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Research Article

Global Behavior for a Diffusive Predator-Prey Model with Stage Structure and Nonlinear Density Restriction-I: The Case in $\mathbb{R}^n$

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This paper deals with a Holling type III diffusive predator-prey model with stage structure and nonlinear density restriction in the space $\mathbb{R}^n$. We first consider the asymptotical stability of equilibrium points for the model of ODE type. Then, the existence and uniform boundedness of global solutions and stability of the equilibrium points for the model of weakly coupled reaction-diffusion type are discussed. Finally, the global existence and the convergence of solutions for the model of cross-diffusion type are investigated when the space dimension is less than 6.

1. Introduction

Population models with stage structure have been investigated by many researchers, and various methods and techniques have been used to study the existence and qualitative properties of solutions [1–9]. However, most of the discussions in these works are devoted to either systems of ODE or weakly coupled systems of reaction-diffusion equations. In this paper we investigate the global existence and convergence of solutions for a strongly coupled cross-diffusion predator-prey model with stage structure and nonlinear density restriction. Nonlinear problems of this kind are quite difficult to deal with since the usual idea to apply maximum principle arguments to get priori estimates cannot be used here [10].

Consider the following predator-prey model with stage-structure:

$$X_1' = BX_2 - r_1X_1 - CX_1 - \eta_1X_1^2 - \eta_2X_1^3 - \frac{EX_1^2X_3}{1+FX_1^2},$$

$$X_2' = CX_1 - r_2X_2,$$
$$X_3' = -r_3X_3 - \eta_3X_3^2 + AX_3\frac{EX_1^2}{1+FX_1^2}, \tag{1.1}$$

where $X_1(t)$, $X_2(t)$ denote the density of the immature and mature population of the prey, respectively, $X_3(t)$ is the density of the predator. For the prey, the immature population is nonlinear density restriction. X_3 is assumed to consume X_1 with Holling type III functional response $EX_1^2/(1+FX_1^2)$ and contributes to its growth with rate $AEX_1^2/(1+FX_1^2)$. For more details on the backgrounds of this model see references [11, 12].

Using the scaling $u = \sqrt{F}X_1$, $v = (r_2\sqrt{F}/C)X_2$, $w = (E/r_2\sqrt{F})X_3$, $d\tau = r_2dt$ and redenoting τ by t, we can reduce the system (1.1) to

$$\begin{aligned} u' &= \beta v - au - bu^2 - cu^3 - \frac{u^2w}{1+u^2} \equiv f_1, \\ v' &= u - v \equiv f_2, \\ w' &= -kw - \gamma w^2 + \frac{\alpha u^2 w}{1+u^2} \equiv f_3, \end{aligned} \tag{1.2}$$

where $\beta = BC/r_2^2$, $a = (r_1+C)/r_2$, $b = \eta_1/r_2\sqrt{F}$, $c = \eta_2/r_2F$, $k = r_3/r_2$, $\alpha = AE/r_2F$, $\gamma = \eta_3\sqrt{F}/E$.

To take into account the natural tendency of each species to diffuse, we are led to the following PDE system of reaction-diffusion type:

$$\begin{gathered} u_t - d_1\Delta u = \beta v - au - bu^2 - cu^3 - \frac{u^2w}{1+u^2}, \quad x\in\Omega,\ t>0, \\ v_t - d_2\Delta v = u - v, \quad x\in\Omega,\ t>0, \\ w_t - d_3\Delta w = -kw - \gamma w^2 + \frac{\alpha u^2 w}{1+u^2}, \quad x\in\Omega,\ t>0, \\ \partial_\eta u = \partial_\eta v = \partial_\eta w = 0, \quad x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x)\geq 0, \quad v(x,0) = v_0(x)\geq 0, \quad w(x,0) = w_0(x)\geq 0, \quad x\in\Omega, \end{gathered} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, η is the outward unit normal vector on $\partial\Omega$, and $\partial_\eta = \partial/\partial\eta$. $u_0(x), v_0(x), w_0(x)$ are nonnegative smooth functions on $\overline{\Omega}$. The diffusion coefficients d_i $(i=1,2,3)$ are positive constants. The homogeneous Neumann boundary condition indicates that system (1.3) is self-contained with zero population flux across the boundary. The knowledge for system (1.3) is limited (see [13–17]).

In the recent years there has been considerable interest to investigate the global behavior for models of interacting populations with linear density restriction by taking into account the effect of self-as well as cross-diffusion [18–26]. In this paper we are led to the following cross-diffusion system:

$$
\begin{aligned}
&u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{13}w)u] + \beta v - au - bu^2 - cu^3 - \frac{u^2 w}{1+u^2}, \quad x \in \Omega,\ t > 0,\\
&v_t = \Delta[(d_2 + \alpha_{22}v)v] + u - v, \quad x \in \Omega,\ t > 0,\\
&w_t = \Delta[(d_3 + \alpha_{33}w)w] - kw - \gamma w^2 + \frac{\alpha u^2 w}{1+u^2}, \quad x \in \Omega,\ t > 0,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x) \ge 0, \quad v(x,0) = v_0(x) \ge 0, \quad w(x,0) = w_0(x) \ge 0, \quad x \in \Omega,
\end{aligned}
\tag{1.4}
$$

where d_1, d_2, d_3 are the diffusion rates of the three species, respectively. α_{ii} $(i = 1,2,3)$ are referred as self-diffusion pressures, and α_{13} is cross-diffusion pressure. The term self-diffusion implies the movement of individuals from a higher to a lower concentration region. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. The value of the cross-diffusion coefficient may be positive, negative, or zero. The term positive cross-diffusion coefficient denotes the movement of the species in the direction of lower concentration of another species and negative cross-diffusion coefficient denotes that one species tends to diffuse in the direction of higher concentration of another species [27]. For $\alpha_{ij} \neq 0$, problem (1.4) becomes strongly coupled with a full diffusion matrix. As far as the authors are aware, very few results are known for cross-diffusion systems with stage-structure.

