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Research Article

Robust Monotone Iterates for Nonlinear Singularly Perturbed Boundary Value Problems

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This paper is concerned with solving nonlinear singularly perturbed boundary value problems. Robust monotone iterates for solving nonlinear difference scheme are constructed. Uniform convergence of the monotone methods is investigated, and convergence rates are estimated. Numerical experiments complement the theoretical results.

1. Introduction

We are interested in numerical solving of two nonlinear singularly perturbed problems of elliptic and parabolic types.

The first one is the elliptic problem

$$\begin{aligned} &-\mu^2 u'' + f(x,u) = 0, \quad x \in \omega = (0,1),\ u(0) = 0,\ u(1) = 0, \\ &f_u \geq c_* = \text{const} > 0, \quad (x,u) \in \overline{\omega} \times (-\infty,\infty), \quad f_u = \partial f/\partial u, \end{aligned} \tag{1.1}$$

where μ is a positive parameter, and f is sufficiently smooth function. For $\mu \ll 1$ this problem is singularly perturbed, and the solution has boundary layers near $x = 0$ and $x = 1$ (see [1] for details).

The second problem is the one-dimensional parabolic problem

$$\begin{gathered} -\mu^2 u_{xx} + u_t + f(x,t,u) = 0, \qquad (x,t) \in Q = \omega \times (0,T], \quad \omega = (0,1), \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = u^0(x), \quad x \in \overline{\omega}, \\ f_u \geq 0, \quad (x,t,u) \in \overline{Q} \times (-\infty,\infty), \end{gathered} \tag{1.2}$$

where μ is a positive parameter. Under suitable continuity and compatibility conditions on the data, a unique solution of this problem exists. For $\mu \ll 1$ problem (1.2) is singularly perturbed and has boundary layers near the lateral boundary of $\overline{Q}$ (see [2] for details).

In the study of numerical methods for nonlinear singularly perturbed problems, the two major points to be developed are: (i) constructing robust difference schemes (this means that unlike classical schemes, the error does not increase to infinity, but rather remains bounded, as the small parameter approaches zero); (ii) obtaining reliable and efficient computing algorithms for solving nonlinear discrete problems.

Our goal is to construct a μ-uniform numerical method for solving problem (1.1), that is, a numerical method which generates μ-uniformly convergent numerical approximations to the solution. We use a numerical method based on the classical difference scheme and the piecewise uniform mesh of Shishkin-type [3]. For solving problem (1.2), we use the implicit difference scheme based on the piecewise uniform mesh in the x-direction, which converges μ-uniformly [4].

A major point about the nonlinear difference schemes is to obtain reliable and efficient computational methods for computing the solution. The reliability of iterative techniques for solving nonlinear difference schemes can be essentially improved by using component-wise monotone globally convergent iterations. Such methods can be controlled every time. A fruitful method for the treatment of these nonlinear schemes is the method of upper and lower solutions and its associated monotone iterations [5]. Since an initial iteration in the monotone iterative method is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution, this method simplifies the search for the initial iteration as is often required in the Newton method. In the context of solving systems of nonlinear equations, the monotone iterative method belongs to the class of methods based on convergence under partial ordering (see [5, Chapter 13] for details).

The purpose of this paper is to construct μ-uniformly convergent monotone iterative methods for solving μ-uniformly convergent nonlinear difference schemes.

The structure of the paper is as follows. In Section 2, we prove that the classical difference scheme on the piecewise uniform mesh converges μ-uniformly to the solution of problem (1.1). A robust monotone iterative method for solving the nonlinear difference scheme is constructed. In Section 3, we construct a robust monotone iterative method for solving problem (1.2). In the final Section 4, we present numerical experiments which complement the theoretical results.

2. The Elliptic Problem

The following lemma from [1] contains necessary estimates of the solution to (1.1).

Lemma 2.1. *If $u(x) \in C^0(\overline{\omega}) \cap C^2(\omega)$ is the solution to (1.1), the following estimates hold:*

$$\max_{x\in\overline{\omega}}|u(x)| \le c_*^{-1}\max_{x\in\overline{\omega}}\left|f(x,0)\right|, \qquad \left|u'(x)\right| \le C\left[1+\mu^{-1}\Pi(x)\right],$$
$$\Pi(x) = \exp\left(-\frac{\sqrt{c_*}}{\mu}\right) + \exp\left(-\frac{\sqrt{c_*}(1-x)}{\mu}\right), \tag{2.1}$$

where constant C is independent of μ.

For $\mu \ll 1$, the boundary layers appear near $x = 0$ and $x = 1$.

2.1. The Nonlinear Difference Scheme

Introduce a nonuniform mesh $\overline{\omega}^h$

$$\overline{\omega}^h = \{x_i,\ 0 \le i \le N;\ x_0 = 0,\ x_N = 1;\ h_i = x_{i+1} - x_i\}. \tag{2.2}$$

For solving (1.1), we use the classical difference scheme

$$\begin{gathered}\mathcal{L}^h v(x) + f(x,v) = 0, \quad x \in \omega^h, \quad v(0) = 0, \quad v(1) = 0, \\ \mathcal{L}^h v_i = -\mu^2 (\hbar_i)^{-1}\Big[(v_{i+1} - v_i)(h_i)^{-1} - (v_i - v_{i-1})(h_{i-1})^{-1}\Big],\end{gathered} \tag{2.3}$$

where $v_i = v(x_i)$ and $\hbar_i = (h_{i-1} + h_i)/2$. We introduce the linear version of this problem

$$\Big(\mathcal{L}^h + c\Big) w(x) = f_0(x), \quad x \in \omega^h, \quad w(0) = 0, \quad w(1) = 0, \tag{2.4}$$

where $c(x) \ge 0$. We now formulate a discrete maximum principle for the difference operator $\mathcal{L}^h + c$ and give an estimate of the solution to (2.4).

