u(t)v(t)\mathrm{d}t \quad (u,v \in L^2[\mathbf{0},\mathbf{2\pi}]), \tag{3.3}$$

and norm $\|u\| = \sqrt{(u,u)} = \left(\int_0^{2\pi} u^2(t)\mathrm{d}t\right)^{1/2}$, respectively. The system of trigonometrical functions,

$$\frac{1}{\sqrt{2\pi}};\ \frac{\mathbf{1}}{\sqrt{\pi}}\mathbf{cos}x,\ \frac{\mathbf{1}}{\sqrt{\pi}}\mathbf{sin}x;\ \frac{\mathbf{1}}{\sqrt{\pi}}\mathbf{cos}2x,\ \frac{\mathbf{1}}{\sqrt{\pi}}\mathbf{sin}2x;\ldots;\ \frac{\mathbf{1}}{\sqrt{\pi}}\mathbf{cos}nx,\ \frac{\mathbf{1}}{\sqrt{\pi}}\mathbf{sin}nx;\ldots, \tag{3.4}$$

is a system of orthonormal functions in $L^2[\mathbf{0},\mathbf{2\pi}]$. Each $v \in L^2[\mathbf{0},\mathbf{2\pi}]$ can be written as the Fourier series

$$v(t) = \left(v, \frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}}\right)\frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}} + \sum_{n=1}^{\infty}\left[\left(v, \frac{\mathbf{cos}nt}{\sqrt{\boldsymbol{\pi}}}\right)\frac{\mathbf{cos}nt}{\sqrt{\boldsymbol{\pi}}} + \left(v, \frac{\mathbf{sin}nt}{\sqrt{\boldsymbol{\pi}}}\right)\frac{\mathbf{sin}nt}{\sqrt{\boldsymbol{\pi}}}\right]. \tag{3.5}$$

Define the linear operator $L = -d^2/dt^2 : D^*(L) \subset L^2[\mathbf{0},\mathbf{2\pi}] \to L^2[\mathbf{0},\mathbf{2\pi}]$,

$$\begin{aligned}
D^*(L) = \Bigg\{ & v \in L^2[\mathbf{0},\mathbf{2\pi}] \mid v(t) = \left(v, \frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}}\right)\frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}} + \sum_{n=1}^{\infty}\left[\left(v, \frac{\mathbf{cos}\,nt}{\sqrt{\boldsymbol{\pi}}}\right)\frac{\mathbf{cos}\,nt}{\sqrt{\boldsymbol{\pi}}} + \left(v, \frac{\mathbf{sin}\,nt}{\sqrt{\boldsymbol{\pi}}}\right)\frac{\mathbf{sin}\,nt}{\sqrt{\boldsymbol{\pi}}}\right], \\
& \sum_{n=1}^{\infty}(n^2+\mathbf{1})\left[\left|\left(v, \frac{\mathbf{cos}\,nt}{\sqrt{\boldsymbol{\pi}}}\right)\right|^2 + \left|\left(v, \frac{\mathbf{sin}\,nt}{\sqrt{\boldsymbol{\pi}}}\right)\right|^2\right] < \infty,\ v(\mathbf{0}) = v(\mathbf{2\pi}) = \mathbf{0}\Bigg\} \subset L^2[\mathbf{0},\mathbf{2\pi}], \\
Lv = {} & \sum_{n=1}^{\infty} n^2\left[\left(v, \frac{\mathbf{cos}\,nt}{\sqrt{\boldsymbol{\pi}}}\right)\frac{\mathbf{cos}\,nt}{\sqrt{\boldsymbol{\pi}}} + \left(v, \frac{\mathbf{sin}\,nt}{\sqrt{\boldsymbol{\pi}}}\right)\frac{\mathbf{sin}\,nt}{\sqrt{\boldsymbol{\pi}}}\right], \\
\boldsymbol{\sigma}(L) = {} & \{n^2 \mid n \in \mathbf{N}\}.
\end{aligned} \tag{3.6}$$