The main purpose of this paper is to study the asymptotic behavior of the solutions for the reaction-diffusion system (1.3), the global existence, and the convergence of solutions for the cross-diffusion system (1.4). The paper will be organized as follows. In Section 2 a linear stability analysis of equilibrium points for the ODE system (1.2) is given. In Section 3 the uniform bound of the solution and stability of the equilibrium points to the weakly coupled system (1.3) are proved. Section 4 deals with the existence and the convergence of global solutions for the strongly coupled system (1.4).

2. Global Stability for System (1.2)

Let $E_0 = (0,0,0)$. If $\beta > a$, then (1.2) has semitrivial equilibria $E_1(m_0, m_0, 0)$, where $m_0 = (\sqrt{b^2 + 4c(\beta - a)} - b)/2c$. To discuss the existence of the positive equilibrium point of (1.2), we give the following assumptions:

$$
\alpha > k, \quad \beta > a, \quad \sqrt{\frac{k}{\alpha - k}} < m_0, \quad \frac{\beta - a - c}{2} + \frac{b^2}{8c} \le \frac{b\sqrt{p_1}}{24c} + \frac{24(\beta - a)c^2}{3b^2 + 4c(\beta - a - c) - b\sqrt{p_1}},
\tag{2.1}
$$

where $p_1 = 9b^2 + 24c(\beta - a - c) \ge 0$. Let one curve l_1: $g_1(u) = ((1+u^2)/u)(\beta - a - bu - cu^2)$, and the other curve l_3: $g_3(u) = k + \gamma w = \alpha u^2/(1+u^2)$. Obviously, l_1 passes the point $(m_0, 0)$. Noting that $(\beta - a - c)u^2 - 2bu^3 - 3cu^4 - \beta + a$ attains its maximum at $u = (\sqrt{p_1} - 3b)/12c$, thus when $(\beta-a-c)/2+b^2/8c \le b\sqrt{p_1}/24c+24(\beta-a)c^2/(3b^2+4c(\beta-a-c)-b\sqrt{p_1})$, $g_1'(u) < 0$ $(0 < u < m_0)$. l_3 has the asymptote $w = \alpha - k/\gamma$ and passes the point $(\sqrt{k/\alpha - k}, 0)$. In this case, l_1 and l_3 have unique intersection (u^*, w^*), as shown in Figure 1. $E^* = (u^*, v^*, w^*)$ is the unique positive equilibrium point of (1.2), where $v^* = u^*$, $w^* = ((1 + u^{*2})/u^*)(\beta - a - bu^* - cu^{*2})$, $k+\gamma w^* = \alpha u^{*2}/(1+u^{*2})$. In addition, the restriction of the existence of the positive equilibrium can be removed, if $\beta < a + c$.

The Jacobian matrix of the equilibrium E_0 is

$$J(E_0) = \begin{pmatrix} -a & \beta & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -k \end{pmatrix}. \tag{2.2}$$

The characteristic equation of $J(E_0)$ is $(\lambda + k)[\lambda^2 + (1 + a)\lambda + a - \beta] = 0$. E_0 is a saddle for $\beta > a$. In addition, the dimensions of the local unstable and stable manifold of E_0 are 1 and 2, respectively. E_0 is locally asymptotically stable for $\beta < a$.

The Jacobian matrix of the equilibrium E_1 is

$$J(E_1) = \begin{pmatrix} a_{11} & \beta & -\dfrac{m_0^2}{1 + m_0^2} \\ 1 & -1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \tag{2.3}$$

where $a_{11} = -a - 2bm_0 - 3cm_0^2$, $a_{33} = -k + \alpha m_0^2/(1 + m_0^2)$. The characteristic equation of $J(E_1)$ is $\lambda^3 + A_1\lambda^2 + B_1\lambda + C_1 = 0$, where

$$\begin{gathered} A_1 = -a_{11} - a_{33} + 1, \\ B_1 = a_{11}a_{33} - a_{33} - (a_{11} + \beta), \\ C_1 = a_{33}(a_{11} + \beta), \\ H_1 = A_1B_1 - C_1 = (a_{11} + a_{33})[a_{33} - a_{11}a_{33} + (a_{11} + \beta)] - a_{33}(1 + \beta) - (a_{11} + \beta). \end{gathered} \tag{2.4}$$

According to Routh-Hurwitz criterion, E_1 is locally asymptotically stable for $a_{11} + \beta < 0$ and $a_{33} < 0$, that is, $m_0^2(\alpha - k) < k$ and $m_0 > (\sqrt{b^2 + 3c(\beta - a)} - b)/3c$.

The Jacobian matrix of the equilibrium E^* is

$$J(E^*) = \begin{pmatrix} a_{11} & \beta & a_{13} \\ 1 & -1 & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}, \tag{2.5}$$

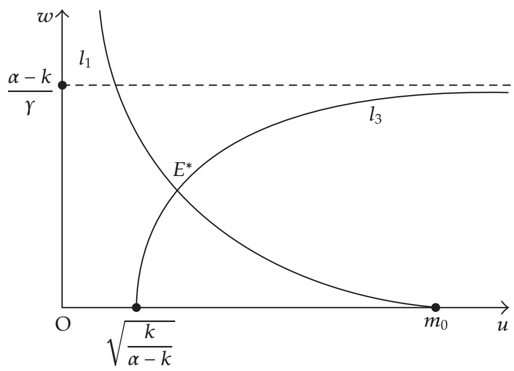

Figure 1

where

$$a_{11} = -a - 2bu^* - 3cu^{*2} - \frac{2u^*w^*}{(1+u^{*2})^2}, \quad a_{13} = -\frac{u^{*2}}{1+u^{*2}},$$
$$a_{31} = \frac{2\alpha u^* w^*}{(1+u^{*2})^2}, \quad a_{33} = -\gamma w^*. \tag{2.6}$$

The characteristic equation of $J(E^*)$ is $\lambda^3 + A_2\lambda^2 + B_2\lambda + C_2 = 0$, where

$$A_2 = -a_{11} - a_{33} + 1,$$
$$B_2 = a_{11}a_{33} - a_{13}a_{31} - a_{33} - (a_{11} + \beta),$$
$$C_2 = a_{33}(a_{11} + \beta) - a_{13}a_{31},$$
$$H_2 = A_2B_2 - C_2 = (a_{11} + a_{33})\left[a_{13}a_{31} + a_{33} - a_{11}a_{33} + (a_{11} + \beta)\right] - a_{33}(1+\beta) - (a_{11} + \beta). \tag{2.7}$$

According to Routh-Hurwitz criterion, E^* is locally asymptotically stable for $a_{11} + \beta < 0$. Obviously, $a_{11} + \beta < 0$ can be checked by (2.1).