Lemma 2.2. (i) *If a mesh function $w(x)$ satisfies the conditions*

$$\Big(\mathcal{L}^h + c\Big) w(x) \ge 0\ (\le 0), \quad x \in \omega^h, \quad w(0), w(1) \ge 0\ (\le 0), \tag{2.5}$$

then $w(x) \ge 0\ (\le 0)$, $x \in \overline{\omega}^h$.

(ii) *If $c(x) \ge c_* = \text{const} > 0$, then the following estimate of the solution to* (2.4) *holds true:*

$$\|w\|_{\overline{\omega}^h} \le \max \|f_0\|_{\omega^h} / c_*, \tag{2.6}$$

where $\|w\|_{\overline{\omega}^h} = \max_{x\in\overline{\omega}^h} |w(x)|$, $\|f_0\|_{\omega^h} = \max_{x\in\omega^h} |f_0(x)|$.

The proof of the lemma can be found in [6].

2.2. Uniform Convergence on the Piecewise Uniform Mesh

We employ a layer-adapted mesh of a piecewise uniform type [3]. The piecewise uniform mesh is formed in the following manner. We divide the interval $\overline{\omega} = [0,1]$ into three parts $[0,\varsigma]$, $[\varsigma, 1-\varsigma]$, and $[1-\varsigma, 1]$. Assuming that N is divisible by 4, in the parts $[0,\varsigma]$, $[1-\varsigma, 1]$ we use the uniform mesh with $N/4+1$ mesh points, and in the part $[\varsigma, 1-\varsigma]$ the uniform mesh with $N/2+1$ mesh points is in use. The transition points ς and $1-\varsigma$ are determined by

$$\varsigma = \min\left\{4^{-1}, \frac{\mu \ln N}{\sqrt{c_*}}\right\}. \tag{2.7}$$

This defines the piecewise uniform mesh. If the parameter μ is small enough, then the uniform mesh inside of the boundary layers with the step size $h_\mu = 4\varsigma N^{-1}$ is fine, and the uniform mesh outside of the boundary layers with the step size $h = 2(1-2\varsigma)N^{-1}$ is coarse, such that

$$N^{-1} < h < 2N^{-1}, \quad h_\mu = 4\mu\left(\sqrt{c_*}N\right)^{-1} \ln N. \tag{2.8}$$

In the following theorem, we give the convergence property of the difference scheme (2.3).

Theorem 2.3. *The difference scheme* (2.3) *on the piecewise uniform mesh* (2.8) *converges μ-uniformly to the solution of* (1.1)*:*

$$\max_{x\in\overline{\omega}^h} |v(x) - u(x)| \le CN^{-1} \ln N, \tag{2.9}$$

where constant C is independent of μ and N.

Proof. Using Green's function G_i of the differential operator $\mu^2 d^2/dx^2$ on $[x_i, x_{i+1}]$, we represent the exact solution $u(x)$ in the form

$$u(x) = u(x_i)\phi_i^I(x) + u(x_{i+1})\phi_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x,s) f(s,u) ds, \tag{2.10}$$

where the local Green function G_i is given by

$$\begin{gathered} G_i(x,s) = \frac{1}{\mu^2 w_i(s)} \begin{cases} \phi_i^I(s)\phi_i^{II}(x), & x \le s, \\ \phi_i^I(x)\phi_i^{II}(s), & x \ge s, \end{cases} \\ w_i(s) = \phi_i^{II}(s)\left[\phi_i^I(x)\right]'_{x=s} - \phi_i^I(s)\left[\phi_i^{II}(x)\right]'_{x=s}, \end{gathered} \tag{2.11}$$

and $\phi_i^I(x)$, $\phi_i^{II}(x)$ are defined by

$$\phi_i^I(x) = \frac{x_{i+1} - x}{h_i}, \quad \phi_i^{II}(x) = \frac{x - x_i}{h_i}, \quad x_i \le x \le x_{i+1}. \tag{2.12}$$

Equating the derivatives $du(x_i - 0)/dx$ and $du(x_i + 0)/dx$, we get the following integral-difference formula:

$$\mathcal{L}^h u(x_i) = \frac{1}{\hbar_i} \int_{x_{i-1}}^{x_i} \phi_{i-1}^{II}(s) f(s) ds + \frac{1}{\hbar_i} \int_{x_i}^{x_{i+1}} \phi_i^I(s) f(s) ds, \tag{2.13}$$

where here and below we suppress variable u in f. Representing $f(x)$ on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ in the forms

$$f(x) = f(x_i - 0) + \int_{x_i}^{x} \frac{df}{ds} ds, \qquad f(x) = f(x_i + 0) + \int_{x_i}^{x} \frac{df}{ds} ds, \tag{2.14}$$

the above integral-difference formula can be written as

$$\mathcal{L}^h u(x) + f(x, u) = \mathrm{Tr}(x), \quad x \in \omega^h, \tag{2.15}$$

where the truncation error $\mathrm{Tr}(x)$ of the exact solution $u(x)$ to (1.1) is defined by

$$\mathrm{Tr}(x_i) \equiv -\frac{1}{\hbar_i} \int_{x_{i-1}}^{x_i} \phi_{i-1}^{II}(s) \left(\int_{x_i}^{s} \frac{df}{d\xi} d\xi \right) ds - \frac{1}{\hbar_i} \int_{x_i}^{x_{i+1}} \phi_i^{I}(s) \left(\int_{x_i}^{s} \frac{df}{d\xi} d\xi \right) ds. \tag{2.16}$$

From here, it follows that

$$|\mathrm{Tr}(x_i)| \leq \frac{1}{\hbar_i} \int_{x_{i-1}}^{x_i} \phi_{i-1}^{II}(s) \left(\int_{x_{i-1}}^{x_i} \left| \frac{df}{d\xi} \right| d\xi \right) ds + \frac{1}{\hbar_i} \int_{x_i}^{x_{i+1}} \phi_i^{I}(s) \left(\int_{x_i}^{x_{i+1}} \left| \frac{df}{d\xi} \right| d\xi \right) ds. \tag{2.17}$$