Denote

$$D(L) = \left\{ u \mid u(t) = v(t) + \boldsymbol{\omega}(t),\ v \in D^*(L),\ \boldsymbol{\omega}(t) = \frac{\left[a(\mathbf{2\pi} - t) + bt\right]}{\mathbf{2\pi}},\ t \in [\mathbf{0}, \mathbf{2\pi}] \right\}. \tag{3.7}$$

Clearly, $L = -d^2/dt^2$ is a self-adjoint operator, and $D^*(L)$ is a Hilbert space for the inner product:

$$\langle u, v \rangle = \int_0^{2\pi} \left[u'(t)v'(t) + u(t)v(t)\right] \mathrm{d}t, \tag{3.8}$$

$u, v \in L^2[\mathbf{0}, \mathbf{2\pi}]$, the norm induced by this inner product is

$$\|v\|^2 = \int_0^{2\pi} \left[v'^2(t) + v^2(t)\right] \mathrm{d}t. \tag{3.9}$$

Note that $D(L)$ is not a space.

Since $g(t,u)$ in (3.1), and hence $g^*(t,v)$ in (3.2), is a potential Carathéodory function, there exists a function $G(t,u)$ such that

$$g(t,u) = \frac{\partial G(t,u)}{\partial u}, \tag{3.10}$$

and hence

$$g^*(t,v) = \frac{\partial G(t, v + \boldsymbol{\omega})}{\partial u}, \tag{3.11}$$

and the mapping g, and hence g^*, generates a Nemytskii operator $N : D(L) \subset L^2[\mathbf{0}, \mathbf{2\pi}] \to L^2[\mathbf{0}, \mathbf{2\pi}]$ by

$$N(u) = g\left(t, u(t)\right), \tag{3.12}$$

and hence

$$N^*(v) = N(v + \boldsymbol{\omega}) = g\left(t, v(t) + \boldsymbol{\omega}(t)\right) = g^*\left(t, v(t)\right). \tag{3.13}$$

Define the functional $f : D(L) \subset L^2[\mathbf{0}, \mathbf{2\pi}] \to \mathbf{R}$ by

$$f(u) = \frac{1}{2}\langle Lu, u \rangle - G(t,u) + e(t)u, \tag{3.14}$$

where G satisfies (3.10) and $e(t)$ is in (3.1). We have

$$f^*(v) = \frac{1}{2}\langle Lv, v \rangle - G^*(t,v) + e(t)v + e(t)\boldsymbol{\omega}. \tag{3.15}$$

It is easy to see that

$$\begin{aligned} \nabla f(v) &= Lu - N(u) + e(t), \\ \nabla f^*(v) &= Lv - N^*(v) + e(t), \end{aligned} \tag{3.16}$$

where $N^*(v) = g^*(t, v(t))$. Clearly, $v_0 \in D^*(L)$ is a critical point of f if and only if v_0 is a solution of the equation $(L - N^*)v = -e(t)$ and hence a solution of (3.2) and thus $u_0 = v_0 + \boldsymbol{\omega} = v_0 + [a(\mathbf{2\pi} - t) + bt]/\mathbf{2\pi},\ t \in [\mathbf{0}, \mathbf{2\pi}]$ is a solution of (3.1).