Now we discuss the global stability of equilibrium points for (1.2).

Theorem 2.1. *(i) Assume that* (2.1),

$$b + cu^* - \frac{u^*(\beta - a - bu^*)}{2 + 2\sqrt{1+u^{*2}}} > \frac{\left(\sqrt{u^{*2}+1} + u^*\right)^2}{8(u^{*2}+1)^2} + \frac{1}{8},$$
$$\frac{\gamma}{\alpha} > \frac{1}{2}, \tag{2.8}$$

hold, then the equilibrium point E^* of (1.2) *is globally asymptotically stable.*

(ii) Assume that $\beta > a, m_0^2(\alpha - k) < k$, and $(\sqrt{b^2 + 3c(\beta - a)} - b)/3c < m_0 < 2k/\alpha$ hold, then the equilibrium point E_1 of (1.2) *is globally asymptotically stable.*

(iii) Assume that $\beta \le a$ holds, then the equilibrium point E_0 of (1.2) *is globally asymptotically stable.*

Proof. (i) Define the Lyapunov function

$$E(t) = \left(u - u^* - u^* \ln \frac{u}{u^*}\right) + \beta\left(v - v^* - v^* \ln \frac{v}{v^*}\right) + \frac{1}{\alpha}\left(w - w^* - w^* \ln \frac{w}{w^*}\right). \tag{2.9}$$

Calculating the derivative of $E(t)$ along the positive solution of (1.2), we have

$$\begin{aligned} E'(t) &= -\frac{\beta}{u^*}\left[\sqrt{\frac{v}{u}}(u - u^*) - \sqrt{\frac{u}{v}}(v - v^*)\right]^2 - (u - u^*)^2\left[b + cu + cu^* + \frac{w^*(1 - u^* u)}{(1 + u^{*2})(1 + u^2)}\right] \\ &\quad - \frac{c}{\alpha}(w - w^*)^2 + (u - u^*)(w - w^*)\left[\frac{u^* + u}{(1 + u^{*2})(1 + u^2)} - \frac{u}{1 + u^2}\right] \\ &\le -(u - u^*)^2\left[b + cu + cu^* + \frac{w^*(1 - u^* u)}{(1 + u^{*2})(1 + u^2)} - \frac{u^2}{2(1 + u^2)^2} - \frac{(u + u^*)^2}{2(1 + u^{*2})^2(1 + u^2)^2}\right. \\ &\quad \left. + \frac{u(u + u^*)}{(1 + u^{*2})(1 + u^2)^2}\right] - \left(\frac{\gamma}{\alpha} - \frac{1}{2}\right)(w - w^*)^2. \end{aligned} \tag{2.10}$$

When $u \in [0, \infty)$, the minimum of $(1 - u^* u)/(1 + u^2)$ and $u(u + u^*)/(1 + u^2)^2$ is $-u^{*2}/(2 + 2\sqrt{1 + u^{*2}})$ and 0, respectively; the maximum of $(u + u^*)/(1 + u^2)$ is $u/(1 + u^2)$ are $(u^* + \sqrt{1 + u^{*2}})/2$ and $1/2$, respectively. Thus, when (2.8) hold, $E'(t) \le 0$. According to the Lyapunov-LaSalle invariance principle [28], E^* is globally asymptotically stable if (2.1)–(2.3) hold.

(ii) Let

$$E(t) = \left(u - m_0 - m_0 \ln \frac{u}{m_0}\right) + \beta\left(v - m_0 - m_0 \ln \frac{v}{m_0}\right) + \frac{1}{\alpha} w. \tag{2.11}$$

Then

$$\begin{aligned} E'(t) &= -\frac{\beta}{m_0}\left[\sqrt{\frac{v}{u}}(u - m_0) - \sqrt{\frac{u}{v}}(v - m_0)\right]^2 \\ &\quad - \left[(b + cu + cm_0)(u - m_0)^2 + \frac{c}{\alpha} w^2 - w\left(\frac{m_0 u}{1 + u^2} - \frac{k}{\alpha}\right)\right]. \end{aligned} \tag{2.12}$$

Noting that the maximum of $u/(1+u^2)$ is $1/2$, and $m_0 < 2k/\alpha$, we find $m_0 u/(1+u^2) - k/\alpha < 0$. Therefore, $E'(t) \le 0$.

(iii) Let

$$E(t) = u + \beta v + \frac{1}{\alpha} w, \tag{2.13}$$

then

$$E'(t) = (\beta - a)u - bu^2 - cu^3 - \frac{k}{\alpha} w - \frac{\gamma}{\alpha} w^2. \tag{2.14}$$

Thus, $E'(t) \leq 0$ for $\beta \leq a$. This completes the proof of Theorem 2.1. □

3. Global Behavior of System (1.3)

In this section we discuss the existence, uniform boundedness of global solutions, and the stability of constant equilibrium solutions for the weakly coupled reaction-diffusion system (1.3). In particular, the unstability results in Section 2 also hold for system (1.3) because solutions of (1.2) are also solutions of (1.3).