From Lemma 2.1, the following estimate on df/dx holds:

$$\left| \frac{df}{dx} \right| \leq C\left[1 + \mu^{-1}\Pi(x)\right]. \tag{2.18}$$

We estimate the truncation error Tr in (2.17) on the interval $(0, 1/2]$. Consider the following three cases: $x_i \in (0, \varsigma)$, $x_i = \varsigma$, and $x_i \in (\varsigma, 1/2]$. If $x_i \in (0, \varsigma)$, then $h_{i-1} = h_i = h_\mu$, and taking into account that $\Pi(x) < 2$ in (2.18), we have

$$|\mathrm{Tr}(x_i)| \leq C h_\mu \left(1 + 2\mu^{-1}\right), \quad x_i \in (0, \varsigma), \tag{2.19}$$

where here and throughout C denotes a generic constant that is independent of μ and N. If $x_i = \varsigma$, then $h_{i-1} = h_\mu$, $h_i = h$. Taking into account that $\varsigma = \mu \ln N / \sqrt{c_*}$, $\Pi(x) < 2$, and $\Pi(x) \leq 2\exp(-\sqrt{c_*}x/\mu)$, we have

$$\begin{aligned} |\mathrm{Tr}(\varsigma)| &\leq \frac{C}{h_\mu + h} \left[h_\mu^2 \left(1 + 2\mu^{-1}\right) + h^2 + 2\left(\sqrt{c_*}N\right)^{-1} \right] \\ &\leq C\left[h_\mu \left(1 + 2\mu^{-1}\right) + h + 2\left(\sqrt{c_*}N\right)^{-1} \right]. \end{aligned} \tag{2.20}$$

If $x_i \in (\varsigma, 1/2]$, then $h_{i-1} = h_i = h$, and we have

$$|\mathrm{Tr}(x_i)| \leq C\left[h + 2\left(\sqrt{c_*}N\right)^{-1} \right], \quad x_i \in (\varsigma, 1/2]. \tag{2.21}$$

Thus,

$$|\mathrm{Tr}(x_i)| \le C\Big[h_\mu\Big(1+2\mu^{-1}\Big)+h+2\big(\sqrt{c_*}N\big)^{-1}\Big], \quad x_i \in (0,1/2]. \tag{2.22}$$

In a similar way we can estimate Tr on $[1/2,1)$ and conclude that

$$|\mathrm{Tr}(x_i)| \le C\Big[h_\mu\Big(1+2\mu^{-1}\Big)+h+2\big(\sqrt{c_*}N\big)^{-1}\Big], \quad x_i \in \omega^h. \tag{2.23}$$

From here and (2.8), we conclude that

$$\max_{x_i\in\omega^h}|\mathrm{Tr}(x_i)| \le CN^{-1}\ln N. \tag{2.24}$$

From (2.3), (2.15), by the mean-value theorem, we conclude that $w=v-u$ satisfies the difference problem

$$\mathcal{L}^h w(x) + f_u w(x) = -\mathrm{Tr}(x), \quad x \in \omega^h, \quad w(0)=w(1)=0. \tag{2.25}$$

Using the assumption on f_u from (1.1) and (2.24), by (2.6), we prove the theorem. □

2.3. The Monotone Iterative Method

In this section, we construct an iterative method for solving the nonlinear difference scheme (2.3) which possesses monotone convergence.

Additionally, we assume that $f(x,u)$ from (1.1) satisfies the two-sided constraint

$$0 < c_* \le f_u \le c^*, \quad c_*, c^* = \mathrm{const}. \tag{2.26}$$

The iterative method is constructed in the following way. Choose an initial mesh function $v^{(0)}$, then the iterative sequence $\{v^{(n)}\}$, $n \ge 1$, is defined by the recurrence formulae

$$\begin{gathered}\Big(\mathcal{L}^h + c^*\Big)z^{(n)}(x) = -\mathcal{R}^h\Big(x, v^{(n-1)}\Big), \quad x \in \omega^h,\\ z^{(1)}(0) = -v^{(0)}(0), \quad z^{(1)}(1) = -v^{(0)}(1), \quad z^{(n)}(0)=z^{(n)}(1)=0, \quad n \ge 2,\\ v^{(n)}(x) = v^{(n-1)}(x) + z^{(n)}(x), \quad x \in \overline{\omega}^h,\\ \mathcal{R}^h\Big(x, v^{(n-1)}\Big) = \mathcal{L}^h v^{(n-1)}(x) + f\Big(x, v^{(n-1)}\Big),\end{gathered} \tag{2.27}$$

where $\mathcal{R}^h(x, v^{(n-1)})$ is the residual of the difference scheme (2.3) on $v^{(n-1)}$.

We say that $\overline{v}(x)$ is an upper solution of (2.3) if it satisfies the inequalities

$$\mathcal{L}^h\overline{v}(x) + f(x,\overline{v}) \ge 0, \quad x \in \omega^h, \quad \overline{v}(0), \overline{v}(1) \ge 0. \tag{2.28}$$

Similarly, $\underline{v}(x)$ is called a lower solution if it satisfies the reversed inequalities. Upper and lower solutions satisfy the inequality

$$\underline{v}(x) \leq \overline{v}(x), \quad x \in \overline{\omega}^h. \tag{2.29}$$

Indeed, by the definition of lower and upper solutions and the mean-value theorem, for $\delta v = \overline{v} - \underline{v}$ we have

$$\mathcal{L}^h \delta v + f_u(x)\delta v(x) \geq 0, \quad x \in \omega^h, \quad \delta v(x) \geq 0, \quad x = 0, 1, \tag{2.30}$$

where $f_u(x) = c_u[x, \underline{v}(x) + \vartheta(x)\delta v(x)]$, $0 < \vartheta(x) < 1$. In view of the maximum principle in Lemma 2.2, we conclude the required inequality.

The following theorem gives the monotone property of the iterative method (2.27).