Now, we suppose that there exists a real-bounded mapping $b(t, u)\ \ (u \in D(L))$ such that

$$g(t, u_2) - g(t, u_1) = b(t, u_1 + \boldsymbol{\tau}(u_2 - u_1))(u_2 - u_1), \quad \boldsymbol{\tau} \in [\mathbf{0}, \mathbf{1}],\ u_1, u_2 \in D(L). \tag{3.17}$$

For $u_1, u_2 \in D(L),\ v_1, v_2 \in D^*(L)$, let

$$b(t, u) = b(t, v + \boldsymbol{\omega}) = b^*(t, v). \tag{3.18}$$

Since

$$\begin{aligned} g^*(t, v_2) - g^*(t, v_1) &= g(t, v_2 + \boldsymbol{\omega}) - g(t, v_1 + \boldsymbol{\omega}) \\ &= b(t, v_1 + \boldsymbol{\omega} + \boldsymbol{\tau}(v_2 - v_1))(v_2 - v_1) \\ &= b(t, v_1 + \boldsymbol{\tau}(v_2 - v_1) + \boldsymbol{\omega})(v_2 - v_1) \\ &= b^*(t, v_1 + \boldsymbol{\tau}(v_2 - v_1))(v_2 - v_1), \end{aligned} \tag{3.19}$$

equation (3.17) is equivalent to

$$g^*(t, v_2) - g^*(t, v_1) = b^*(t, v_1 + \boldsymbol{\tau}(v_2 - v_1))(v_2 - v_1), \quad \boldsymbol{\tau} \in [\mathbf{0}, \mathbf{1}],\ v_1, v_2 \in D^*(L). \tag{3.20}$$

Suppose that for $v \in D^*(L)$,

$$n^2 < b^*(t, v) < (n + \mathbf{1})^2 \quad (n \in \mathbf{N}), \tag{3.21}$$

and for $v_1, v_2, v \in D^*(L)$, define

$$\boldsymbol{\alpha}^*(\|v_1 - v_2\|) = \min_{\|v\| \le \|v_1 - v_2\|} \min_{n \in \mathbf{N}} \left\{ (n + \mathbf{1})^2 - \max_{0 \le t \le 2\pi} b^*(t, v) > 0,\ \min_{1 \le t \le 2\pi} b^*(t, v) - n^2 > \mathbf{0} \right\}, \tag{3.22}$$

which is equivalent to

$$\boldsymbol{\alpha}(\|u_1 - u_2\|) = \min_{\|u\| \le \|u_1 - u_2\|} \min_{n \in \mathbf{N}} \left\{ (n + \mathbf{1})^2 - \max_{0 \le t \le 2\pi} b(t, v) > 0,\ \min_{0 \le t \le 2\pi} b(t, v) - n^2 > \mathbf{0} \right\}. \tag{3.23}$$

Since $L = -d^2/dt^2$ is a self-adjoint operator, it possesses spectral resolution

$$L = \int_{-\infty}^{+\infty} \boldsymbol{\lambda} \mathrm{d}E_{\boldsymbol{\lambda}}, \quad \boldsymbol{\lambda} \in \boldsymbol{\sigma}(L), \tag{3.24}$$

with a right continuous spectral family $\{E_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathbf{R}\}$; and we let

$$E(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} \mathrm{d}E_{\boldsymbol{\lambda}} \tag{3.25}$$

for all $\alpha, \beta \in \rho(L) \cap \{\pm\infty\}$ with $\alpha < \beta$. Then, the operator $L - b^*(t,v)I$ has the spectral resolution:

$$L - b^*(t,v)I = \int_{-\infty}^{+\infty} (\lambda - b^*(t,v))\mathrm{d}E_\lambda, \quad v \in D^*(L), \tag{3.26}$$

where I is an identity operator.

Define X and Y by

$$X = E(-\infty, b^*(t,v))D^*(L), \qquad Y = E(b^*(t,v), +\infty)D^*(L). \tag{3.27}$$

By (3.21), we have

$$E(-\infty, b^*(t,v)) = I - E(b^*(t,v), +\infty), \quad v \in D^*(L), \tag{3.28}$$

and hence

$$D^*(L) = X \oplus Y, \quad X \text{ and } Y \text{ are orthogonal.} \tag{3.29}$$

We need to prove a lemma before presenting our main theorem.