Theorem 3.1. *Let $u_0(x), v_0(x), w_0(x)$ be nonnegative smooth functions on $\overline{\Omega}$. Then system* (1.3) *has a unique nonnegative solution $(u(x,t), v(x,t), w(x,t)) \in [C(\overline{\Omega} \times [0,\infty)) \bigcap C^{2,1}(\Omega \times (0,\infty))]^3$, and*

$$0 \leq u \leq \widehat{M_1} = \max\left\{ \sup_{\Omega} u_0, \sup_{\Omega} v_0, \frac{\sqrt{b^2 + 4c(\beta - a)} - b}{2c} \right\},$$
$$0 \leq v \leq \widehat{M_2} = \widehat{M_1}, \tag{3.1}$$
$$0 \leq w \leq \widehat{M_3} = \max\left\{ \sup_{\Omega} w_0, \frac{\alpha \widehat{M_1}^2}{\gamma\left(1 + \widehat{M_1}^2\right)} - \frac{k}{\gamma} \right\}$$

on $\overline{\Omega} \times [0,\infty)$. In particular, if $u_0, v_0, w_0 \geq (\not\equiv) 0$, then $u, v, w > 0$ for all $t > 0, x \in \overline{\Omega}$.

Proof. It is easily seen that (f_1, f_2, f_3) is sufficiently smooth in $\mathbb{R}_+^3$ and possesses a mixed quasimonotone property in $\mathbb{R}_+^3$. In addition, $(0,0,0)$ and $(\widehat{M_1}, \widehat{M_2}, \widehat{M_3})$ are a pair of lower-upper solutions of problem (1.3) (cf. $(\widehat{M_1}, \widehat{M_2}, \widehat{M_3})$ in (3.1)). From [29, Theorem 5.3.4], we conclude that (1.3) exists a unique classical solution (u, v, w) satisfying (3.1). According to strong maximum principle, it follows that $u(x,t), v(x,t), w(x,t) > 0$, $\forall t > 0, x \in \overline{\Omega}$. So the proof of the Theorem is completed. □

Remark 3.2. When $c = 0$ (namely $\eta_2 = 0$), system (1.3) reduces to a system in which the immature population of the prey is linear density restriction. Similar to the proof of Theorem 3.1, we have

$$\widehat{M_1} = \widehat{M_2} = \max\left\{ \sup_{\Omega} u_0, \sup_{\Omega} v_0, \frac{\beta - a}{b} \right\},$$
$$\widehat{M_3} = \max\left\{ \sup_{\Omega} w_0, \frac{\alpha \widehat{M_1}^2}{\gamma\left(1 + \widehat{M_1}^2\right)} - \frac{k}{\gamma} \right\}. \tag{3.2}$$

Now we show the local and global stability of constant equilibrium solutions E_0, E_1, E^* for (1.3), respectively.

Theorem 3.3. (i) *Assume that* (2.1) *holds, then the equilibrium point* E^* *of* (1.3) *is locally asymptotically stable.*

(ii) *Assume that* $\beta > a$, $m_0^2(\alpha - k) < k$, *and* $m_0 > \sqrt{b^2 + 3c(\beta - a)} - b/3c$ *hold, then the equilibrium point* E_1 *of* (1.3) *is locally asymptotically stable.*

(iii) *Assume that* $\beta < a$ *holds, then the equilibrium point* E_0 *of* (1.3) *is locally asymptotically stable.*

Proof. Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with Neumann boundary condition, and let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$. Let

$$X = \left\{ U \in \left[C^1\left(\overline{\Omega}\right)\right]^3, \partial_\eta U = 0, x \in \partial\Omega \right\}, \qquad X_{ij} = \left\{ c \cdot \phi_{ij} : c \in \mathbb{R}^3 \right\}, \tag{3.3}$$

where $\{\phi_{ij}; j = 1, \ldots, \dim E(\mu_i)\}$ is an orthonormal basis of $E(\mu_i)$, then

$$X = \oplus_{i=1}^{\infty} X_i, \qquad X_i = \oplus_{j=1}^{\dim E(\mu_i)} X_{ij}. \tag{3.4}$$

(i) Let $D = \operatorname{diag}(d_1, d_2, d_3)$, $L = D\Delta + F_U(E^*) = D\Delta + \{a_{ij}\}$, where

$$a_{11} = -a - 2bu^* - 3cu^{*2} - \frac{2u^* w^*}{(1 + u^{*2})^2}, \qquad a_{12} = \beta, \qquad a_{13} = -\frac{u^{*2}}{1 + u^{*2}},$$
$$a_{21} = 1, \qquad a_{22} = -1, \qquad a_{23} = 0, \tag{3.5}$$
$$a_{31} = \frac{2\alpha u^* w^*}{(1 + u^{*2})^2}, \qquad a_{32} = 0, \qquad a_{33} = -\gamma w^*.$$

The linearization of (1.3) is $U_t = LU$ at E^*. For each $i \geq 1$, X_i is invariant under the operator L, and λ is an eigenvalue of L on X_i, if and only if λ is an eigenvalue of the matrix $-\mu_i D + F_U(E^*)$. The characteristic equation is $\varphi_i(\lambda) = \lambda^3 + A_i\lambda^2 + B_i\lambda + C_i = 0$, where