Theorem 2.4. *Let* $\overline{v}^{(0)}$, $\underline{v}^{(0)}$ *be upper and lower solutions of* (2.3) *and* f *satisfy* (2.26)*. Then the upper sequence* $\{\overline{v}^{(n)}\}$ *generated by* (2.27) *converges monotonically from above to the unique solution* v *of* (2.3)*, the lower sequence* $\{\underline{v}^{(n)}\}$ *generated by* (2.27) *converges monotonically from below to* v*:*

$$\underline{v}^{(n)}(x) \leq \underline{v}^{(n+1)}(x) \leq v(x) \leq \overline{v}^{(n+1)}(x) \leq \overline{v}^{(n)}(x), \quad x \in \overline{\omega}^h, \tag{2.31}$$

and the sequences converge at the linear rate $q = 1 - c_*/c^*$.

Proof. We consider only the case of the upper sequence. If $\overline{v}^{(0)}$ is an upper solution, then from (2.27) we conclude that

$$\left(\mathcal{L}^h + c^*\right) z^{(1)}(x) \leq 0, \quad x \in \omega^h, \quad z^{(1)}(0), z^{(1)}(1) \leq 0. \tag{2.32}$$

From Lemma 2.2, by the maximum principle for the difference operator $\mathcal{L}^h + c^*$, it follows that $z^{(1)}(x) \leq 0$, $x \in \overline{\omega}^h$. Using the mean-value theorem and the equation for $z^{(1)}$, we represent $\mathcal{R}^h(x, v^{(1)})$ in the form

$$\mathcal{R}^h\left(x, v^{(1)}\right) = -\left(c^* - f_u^{(1)}(x)\right) z^{(1)}(x), \quad x \in \omega^h, \tag{2.33}$$

where $f_u^{(1)}(x) = f_u[x, \overline{v}^{(0)}(x) + \vartheta^{(1)}(x) z^{(1)}(x)]$, $0 < \vartheta^{(1)}(x) < 1$. Since the mesh function $z^{(1)}$ is nonpositive on ω^h and taking into account (2.26), we conclude that $\overline{v}^{(1)}$ is an upper solution. By induction on n, we obtain that $z^{(n)}(x) \leq 0$, $x \in \overline{\omega}^h$, $n \geq 1$, and prove that $\{\overline{v}^{(n)}\}$ is a monotonically decreasing sequence of upper solutions.

We now prove that the monotone sequence $\{\overline{v}^{(n)}\}$ converges to the solution of (2.3). Similar to (2.33), we obtain

$$\mathcal{R}\left(x, \overline{v}^{(n-1)}\right) = -\left(c^* - f_u^{(n-1)}(x)\right) z^{(n-1)}(x), \quad x \in \omega^h, \; n \geq 2, \tag{2.34}$$

and from (2.27), it follows that $z^{(n+1)}$ satisfies the difference equation

$$(\mathcal{L} + c^*)z^{(n)}(x) = \left(c^* - f_u^{(n-1)}(x)\right)z^{(n-1)}(x), \quad x \in \omega^h. \tag{2.35}$$

Using (2.26) and (2.6), we have

$$\|z^{(n)}\|_{\overline{\omega}^h} \leq q^{n-1}\|z^{(1)}\|_{\overline{\omega}^h}. \tag{2.36}$$

This proves the convergence of the upper sequence at the linear rate q. Now by linearity of the operator $\mathcal{L}^h$ and the continuity of f, we have also from (2.27) that the mesh function v defined by

$$v(x) = \lim_{n\to\infty} \overline{v}^{(n)}(x), \quad x \in \overline{\omega}^h, \tag{2.37}$$

is the exact solution to (2.3). The uniqueness of the solution to (2.3) follows from estimate (2.6). Indeed, if by contradiction, we assume that there exist two solutions v_1 and v_2 to (2.3), then by the mean-value theorem, the difference $\delta v = v_1 - v_2$ satisfies the difference problem

$$\mathcal{L}^h\delta v + f_u\delta v = 0, \quad x \in \omega^h, \quad \delta v(0) = \delta v(1) = 0. \tag{2.38}$$

By (2.6), $\delta v = 0$ which leads to the uniqueness of the solution to (2.3). This proves the theorem. □

Consider the following approach for constructing initial upper and lower solutions $\overline{v}^{(0)}$ and $\underline{v}^{(0)}$. Introduce the difference problems

$$\begin{gathered}\left(\mathcal{L}^h + c_*\right)v_\nu^{(0)} = \nu\left|f(x,0)\right|, \quad x \in \omega^h,\\ v_\nu^{(0)}(0) = v_\nu^{(0)}(1) = 0, \quad \nu = 1, -1,\end{gathered} \tag{2.39}$$

where c_* from (2.26). Then the functions $v_1^{(0)}, v_{-1}^{(0)}$ are upper and lower solutions, respectively. We check only that $v_1^{(0)}$ is an upper solution. From the maximum principle in Lemma 2.2, it follows that $v_1^{(0)} \geq 0$ on $\overline{\omega}^h$. Now using the difference equation for $v_1^{(0)}$ and the mean-value theorem, we have

$$\mathcal{R}^h\left(x, v_1^{(0)}\right) = f(x,0) + \left|f(x,0)\right| + \left(f_u^{(0)} - c_*\right)v_1^{(0)}. \tag{2.40}$$

Since $f_u^{(0)} \geq c_*$ and $v_1^{(0)}$ is nonnegative, we conclude that $v_1^{(0)}$ is an upper solution.