Lemma 3.1. *Suppose that* $g^* : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ *in* (3.2) *satisfies* (3.20), $b^*(t,v)$ $(v \in D^*(L))$ *commutes with the linear operator* $L = -d^2/dt^2$ *and satisfies* (3.21), $\alpha^*(s)$ *is a continuous function defined in* (3.22). *Then, for* $v_1 = x_1 + y_1 \in D^*(L)$, $x_1 \in X$, $y_1 \in Y$, $v_2 = x_2 + y_2 \in D^*(L)$, $x_2 \in X$, $y_2 \in Y$, $\tau \in [0,1]$, $z = v_1 + \tau(v_2 - v_1) \in D^*(L)$,

$$\begin{gathered} \langle (L - b^*(t,z)I)(x_1 - x_2), x_1 - x_2 \rangle \le -\alpha^*(\|v_1 - v_2\|)\|x_1 - x_2\|^2, \\ \langle (L - b^*(t,z)I)(y_1 - y_2), y_1 - y_2 \rangle \ge \alpha^*(\|v_1 - v_2\|)\|y_1 - y_2\|^2. \end{gathered} \tag{3.30}$$

Proof. For $z = v_1 + \tau(v_2 - v_1) \in D^*(L)$, $v_1 \in D^*(L)$, $v_2 \in D^*(L)$, $v \in D^*(L)$, let

$$x = E(-\infty, b^*(t,z))v, \qquad y = E(b^*(t,z), +\infty)v. \tag{3.31}$$

Note that $b^*(t,z)$ $(z \in D^*(L))$ commutes with the linear operator L and

$$\begin{aligned} (L - b^*(t,z)I)(x_1 - x_2) &= \int_{-\infty}^{+\infty} (\lambda - b^*(t,z))\mathrm{d}E_\lambda \circ E(-\infty, b^*(t,z))(v_1 - v_2) \\ &= \int_{-\infty}^{b^*(t,z)} (\lambda - b^*(t,z))\mathrm{d}E_\lambda(v_1 - v_2), \\ (L - b^*(t,z)I)(y_1 - y_2) &= \int_{-\infty}^{+\infty} (\lambda - b^*(t,z))\mathrm{d}E_\lambda \circ E(b^*(t,z), +\infty)(v_1 - v_2) \\ &= \int_{b^*(t,z)}^{+\infty} (\lambda - b^*(t,z))\mathrm{d}E_\lambda(v_1 - v_2). \end{aligned} \tag{3.32}$$

By (3.22),

$$\begin{aligned}\langle (L - b^*(t,z)I)(x_1 - x_2), x_1 - x_2\rangle &= \int_{-\infty}^{b^*(t,z)} (\lambda - b^*(t,z))\,\mathrm{d}\|E_\lambda(v_1 - v_2)\|^2 \\ &\le -\alpha^*(\|v_1 - v_2\|)\int_{-\infty}^{b^*(t,z)} \mathrm{d}\|E_\lambda(v_1 - v_2)\|^2 \\ &= -\alpha^*(\|v_1 - v_2\|)\|x_1 - x_2\|^2, \quad v_1, v_2 \in D^*(L),\ x_1, x_2 \in X, \\ \langle (L - b^*(t,z)I)(y_1 - y_2), y_1 - y_2\rangle &= \int_{b^*(t,z)}^{+\infty} (\lambda - b^*(t,z))\,\mathrm{d}\|E_\lambda(v_1 - v_2)\|^2 \\ &\ge \alpha^*(\|v_1 - v_2\|)\int_{b^*(t,z)}^{+\infty} \mathrm{d}\|E_\lambda(v_1 - v_2)\|^2 \\ &= \alpha^*(\|v_1 - v_2\|)\|y_1 - y_2\|^2, \quad v_1, v_2 \in D^*(L),\ y_1, y_2 \in Y.\end{aligned} \tag{3.33}$$

□

Now, we show our main theorem dealing with (3.1).