$$\begin{aligned}
A_i &= \mu_i(d_1 + d_2 + d_3) - a_{11} - a_{33} + 1,\\
B_i &= \mu_i^2(d_1d_2 + d_1d_3 + d_2d_3)\\
&\quad + \mu_i[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})]\\
&\quad + a_{11}a_{33} - a_{13}a_{31} - a_{33} - (a_{11} + \beta),\\
C_i &= \mu_i^3 d_1d_2d_3 + \mu_i^2(d_1d_3 - a_{33}d_1d_2 - a_{11}d_2d_3)\\
&\quad - \mu_i[d_1a_{33} - d_2(a_{11}a_{33} - a_{13}a_{31}) + d_3(a_{11} + \beta)]\\
&\quad + a_{33}(a_{11} + \beta) - a_{13}a_{31},\\
H_i &= A_iB_i - C_i = P_3\mu_i^3 + P_2\mu_i^2 + P_1\mu_i + P_0,\\
P_3 &= (d_1 + d_2)(d_1d_2 + d_1d_3 + d_2d_3) + d_3^2(d_1 + d_2),\\
P_2 &= (d_1 + d_2 + d_3)[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})]\\
&\quad - a_{11}d_1(d_2 + d_3) + d_2(d_1 + d_3) - a_{33}d_3(d_1 + d_2),\\
P_1 &= d_1[a_{11}a_{33} - a_{13}a_{31} - (a_{11} + \beta)] - d_2[(a_{11} + \beta) + a_{33}]\\
&\quad + d_3(a_{11}a_{33} - a_{33} - a_{13}a_{31})\\
&\quad - (a_{11} + a_{33} - 1)[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})],\\
P_0 &= (a_{11} + a_{33})[a_{13}a_{31} + a_{33} - a_{11}a_{33} + (a_{11} + \beta)]\\
&\quad - a_{33}(1 + \beta) - (a_{11} + \beta).
\end{aligned} \tag{3.6}$$

From Routh-Hurwitz criterion, we can see that three eigenvalues (denoted by $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$) all have negative real parts if and only if $A_i > 0, C_i > 0, H_i > 0$. Noting that $a_{11}, a_{13}, a_{33} < 0, a_{31} > 0$, we must have $a_{11} + \beta < 0$. It is easy to check that $a_{11} + \beta < 0$ if $g_1'(u_1) < 0$ (see Section 2).

We can conclude that there exists a positive constant δ, such that

$$\operatorname{Re}\{\lambda_{i,1}\}, \operatorname{Re}\{\lambda_{i,2}\}, \operatorname{Re}\{\lambda_{i,3}\} \leq -\delta, \quad i \geq 1. \tag{3.7}$$

In fact, let $\lambda = \mu_i\xi$, then

$$\varphi_i(\lambda) = \mu_i^3\xi_i^3 + A_i\mu_i^2\xi_i^2 + B_i\mu_i\xi + C_i \triangleq \widetilde{\varphi}_i(\xi). \tag{3.8}$$

Since $\mu_i \to \infty$ as $i \to \infty$, it follows that

$$\lim_{i\to\infty} \frac{\widetilde{\varphi}_i(\xi)}{\mu_i^3} = \xi^3 + (d_1 + d_2 + d_3)\xi^2 + (d_1d_2 + d_2d_3 + d_1d_3)\xi + d_1d_2d_3 \triangleq \widetilde{\varphi}(\xi). \tag{3.9}$$

Clearly, $\widetilde{\varphi}(\xi)$ has the three roots $-d_1, -d_2, -d_3$. Let $d = \min\{d_1, d_2, d_3\}$. By continuity, there exists i_0 such that the three roots $\xi_{i1}, \xi_{i2}, \xi_{i3}$ of $\widetilde{\varphi}_i(\xi) = 0$ satisfy

$$\operatorname{Re}\{\xi_{i1}\}, \operatorname{Re}\{\xi_{i2}\}, \operatorname{Re}\{\xi_{i3}\} \leq -\frac{d}{2}, \quad i \geq i_0. \tag{3.10}$$

Let $-\widetilde{\delta} = \max_{0\leq i\leq i_0}\{\operatorname{Re}\{\lambda_{i1}\}, \operatorname{Re}\{\lambda_{i2}\}, \operatorname{Re}\{\lambda_{i3}\}\}$, then $\widetilde{\delta} > 0$. Let $\delta = \min\{\widetilde{\delta}, d/2\}$, then (3.7) holds. According to [30, Theorem 5.1.1], we have the locally asymptotically stability of E^*.

(ii) The linearization of (1.4) is $U_t = LU$ at E_1, where $L = D\Delta + F_U(E_1) = D\Delta + \{a_{ij}\}$, and

$$\begin{gathered} a_{11} = -a - 2bm_0 - 3cm_0^2, \qquad a_{12} = \beta, \qquad a_{13} = -\frac{m_0^2}{1+m_0^2}, \\ a_{21} = 1, \qquad a_{22} = -1, \qquad a_{23} = 0, \\ a_{31} = 0, \qquad a_{32} = 0, \qquad a_{33} = -k + \frac{\alpha m_0^2}{1+m_0^2}. \end{gathered} \tag{3.11}$$

The characteristic equation of $-\mu_i D + F_U(E_1)$ is $\varphi_i(\lambda) = \lambda^3 + A_i\lambda^2 + B_i\lambda + C_i = 0$, where

$$\begin{aligned} A_i &= \mu_i(d_1 + d_2 + d_3) - a_{11} - a_{33} + 1, \\ B_i &= \mu_i^2(d_1d_2 + d_1d_3 + d_2d_3) \\ &\quad + \mu_i[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})] \\ &\quad + a_{11}a_{33} - a_{33} - (a_{11} + \beta), \\ C_i &= \mu_i^3 d_1d_2d_3 + \mu_i^2(d_1d_3 - a_{33}d_1d_2 - a_{11}d_2d_3) \\ &\quad - \mu_i\left[d_1a_{33} - d_2a_{11}a_{33} + d_3(a_{11} + \beta)\right] + a_{33}(a_{11} + \beta), \\ H_i &= A_iB_i - C_i = P_3\mu_i^3 + P_2\mu_i^2 + P_1\mu_i + P_0, \\ P_3 &= (d_1 + d_2)(d_1d_2 + d_1d_3 + d_2d_3) + d_3^2(d_1 + d_2), \\ P_2 &= (d_1 + d_2 + d_3)[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})] \\ &\quad - a_{11}d_1(d_2 + d_3) + d_2(d_1 + d_3) - a_{33}d_3(d_1 + d_2), \\ P_1 &= d_1\left[a_{11}a_{33} - (a_{11} + \beta)\right] - d_2\left[(a_{11} + \beta) + a_{33}\right] + d_3(a_{11}a_{33} - a_{33}) \\ &\quad - (a_{11} + a_{33} - 1)[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})], \\ P_0 &= (a_{11} + a_{33})\left[a_{33} - a_{11}a_{33} + (a_{11} + \beta)\right] - a_{33}(1 + \beta) - (a_{11} + \beta). \end{aligned} \tag{3.12}$$

The three roots of $\varphi_i(\lambda) = 0$ all have negative real parts for $a_{11} + \beta < 0$ and $a_{33} < 0$. Namely, E_1 is the locally asymptotically stable, if $m_0^2(\alpha - k) < k$ and $m_0 > (\sqrt{b^2 + 3c(\beta - a)} - b)/3c$.