Theorem 2.5. *If the initial upper or lower solution* $v^{(0)}$ *is chosen in the form of* (2.39), *then the monotone iterative method* (2.27) *converges* μ*-uniformly to the solution* v *of the nonlinear difference scheme* (2.3)

$$\left\|v^{(n)} - v\right\|_{\overline{\omega}^h} \le \frac{c_0 q^n}{1-q}\|f(x,0)\|_{\overline{\omega}^h},$$
$$q = 1 - c_*/c^* < 1, \quad c_0 = (3c_* + c^*)/(c_* c^*). \tag{2.41}$$

Proof. From (2.27), (2.39), and the mean-value theorem, by (2.6),

$$\begin{aligned}\left\|z^{(1)}\right\|_{\overline{\omega}^h} &\le \frac{1}{c^*}\left\|\mathcal{L}^h v^{(0)}\right\|_{\omega^h} + \frac{1}{c^*}\left\|f\left(x, v^{(0)}\right)\right\|_{\omega^h} \\ &\le \frac{1}{c^*}\left(c_*\left\|v^{(0)}\right\|_{\omega^h} + \|f(x,0)\|_{\omega^h}\right) \\ &\quad + \frac{1}{c^*}\|f(x,0)\|_{\omega^h} + \left\|v^{(0)}\right\|_{\omega^h}.\end{aligned} \tag{2.42}$$

From here and estimating $v^{(0)}$ from (2.39) by (2.6),

$$\left\|v^{(0)}\right\|_{\overline{\omega}^h} \le \frac{1}{c_*}\|f(x,0)\|_{\omega^h}, \tag{2.43}$$

we conclude the estimate on $z^{(1)}$ in the form

$$\left\|z^{(1)}\right\|_{\overline{\omega}^h} \le c_0\|f(x,0)\|_{\overline{\omega}^h}, \tag{2.44}$$

where c_0 is defined in the theorem. From here and (2.36), we conclude that

$$\left\|z^{(n)}\right\|_{\overline{\omega}^h} \le c_0 q^{n-1}\|f(x,0)\|_{\overline{\omega}^h}. \tag{2.45}$$

Using this estimate, we have

$$\begin{aligned}\left\|v^{(n+k)} - v^{(n)}\right\|_{\overline{\omega}^h} &\le \sum_{i=n}^{n+k-1}\left\|v^{(i+1)} - v^{(i)}\right\|_{\overline{\omega}^h} = \sum_{i=n}^{n+k-1}\left\|z^{(i+1)}\right\|_{\overline{\omega}^h} \\ &\le \frac{q}{1-q}\left\|z^{(n)}\right\|_{\overline{\omega}^h} \le \frac{c_0 q^n}{1-q}\|f(x,0)\|_{\overline{\omega}^h}.\end{aligned} \tag{2.46}$$

Taking into account that $\lim v^{(n+k)} = v$ as $k \to \infty$, where v is the solution to (2.3), we conclude the theorem. □

From Theorems 2.3 and 2.5 we conclude the following theorem.

Theorem 2.6. *Suppose that the initial upper or lower solution $v^{(0)}$ is chosen in the form of* (2.39)*. Then the monotone iterative method* (2.27) *on the piecewise uniform mesh* (2.8) *converges μ-uniformly to the solution of problem* (1.1)*:*

$$\left\| v^{(n)} - u \right\|_{\overline{\omega}^h} \leq C\left(N^{-1} \ln N + q^n\right), \tag{2.47}$$

where $q = 1 - c_*/c^*$, and constant C is independent of μ and N.

3. The Parabolic Problem

3.1. The Nonlinear Difference Scheme

Introduce uniform mesh $\overline{\omega}^\tau$ on $[0, T]$

$$\overline{\omega}^\tau = \{t_k = k\tau,\ 0 \leq k \leq N_\tau,\ N_\tau \tau = T\}. \tag{3.1}$$

For approximation of problem (1.2), we use the implicit difference scheme

$$\begin{gathered} \mathcal{L}v(x,t) - \tau^{-1} v(x, t-\tau) = -f(x,t,v), \quad (x,t) \in \omega^h \times \overline{\omega}^\tau \setminus \{\emptyset\}, \\ v(0,t) = 0, \quad v(1,t) = 0, \quad v(x,0) = u^0(x), \quad x \in \overline{\omega}^h, \\ \mathcal{L} = \mathcal{L}^h + \tau^{-1}, \end{gathered} \tag{3.2}$$

where $\overline{\omega}^h$ and $\mathcal{L}^h$ are defined in (2.2) and (2.3), respectively. We introduce the linear version of problem (3.2)

$$\begin{gathered} (\mathcal{L} + c) w(x,t) = f_0(x,t), \quad x \in \omega^h, \\ w(0,t) = 0, \quad w(1,t) = 0, \quad c(x,t) \geq 0, \quad x \in \overline{\omega}^h. \end{gathered} \tag{3.3}$$

We now formulate a discrete maximum principle for the difference operator $\mathcal{L} + c$ and give an estimate of the solution to (3.3).

Lemma 3.1. (i) *If a mesh function $w(x,t)$ on a time level $t \in \overline{\omega}^\tau \setminus \{\emptyset\}$ satisfies the conditions*

$$(\mathcal{L} + c) w(x,t) \geq 0 \ (\leq 0), \quad x \in \omega^h, \quad w(0,t),\ w(1,t) \geq 0 \ \ (\leq 0), \tag{3.4}$$

then $w(x,t) \geq 0$ (≤ 0), $x \in \overline{\omega}^h$.

(ii) *If* $c(x,t) \geq c_* = const > 0$, *then the following estimate of the solution to* (3.3) *holds true:*

$$\|w(t)\|_{\overline{\omega}^h} \leq \|f_0(t)\|_{\omega^h} / \left(c_* + \tau^{-1}\right), \tag{3.5}$$

where $\|w(t)\|_{\overline{\omega}^h} = \max_{x\in\overline{\omega}^h}|w(x,t)|$, $\|f_0(t)\|_{\omega^h} = \max_{x\in\omega^h}|f_0(x,t)|$.

The proof of the lemma can be found in [6].