Theorem 3.2. *Let* $g : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ *be a potential Carathéodory function satisfying* (3.17). *Suppose that* $b(t,u)$ $(u \in D(L),\ t \in [0, 2\pi])$ *commutes with the linear operator* $L = -d^2/dt^2$ *and satisfies*

$$n^2 < b(t,u) < (n+1)^2, \quad n \in \mathbf{N},\ u \in D(L), \tag{3.34}$$

and the continuous function $\alpha(s)$ *defined by* (3.23) *satisfies the conditions*

$$\alpha : [0, +\infty) \longrightarrow (0, +\infty), \qquad s \cdot \alpha(s) \longrightarrow +\infty \quad \text{as } s \longrightarrow \infty. \tag{3.35}$$

Then, (3.1) *has a unique solution* $u_0 \in D(L)$ *such that*

$$\nabla f(u_0) = 0, \qquad f(u_0) = \max_{x\in X}\min_{y\in Y} f(x + y + \omega) = \min_{y\in Y}\max_{x\in X} f(x + y + \omega), \tag{3.36}$$

where f *is a functional defined in* (3.14) *and* $\omega(t) = [a(2\pi - t) + bt]/2\pi$, $t \in [0, 2\pi]$.

Proof. For $v_1 = x_1 + y_1 \in D^*(L)$, $v_2 = x_2 + y_2 \in D^*(L)$, $x_1, x_2 \in X$, $y_1, y_2 \in Y$, by Lemma 3.1, we have

$$\begin{aligned}\langle \nabla f^*(v_2) - \nabla f^*(v_1), x_2 - x_1\rangle &= \langle Lv_2 - N^*(v_2) + e(t) - Lv_1 + N^*(v_1) - e(t), x_2 - x_1\rangle \\ &= \langle L(v_2 - v_1) - (g^*(t, v_2(t)) - g^*(t, v_1(t)), x_2 - x_1\rangle \\ &= \langle L(v_2 - v_1) - b^*(t, v_1 + \tau(v_2 - v_1))(v_2 - v_1), x_2 - x_1\rangle \\ &= \langle [L - b^*(t, v_1 + \tau(v_2 - v_1))I](v_2 - v_1), x_2 - x_1\rangle \\ &= \langle [L - b^*(t, v_1 + \tau(v_2 - v_1))I](x_2 - x_1), x_2 - x_1\rangle \\ &\le -\alpha^*(\|v_2 - v_1\|)\|x_1 - x_2\|^2, \quad \tau \in [0, 1],\end{aligned}$$

$$\begin{aligned}
\langle \nabla f^*(v_2) - \nabla f^*(v_1), y_2 - y_1 \rangle &= \langle L v_2 - N^*(v_2) + e(t) - L v_1 + N^*(v_1) - e(t), y_2 - y_1 \rangle \\
&= \langle L(v_2 - v_1) - (g^*(t, v_2(t)) - g^*(t, v_1(t))), y_2 - y_1 \rangle \\
&= \langle [L - b^*(t, v_1 + \boldsymbol{\tau}(v_2 - v_1)) I](y_2 - y_1), y_2 - y_1 \rangle \\
&\geq \boldsymbol{\alpha}^*(\|v_2 - v_1\|) \|y_1 - y_2\|^2, \quad \boldsymbol{\tau} \in [\mathbf{0}, \mathbf{1}].
\end{aligned} \tag{3.37}$$