(iii) The linearization of (1.3) is $U_t = LU$ at E_0, where $L = D\Delta + F_U(E_0) = D\Delta + \{a_{ij}\}$, and

$$\begin{aligned} &a_{11} = -a, \quad a_{12} = \beta, \quad a_{13} = 0,\\ &a_{21} = 1, \quad a_{22} = -1, \quad a_{23} = 0,\\ &a_{31} = 0, \quad a_{32} = 0, \quad a_{33} = -k. \end{aligned} \tag{3.13}$$

Similar to (i), E_1 is locally asymptotically stable, when $\beta < a$. □

Remark 3.4. When $c = 0$, denote $E_0 = (0,0,0)$. If $\beta > a$, then (1.3) has the semitrivial equilibrium point $E_1 = (m_0, m_0, 0)$, where $m_0 = (\beta - a)/b$. If $\alpha > k, \beta > a, kb^2 < (\alpha - k)(\beta - a)^2 < 27b^2(\alpha - k)$, then (1.3) has a unique positive equilibrium point $E^* = (u^*, v^*, w^*)$. Similar as Theorem 3.3, we have the following.

(i) If $\beta > a$, $\alpha > k$, and $kb^2 < (\alpha - k)(\beta - a)^2 < 27b^2(\alpha - k)$ (namely, $\alpha > k$, $\beta > a$, $\sqrt{k/(\alpha - k)} < (\beta - a)/b < 3\sqrt{3}$), then E^* is locally asymptotically stable.

(ii) If $\beta > a$ and $(\alpha - k)(\beta - a)^2 < kb^2$, then E_1 is locally asymptotically stable.

(iii) If $\beta < a$, then E_0 is locally asymptotically stable.

Before discussing the global stability, we give an important lemma which has been proved in [31, Lemma 4.1] or in [32, Lemma 2.5.3].

Lemma 3.5. *Let a, b be positive constants. Assume that $\phi, \psi \in C^1([a, \infty))$, $\psi(t) \geq 0$, and ϕ is bounded from below. If $\phi'(t) \leq -b\psi(t)$ and $\psi'(t) \leq K$ ($\forall t \geq a$) for some positive constant K, then* $\lim_{t\to\infty}\psi(t) = 0$.

Theorem 3.6. (i) *Assume that* (2.1),

$$\begin{gathered} b + cu^* - \frac{u^*(\beta - a - bu^*)}{2 + 2\sqrt{1 + u^{*2}}} > \frac{\left(\sqrt{u^{*2} + 1} + u^*\right)^2}{8(u^{*2} + 1)^2} + \frac{1}{8},\\ \frac{\gamma}{\alpha} > \frac{1}{2}, \end{gathered} \tag{3.14}$$

hold, then the equilibrium point E^* of system (1.3) *is globally asymptotically stable.*

(ii) *Assume that $\beta > a, m_0^2(\alpha - k) < k$, and $(\sqrt{b^2 + 3c(\beta - a)} - b)/3c < m_0 < 2k/\alpha$ hold, then the equilibrium point E_1 of system* (1.3) *is globally asymptotically stable.*

(iii) *Assume that $\beta < a$ and $k > \alpha$ hold, then the equilibrium point E_0 of system* (1.3) *is globally asymptotically stable.*

Proof. Let (u,v,w) be the unique positive solution of (1.3). By Theorem 3.1, there exists a positive constant C which is independent of $x \in \overline{\Omega}$ and $t \geq 0$ such that $\|u(\cdot,t)\|_\infty, \|v(\cdot,t)\|_\infty, \|w(\cdot,t)\|_\infty \leq C$, for $t \geq 0$. By [33, Theorem A_2],

$$\|u(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})}, \|v(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})}, \|w(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C, \quad \forall t \geq t_0, \forall t_0 > 0. \tag{3.15}$$

(i) Define the Lyapunov function

$$\begin{aligned} E(t) = &\int_\Omega \left(u - u^* - u^* \ln \frac{u}{u^*}\right) \mathrm{d}x + \beta \int_\Omega \left(v - v^* - v^* \ln \frac{v}{v^*}\right) \mathrm{d}x \\ &+ \frac{1}{\alpha}\int_\Omega \left(w - w^* - w^* \ln \frac{w}{w^*}\right) \mathrm{d}x. \end{aligned} \tag{3.16}$$

By Theorem 3.1, $E(t)$ $(t > 0)$ is defined well for all solutions of (1.3) with the initial functions $u_0, v_0, w_0 \geq (\not\equiv) 0$. It is easily see that $E(t) \geq 0$ and $E(t) = 0$ if and only if $u = u^*$.