3.2. The Monotone Iterative Method

Assume that $f(x,t,u)$ from (3.2) satisfies the two-sided constraint

$$0 \leq f_u(x,t,u) \leq c^*, \quad c^* = \text{const}. \tag{3.6}$$

We consider the following iterative method for solving (3.2). Choose an initial mesh function $v^{(0)}(x,t)$. On each time level, the iterative sequence $\{v^{(n)}(x,t)\}$, $n = 1,\dots,n_*$, is defined by the recurrence formulae

$$\begin{gathered}
(\mathcal{L} + c^*)z^{(n)}(x,t) = -\mathcal{R}\left(x,t,v^{(n-1)}\right), \quad x \in \omega^h, \\
z^{(1)}(0,t) = -v^{(0)}(0,t), \quad z^{(1)}(1,t) = -v^{(0)}(1,t), \\
z^{(n)}(0,t) = z^{(n)}(1,t) = 0, \quad n \geq 2, \quad v^{(n)}(x,t) = v^{(n-1)}(x,t) + z^{(n)}(x,t), \\
\mathcal{R}\left(x,t,v^{(n-1)}\right) = \mathcal{L}v^{(n-1)}(x,t) + f\left(x,t,v^{(n-1)}\right) - \tau^{-1}v(x,t-\tau), \\
v(x,t) = v^{(n_*)}(x,t), \quad x \in \overline{\omega}^h, \quad v(x,0) = u^0(x), \quad x \in \overline{\omega}^h,
\end{gathered} \tag{3.7}$$

where $\mathcal{R}(x,t,v^{(n-1)})$ is the residual of the difference scheme (3.2) on $v^{(n-1)}$.

On a time level $t \in \overline{\omega}^\tau \setminus \{\emptyset\}$, we say that $\overline{v}(x,t)$ is an upper solution of (3.2) with respect to $v(x,t-\tau)$ if it satisfies the inequalities

$$\begin{gathered}
\mathcal{L}\overline{v}(x,t) + f(x,t,\overline{v}) - \tau^{-1}v(x,t-\tau) \geq 0, \quad x \in \omega^h, \\
\overline{v}(0,t) \geq 0, \quad \overline{v}(1,t) \geq 0.
\end{gathered} \tag{3.8}$$

Similarly, $\underline{v}(x,t)$ is called a lower solution if it satisfies all the reversed inequalities. Upper and lower solutions satisfy the inequality

$$\underline{v}(x,t) \leq \overline{v}(x,t), \quad p \in \overline{\omega}^h. \tag{3.9}$$

This result can be proved in a similar way as for the elliptic problem.

The following theorem gives the monotone property of the iterative method (3.7).

Theorem 3.2. *Assume that $f(x,t,u)$ satisfies* (3.6)*. Let $v(x,t-\tau)$ be given and $\overline{v}^{(0)}(x,t)$, $\underline{v}^{(0)}(x,t)$ be upper and lower solutions of* (3.2) *corresponding $v(x,t-\tau)$. Then the upper sequence $\{\overline{v}^{(n)}(x,t)\}$ generated by* (3.7) *converges monotonically from above to the unique solution $v(x,t)$ of the problem*

$$\mathcal{L}v(x,t) + f(x,t,v) - \tau^{-1}v(x,t-\tau) = 0, \quad x \in \partial\omega^h, \\ v(0,t) = 0, \quad v(1,t) = 0, \tag{3.10}$$

the lower sequence $\{\underline{v}^{(n)}(x,t)\}$ generated by (3.7) *converges monotonically from below to $v(x,t)$ and the following inequalities hold*

$$\underline{v}^{(n-1)}(x,t) \le \underline{v}^{(n)}(x,t) \le v(x,t) \le \overline{v}^{(n)}(x,t) \le \overline{v}^{(n-1)}(x,t), \quad x \in \overline{\omega}^h. \tag{3.11}$$

Proof. We consider only the case of the upper sequence, and the case of the lower sequence can be proved in a similar way.

If $\overline{v}^{(0)}$ is an upper solution, then from (3.7) we conclude that

$$\mathcal{L}z^{(1)}(x,t) \le 0, \quad x \in \omega^h, \quad z^{(1)}(0,t) \le 0, \quad z^{(1)}(1,t) \le 0. \tag{3.12}$$

From Lemma 3.1, it follows that

$$z^{(1)}(x,t) \le 0, \quad x \in \overline{\omega}^h, \tag{3.13}$$

and from (3.7), it follows that $v^{(1)}$ satisfies the boundary conditions.

Using the mean-value theorem and the equation for $z^{(1)}$ from (3.7), we represent $\mathcal{R}(x,t,v^{(1)})$ in the form

$$\mathcal{R}\left(x,t,v^{(1)}\right) = -\left(c^* - f_u^{(1)}(x,t)\right)z^{(1)}(x,t), \tag{3.14}$$

where $f_u^{(1)}(x,t) = f_u[x,t,\overline{v}^{(0)}(x,t) + \vartheta^{(1)}(x,t)z^{(1)}(x,t)]$, $0 < \vartheta^{(1)}(x,t) < 1$. Since the mesh function $z^{(1)}$ is nonpositive on ω^h and taking into account (3.6), we conclude that $v^{(1)}$ is an upper solution to (3.2). By induction on n, we obtain that $z^{(n)}(x,t) \le 0$, $x \in \overline{\omega}^h$, $n \ge 1$, and prove that $\{\overline{v}^{(n)}(x,t)\}$ is a monotonically decreasing sequence of upper solutions.