Employing Theorem 2.1, we can know that there exists a unique $v_0 \in D^*(L)$ such that $\nabla f^*(v_0) = \mathbf{0}$, where $v_0 \in D^*(L)$ is a solution of (3.2) and this means that (3.1) has a unique solution $u_0 = v_0 + \boldsymbol{\omega} \in D(L)$ such that

$$\begin{aligned}
\nabla f^*(v_0) &= \nabla f(v_0 + \boldsymbol{\omega}) = \nabla f(u_0) = \mathbf{0}, \\
f(u_0) = f(v_0 + \boldsymbol{\omega}) = f^*(v_0) &= \max_{x \in X} \min_{y \in Y} f^*(x + y) = \min_{y \in Y} \max_{x \in X} f^*(x + y) \\
&= \max_{x \in X} \min_{y \in Y} f(x + y + \boldsymbol{\omega}) = \min_{y \in Y} \max_{x \in X} f(x + y + \boldsymbol{\omega}),
\end{aligned} \tag{3.38}$$

where f is a functional defined in (3.14) and $\boldsymbol{\omega}(t) = [a(\mathbf{2\pi} - t) + bt]/\mathbf{2\pi}$, $t \in [\mathbf{0}, \mathbf{2\pi}]$.

If the perturbation function $G(t, u)$ in (3.10) is a second Gâteaux differential, (3.17), (3.34), and (3.23) become

$$g(t, u_2) - g(t, u_1) = g_u'(t, u_1 + \boldsymbol{\tau}(u_2 - u_1))(u_2 - u_1), \quad \boldsymbol{\tau} \in (\mathbf{0}, \mathbf{1}),\ u_1, u_2 \in D(L), \tag{3.39}$$

$$n^2 < g_u'(t, u) < (n + \mathbf{1})^2, \quad n \in \mathbf{N},\ u \in D(L), \tag{3.40}$$

$$\boldsymbol{\alpha}(\|u_1 - u_2\|) = \min_{\|u\| \leq \|u_1 - u_2\|} \min_{n \in \mathbf{N}} \left\{ (n + \mathbf{1})^2 - \max_{0 \leq t \leq 2\pi} g_u'(t, u) > \mathbf{0},\ \min_{0 \leq t \leq 2\pi} g_u'(t, u) - n^2 > \mathbf{0} \right\}, \tag{3.41}$$

respectively. By (3.30) in Lemma 3.1, we have

$$\begin{aligned}
\langle (L - g_u^*(t, v) I)(x_2 - x_1), x_2 - x_1 \rangle &\leq -\boldsymbol{\alpha}^*(\|v_1 - v_2\|) \|x_1 - x_2\|^2, \\
\langle (L - g_u^*(t, v) I)(y_2 - y_1), y_2 - y_1 \rangle &\geq \boldsymbol{\alpha}^*(\|v_1 - v_2\|) \|y_1 - y_2\|^2,
\end{aligned} \tag{3.42}$$

where $v = v_1 + \boldsymbol{\tau}(v_2 - v_1) \in D^*(L)$, $v_1, v_2 \in D^*(L)$, $\boldsymbol{\tau} \in (\mathbf{0}, \mathbf{1})$, $x_1, x_2 \in X$, $y_1, y_2 \in Y$. □

We then have the following corollary of Theorem 3.2.

Corollary 3.3. *Let $g : [\mathbf{0}, \mathbf{2\pi}] \times \mathbf{R} \to \mathbf{R}$ be a potential Carathéodory function with first Gâteaux derivative g_u' satisfying* (3.39) *and* (3.40) *and $g_u'(t, u)$ ($u \in D(L)$, $t \in [\mathbf{0}, \mathbf{2\pi}]$) commutes with the linear operator $L = -d^2/dt^2$. If the continuous function $\boldsymbol{\alpha}(s)$ defined by* (3.41) *satisfies* (3.35), *then* (3.1) *has a unique solution $u_0 \in D(L)$ and*

$$f(u_0) = \max_{x \in X} \min_{y \in Y} f(x + y + \boldsymbol{\omega}) = \min_{y \in Y} \max_{x \in X} f(x + y + \boldsymbol{\omega}), \tag{3.43}$$

where f is a functional defined in (3.14) *and $\boldsymbol{\omega}(t) = [a(\mathbf{2\pi} - t) + bt]/\mathbf{2\pi}$, $t \in [\mathbf{0}, \mathbf{2\pi}]$.*

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