Calculating the derivative of $E(t)$ along positive solution of (1.3) by integration by parts and the Cauchy inequality, we have

$$\begin{aligned} E'(t) = &-\int_\Omega \left(\frac{d_1 u^*}{u^2}|\nabla u|^2 + \beta \frac{d_2 v^*}{v^2}|\nabla v|^2 + \frac{d_3 w^*}{\alpha w^2}|\nabla w|^2\right) \mathrm{d}x \\ &+ \int_\Omega \left[(u-u^*)\frac{f_1(u,v,w)}{u} + \beta(v-v^*)\frac{f_2(u,v,w)}{v} + \frac{1}{\alpha}(w-w^*)\frac{f_3(u,v,w)}{w}\right] \mathrm{d}x \\ \leq &-\int_\Omega (u-u^*)^2 \left[b + cu + cu^* + \frac{w^*(1-u^*u)}{(1+u^{*2})(1+u^2)} - \frac{u^2}{2(1+u^2)^2} - \frac{(u+u^*)^2}{2(1+u^{*2})^2(1+u^2)^2}\right. \\ &\left. + \frac{u(u+u^*)}{(1+u^{*2})(1+u^2)^2}\right] \mathrm{d}x - \left(\frac{\gamma}{\alpha} - \frac{1}{2}\right)\int_\Omega (w-w^*)^2 \mathrm{d}x. \end{aligned} \tag{3.17}$$

It is not hard to verify that

$$E'(t) \leq -l_1 \int_\Omega (u-u^*)^2 \mathrm{d}x - l_3 \int_\Omega (w-w^*)^2 \mathrm{d}x, \tag{3.18}$$

if (3.14) hold. Applying Lemma 3.5, we can obtain

$$\lim_{t\to\infty} \int_\Omega (u-u^*)^2 \mathrm{d}x = 0, \qquad \lim_{t\to\infty} \int_\Omega (w-w^*)^2 \mathrm{d}x = 0. \tag{3.19}$$

Recomputing $E'(t)$, we find

$$\begin{aligned} E'(t) &\leq -\int_\Omega \left(\frac{d_1 u^*}{u^2} |\nabla u|^2 + \beta \frac{d_2 v^*}{v^2} |\nabla v|^2 + \frac{d_3 w^*}{\alpha w^2} |\nabla w|^2 \right) \mathrm{d}x \\ &\leq -C \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \right) \mathrm{d}x \triangleq -g(t). \end{aligned} \tag{3.20}$$

From (3.15), we can see that $g'(t)$ is bounded in $[t_0, \infty)$, $t_0 > 0$. It follows from Lemma 3.5 and (3.15) that $g(t) \to 0$ as $t \to \infty$. Namely,

$$\lim_{t \to \infty} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \right) \mathrm{d}x = 0. \tag{3.21}$$

Using the Pioncaré inequality, we have

$$\lim_{t \to \infty} \int_\Omega (u - \overline{u})^2 \mathrm{d}x = \lim_{t \to \infty} \int_\Omega (v - \overline{v})^2 \mathrm{d}x = \lim_{t \to \infty} \int_\Omega (w - \overline{w})^2 \mathrm{d}x = 0, \tag{3.22}$$

where $\overline{u}(t) = (1/|\Omega|) \int_\Omega u \ \mathrm{d}x, \overline{v}(t) = (1/|\Omega|) \int_\Omega v \ \mathrm{d}x, \overline{w}(t) = (1/|\Omega|) \int_\Omega w \ \mathrm{d}x$. Noting that

$$\begin{aligned} |\Omega| |\overline{u}(t) - u^*|^2 &= \int_\Omega (\overline{u} - u^*)^2 \mathrm{d}x \leq 2 \int_\Omega (\overline{u} - u)^2 \mathrm{d}x + 2 \int_\Omega (u - u^*)^2 \mathrm{d}x, \\ |\Omega| |\overline{w}(t) - w^*|^2 &= \int_\Omega (\overline{w} - w^*)^2 \mathrm{d}x \leq 2 \int_\Omega (\overline{w} - w)^2 \mathrm{d}x + 2 \int_\Omega (w - w^*)^2 \mathrm{d}x, \end{aligned} \tag{3.23}$$

according to (3.19) and (3.22), we can see

$$\overline{u}(t) \to u^*, \quad \overline{w}(t) \to w^* \quad (t \to \infty). \tag{3.24}$$

Thus, there exists $\{t_m\}, \overline{u}'(t_m) \to 0$ as $t_m \to \infty$. Applying the boundness of $\{\overline{v}(t_m)\}$, there exists a subsequence of $\{\overline{v}(t_m)\}$, denoted still by $\{\overline{v}(t_m)\}$, such that $\overline{v}(t_m) \to \widehat{v}$. On the one hand

$$\int_\Omega u_t \ \mathrm{d}x \Big|_{t_m} = |\Omega| \overline{u}'(t_m) \longrightarrow 0, \quad t_m \longrightarrow \infty. \tag{3.25}$$

On the other hand

$$\begin{aligned} \int_\Omega u_t \mathrm{d}x \Big|_{t_m} &= \int_\Omega (d_1 \Delta u + f_1(u, v, w)) \mathrm{d}x \Big|_{t_m} = \int_\Omega f_1(u, v, w) \mathrm{d}x \Big|_{t_m} \\ &= \int_\Omega \left[\beta(v - v^*) - \left(a + b(u + u^*) + c \left(u^2 + u u^* + u^{*2} \right) \right) (u - u^*) - du(w - w^*) \right] \mathrm{d}x \Big|_{t_m}. \end{aligned} \tag{3.26}$$

According to (3.19) to compute the limit of the previous equation and using the uniqueness of the limit, we have $\hat{v} = v^*$, and

$$\lim_{t_m \to \infty} \overline{v}(t_m) = v^*. \tag{3.27}$$

It follows from (3.15) that there exists a subsequence of $\{t_m\}$, denoted still by $\{t_m\}$, and nonnegative functions $g_i \in C^2(\overline{\Omega}), i = 1, 2, 3$, such that

$$u(\cdot, t_m) \longrightarrow g_1(\cdot), \quad v(\cdot, t_m) \longrightarrow g_2(\cdot), \quad w(\cdot, t_m) \longrightarrow g_3(\cdot) \quad \text{in } C^2\left(\overline{\Omega}\right). \tag{3.28}$$

Applying (3.19)–(3.27), we obtain that $g_1 = u^*, g_2 = v^*, g_3 = w^*$, and

$$u(\cdot, t_m) \longrightarrow u^*, \quad v(\cdot, t_m) \longrightarrow v^*, \quad w(\cdot, t_m) \longrightarrow w^* \quad \text{in } C^2\left(\overline{\Omega}\right). \tag{3.29}$$

In view of Theorem 3.3, we can conclude that E^* is globally asymptotically stable.