We now prove that the monotone sequence $\{\overline{v}^{(n)}\}$ converges to the solution of (3.2). The sequence $\{\overline{v}^{(n)}\}$ is monotonically decreasing and bounded below by $\underline{v}$, where $\underline{v}$ is any lower solution (3.9). Now by linearity of the operator $\mathcal{L}$ and the continuity of f, we have also from (3.7) that the mesh function v defined by

$$v(x,t) = \lim_{n\to\infty} \overline{v}^{(n)}(x,t), \quad x \in \overline{\omega}^h, \tag{3.15}$$

is an exact solution to (3.2). If by contradiction, we assume that there exist two solutions v_1 and v_2 to (3.2), then by the mean-value theorem, the difference $\delta v = v_1 - v_2$ satisfies the system

$$\mathcal{L}\delta v(x,t) + f_u \delta v(x,t) = 0, \quad x \in \omega^h, \quad \delta v(0,t) = v(1,t) = 0. \tag{3.16}$$

By Lemma 3.1, $\delta v = 0$ which leads to the uniqueness of the solution to (3.2). This proves the theorem. □

Consider the following approach for constructing initial upper and lower solutions $\overline{v}^{(0)}(x,t)$ and $\underline{v}^{(0)}(x,t)$. Introduce the difference problems

$$\begin{gathered}\mathcal{L}v_\nu^{(0)}(x,t) = \nu \left| f(x,t,0) - \tau^{-1} v(x,t-\tau) \right|, \quad x \in \omega^h, \\ v_\nu^{(0)}(0,t) = v^{(0)}(1,t) = 0, \quad \nu = 1, -1.\end{gathered} \tag{3.17}$$

The functions $v_1^{(0)}(x,t)$, $v_{-1}^{(0)}(x,t)$ are upper and lower solutions, respectively. This result can be proved in a similar way as for the elliptic problem.

Theorem 3.3. *Let initial upper or lower solution be chosen in the form of* (3.17)*, and let f satisfy* (3.6)*. Suppose that on each time level the number of iterates $n_* \geq 2$. Then for the monotone iterative methods* (3.7)*, the following estimate on convergence rate holds:*

$$\max_{1 \leq k \leq N_\tau} \| v(t_k) - v_*(t_k) \|_{\overline{\omega}^h} \leq C \eta^{n_*-1}, \quad \eta = \frac{c^*}{\left(c^* + \tau^{-1}\right)}, \tag{3.18}$$

where $v_*(x,t)$ is the solution to (3.2)*, $v(x,t) = v^{(n_*)}(x,t)$, and constant C is independent of μ, N, and τ.*

Proof. Similar to (3.14), using the mean-value theorem and the equation for $z^{(n)}$ from (3.7), we have

$$\begin{gathered}\mathcal{L}v^{(n)}(x,t) + f\left(x,t,v^{(n)}\right) - \tau^{-1} v(x,t-\tau) = -\left[c^* - f_u^{(n)}(x,t)\right] z^{(n)}(x,t), \\ f_u^{(n)}(x,t) \equiv f_u\left[x,t,v^{(n-1)}(x,t) + \vartheta^{(n)}(x,t) z^{(n)}(x,t)\right], \quad 0 < \vartheta^{(n)}(x,t) < 1.\end{gathered} \tag{3.19}$$

From here and (3.7), we have

$$\left(\mathcal{L} + c^*\right) z^{(n)}(x,t) = \left(c^* - f_u^{(n)}\right) z^{(n-1)}(x,t), \quad x \in \omega^h. \tag{3.20}$$

Using (3.5) and (3.6), we have

$$\left\|z^{(n)}\right\|_{\overline{\omega}^h} \le \eta^{n-1}\left\|z^{(1)}\right\|_{\overline{\omega}^h}, \tag{3.21}$$

where η is defined in (3.18).

Introduce the notation

$$w(x,t) = v_*(x,t) - v(x,t), \tag{3.22}$$

where $v(x,t) = v^{(n_*)}(x,t)$. Using the mean-value theorem, from (3.2) and (3.19), we conclude that $w(x,\tau)$ satisfies the problem

$$\begin{gathered}\mathcal{L}w(x,\tau) + f_u(x,\tau)w(x,\tau) = \left(c^* - f_u^{(n_*)}(x,\tau)\right)z^{(n_*)}(x,\tau), \quad x \in \omega^h,\\ w(0,\tau) = w(1,\tau) = 0, \quad x \in \partial\omega^h,\end{gathered} \tag{3.23}$$

where $f_u^{(n_*)}(x,\tau) = f_u[x,\tau,v(x,\tau) + \vartheta(x,\tau)w(x,\tau)]$, $0 < \vartheta(x,\tau) < 1$, and we have taken into account that $v(x,0) = v_*(x,0) = u^0(x)$. By (3.5), (3.6), and (3.21),

$$\|w(\tau)\|_{\overline{\omega}^h} \le c^*\tau\eta^{n_*-1}\left\|z^{(1)}(\tau)\right\|_{\overline{\omega}^h}. \tag{3.24}$$

Using (3.6), (3.17), and the mean-value theorem, estimate $z^{(1)}(x,\tau)$ from (3.7) by (3.5),

$$\begin{aligned}\left\|z^{(1)}(\tau)\right\|_{\overline{\omega}^h} &\le \tau\left\|\mathcal{L}v^{(0)}(\tau)\right\|_{\overline{\omega}^h} + c^*\tau\left\|v^{(0)}(\tau)\right\|_{\overline{\omega}^h}\\ &\quad + \tau\left\|f(x,\tau,0) - \tau^{-1}u^0\right\|_{\overline{\omega}^h}\\ &\le \left(2\tau + c^*\tau^2\right)\left\|f(x,\tau,0) - \tau^{-1}u^0\right\|_{\overline{\omega}^h}\\ &\le (2 + c^*\tau)\left(\tau\|f(x,\tau,0)\|_{\overline{\omega}^h} + \left\|u^0\right\|_{\overline{\omega}^h}\right) \le C_1,\end{aligned} \tag{3.25}$$

where C_1 is independent of τ $(\tau \le T)$, μ, and N. Thus,

$$\|w(\tau)\|_{\overline{\omega}^h} \le c^*C_1\tau\eta^{n_*-1}. \tag{3.26}$$

Similarly, from (3.2) and (3.19), it follows that

$$\mathcal{L}w(x,2\tau) + f_u(x,2\tau)w(x,2\tau) = \left(c^* - f_u^{(n_*)}(x,2\tau)\right)z^{(n_*)}(x,2\tau) + \tau^{-1}w(x,\tau), \quad x \in \omega^h. \tag{3.27}$$

Using (3.21), by (3.5),

$$\|w(2\tau)\|_{\overline{\omega}^h} \le \|w(\tau)\|_{\overline{\omega}^h} + c^*\tau\eta^{n_*-1}\left\|z^{(1)}(2\tau)\right\|_{\overline{\omega}^h}. \tag{3.28}$$

Using (3.17), estimate $z^{(1)}(x,2\tau)$ from (3.7) by (3.5),

$$\left\|z^{(1)}(2\tau)\right\|_{\overline{\omega}^h} \leq (2+c^*\tau)\left(\tau\left\|f(x,2\tau,0)\right\|_{\overline{\omega}^h} + \|v(\tau)\|_{\overline{\omega}^h}\right) \leq C_2, \tag{3.29}$$

where $v(x,\tau) = v^{(n_*)}(x,\tau)$. As follows from Theorem 3.2, the monotone sequences $\{\overline{v}^{(n)}(x,\tau)\}$ and $\{\underline{v}^{(n)}(x,\tau)\}$ are bounded from above and below by, respectively, $\overline{v}^{(0)}(x,\tau)$ and $\underline{v}^{(0)}(x,\tau)$. Applying (3.5) to problem (3.17) at $t=\tau$, we have

$$\left\|v^{(0)}(\tau)\right\|_{\overline{\omega}^h} \leq \tau\left\|f(x,\tau,0) - \tau^{-1}u^0(x)\right\|_{\overline{\omega}^h} \leq K_1, \tag{3.30}$$

where constant K_1 is independent of μ, N, and τ. Thus, we prove that C_2 is independent of μ, N, and τ. From (3.26) and (3.28), we conclude

$$\|w(2\tau)\|_{\overline{\omega}^h} \leq c^*(C_1+C_2)\tau\eta^{n_*-1}. \tag{3.31}$$

By induction on k, we prove

$$\|w(t_k)\|_{\overline{\omega}^h} \leq c^*\left(\sum_{l=1}^{k} C_l\right)\tau\eta^{n_*-1}, \quad k=1,\ldots,N_\tau, \tag{3.32}$$

where all constants C_l are independent of μ, N, and τ. Taking into account that $N_\tau\tau = T$, we prove the estimate (3.18) with $C = c^*T\max_{1\leq l\leq N_\tau} C_l$. □

In [4], we prove that the difference scheme (3.2) on the piecewise uniform mesh (2.8) converges μ-uniformly to the solution of problem (1.2):

$$\max_{1\leq k\leq N_\tau} \|v_*(t_k) - u(t_k)\|_{\overline{\omega}^h} \leq C\left(N^{-1}\ln N + \tau\right), \tag{3.33}$$

where $v_*(x,t)$ is the exact solution to (3.2), and constant C is independent of μ, N, and τ. From here and Theorem 3.3, we conclude the following theorem.

Theorem 3.4. *Suppose that on each time level the initial upper or lower solution $v^{(0)}$ is chosen in the form of* (3.17) *and $n_* \geq 2$. Then the monotone iterative method* (3.7) *on the piecewise uniform mesh* (2.8) *converges μ-uniformly to the solution of problem* (1.2)*:*

$$\|v(t_k) - u(t_k)\|_{\overline{\omega}^h} \leq C\left(N^{-1}\ln N + \tau + \eta^{n_*-1}\right), \tag{3.34}$$

where $\eta = c^*/(c^*+\tau^{-1})$, and constant C is independent of μ, N, and τ.

4. Numerical Experiments

It is found that in all numerical experiments the basic feature of monotone convergence of the upper and lower sequences is observed. In fact, the monotone property of the sequences

Table 1: Numbers of iterations for the Newton iterative method.

$v^{(0)} \setminus N$	128	256	512	1024
−1	7	7	9	*
1	8	11	18	*
3	73	*	*	*

holds at every mesh point in the domain. This is, of course, to be expected from the analytical consideration.

4.1. The Elliptic Problem

Consider problem (1.1) with $f(u) = (u-3)/(4-u)$. We mention that $u_r = 3$ is the solution of the reduced problem, where $\mu = 0$. This problem gives $c_* = 1/25$, $c^* = 1$, and initial lower and upper solutions are chosen in the form of (2.39). The stopping criterion for the monotone iterative method (2.27) is

$$\left\| v^{(n)} - v^{(n-1)} \right\|_{\overline{\omega}^h} \leq 10^{-5}. \tag{4.1}$$

Our numerical experiments show that for $10^{-1} \leq \mu \leq 10^{-6}$ and $32 \leq N \leq 1024$, iteration counts for monotone method (2.27) on the piecewise uniform mesh are independent of μ and N, and equals 12 and 8 for the lower and upper sequences, respectively. These numerical results confirm our theoretical results stated in Theorem 2.5.

In Table 1, we present numbers of iterations for solving the test problem by the Newton iterative method with the initial iterations $v^{(0)}(x) = -1,\ \ 1,\ \ 3$, $x \in \omega^h$. Here $\mu = 10^{-3}$ is in use, and we denote by an "$*$" if more than 100 iterations is needed to satisfy the stopping criterion, or if the method diverges. The numerical results indicate that the Newton method cannot be used successfully for this test problem.

4.2. The Parabolic Problem

For the parabolic problem (1.2), we consider the test problem with $f(u) = \exp(-1) - \exp(-u)$ and $u^0 = 0$. This problem gives $c_* = \exp(-1)$, $c^* = 1$, and the initial lower and upper solutions are chosen in the form of (3.17).

The stopping test for the monotone method (3.7) is defined by

$$\left\| v^{(n)}(t) - v^{(n-1)}(t) \right\|_{\overline{\omega}^h} \leq 10^{-5}. \tag{4.2}$$

Our numerical experiments show that for $10^{-1} \leq \mu \leq 10^{-6}$ and $32 \leq N \leq 1024$, on each time level the number of iterations for monotone method (3.7) on the piecewise uniform mesh is independent of μ and N and equal 4, 4, and 3 for $\tau = 0.1,\ \ 0.05,\ \ 0.01$, respectively. These numerical results confirm our theoretical results stated in Theorem 3.3.

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