(ii) Let

$$E(t) = \int_{\Omega} \left(u - m_0 - m_0 \ln \frac{u}{m_0} \right) \mathrm{d}x + \beta \int_{\Omega} \left(v - m_0 - m_0 \ln \frac{v}{m_0} \right) \mathrm{d}x + \frac{1}{\alpha} \int_{\Omega} w \, \mathrm{d}x. \tag{3.30}$$

Then

$$\begin{aligned} E'(t) &= -m_0 \int_{\Omega} \left(\frac{d_1}{u^2} |\nabla u|^2 + \beta \frac{d_2}{v^2} |\nabla v|^2 \right) \mathrm{d}x \\ &\quad + \int_{\Omega} \left[(u - u^*) \frac{f_1(u, v, w)}{u} + \beta (v - v^*) \frac{f_2(u, v, w)}{v} + \frac{1}{\alpha} f_3(u, v, w) \right] \mathrm{d}x \\ &\leq -\int_{\Omega} \frac{\beta}{m_0} \left[\sqrt{\frac{v}{u}} (u - m_0) - \sqrt{\frac{u}{v}} (v - m_0) \right]^2 \\ &\quad - \int_{\Omega} \left[(b + cu + cm_0)(u - m_0)^2 + \frac{\gamma}{\alpha} w^2 - w \left(\frac{m_0 u}{1 + u^2} - \frac{k}{\alpha} \right) \right] \mathrm{d}x. \end{aligned} \tag{3.31}$$

Therefore, $E'(t) \leq -(b + cm_0) \int_{\Omega} (u - m_0)^2 \mathrm{d}x - \frac{\gamma}{\alpha} \int_{\Omega} w^2 \mathrm{d}x$. It follows that the equilibrium point E_1 of (1.3) is globally asymptotically stable.

(iii) Define

$$E(t) = \frac{1}{2} \int_{\Omega} \left(u^2 + \beta v^2 + w^2 \right) \mathrm{d}x. \tag{3.32}$$

Then

$$E'(t) = -\int_\Omega \left(d_1|\nabla u|^2 + \beta d_2|\nabla v|^2 + d_3|\nabla w|^2\right)\mathrm{d}x \\ + \int_\Omega (uf_1(u,v,w) + \beta v f_2(u,v,w) + w f_3(u,v,w))\mathrm{d}x. \tag{3.33}$$

When $a > \beta, k > \alpha$,

$$E'(t) \le -\int_\Omega \left[au^2 + \beta v^2 + (k-\alpha)w^2\right]\mathrm{d}x. \tag{3.34}$$

The following proof is similar to (i). □

Remark 3.7. When $c = 0$, Theorem 3.6 shows the following.

(i) Assume that $\beta > a, \alpha > k, \sqrt{k/(\alpha - k)} < (\beta - a)/b < 3\sqrt{3}$,

$$b - \frac{u^*(\beta - a - bu^*)}{2 + 2\sqrt{1 + u^{*2}}} > \frac{\left(\sqrt{u^{*2}+1} + u^{*2}\right)^2}{8(u^{*2}+1)^2} + \frac{1}{8}, \quad \frac{\gamma}{\alpha} > \frac{1}{2}, \tag{3.35}$$

hold, then the equilibrium point E^* of (1.3) is globally asymptotically stable.

(ii) Assume that $\beta > a$ and $b^2k/(\beta - a) > \max\{(\alpha - k)(\beta - a), b\alpha/2\}$ hold, then the equilibrium point E_1 of (1.3) is globally asymptotically stable.

(iii) Assume that $\beta < a$ and $k > \alpha$ hold, then the equilibrium point E_0 of (1.3) is globally asymptotically stable.

Example 3.8. Consider the following system:

$$\begin{aligned} X_{1t} - D_1\Delta X_1 &= 5X_2 - 0.6X_1 - 1.4X_1 - 2X_1^2 - 6X_1^3 - X_3\frac{2X_1^2}{1+2X_1^2}, \quad x \in \Omega,\ t > 0, \\ X_{2t} - D_2\Delta X_2 &= 1.4X_1 - X_2, \quad x \in \Omega,\ t > 0, \\ X_{3t} - D_3\Delta X_3 &= -X_3 - \sqrt{2}X_3^2 + X_3\frac{2X_1^2}{1+2X_1^2}, \quad x \in \Omega,\ t > 0, \\ \partial_\eta X_i &= 0, \quad i = 1,2,3,\ x \in \partial\Omega,\ t > 0, \\ X_i(x,0) &= X_{i0}(x) \ge 0, \quad i = 1,2,3,\ x \in \Omega. \end{aligned} \tag{3.36}$$

Using the software Matlab, one can obtain $u^* = v^* = 1.1274$, $w^* = 0.1199$. It is easy to see that the previous system satisfies the all conditions of Theorem 3.6(i). So the positive equilibrium point (0.5637,0.5637,0.1199) of the previous system is globally asymptotically stable.

4. Global Existence and Stability of Solutions for the System (1.4)

By [34–36], we have the following result.

Theorem 4.1. *If $u_0, v_0, w_0 \in W_p^1(\Omega), p > n$, then* (1.4) *has a unique nonnegative solution $u, v, w \in C([0,T), W_p^1(\Omega)) \bigcap C^\infty((0,T), C^\infty(\Omega))$, where $T \le +\infty$ is the maximal existence time of the solution. If the solution (u, v, w) satisfies the estimate*

$$\sup\left\{\|u(\cdot,t)\|W_p^1(\Omega), \|v(\cdot,t)\|W_p^1(\Omega), \|w(\cdot,t)\|W_p^1(\Omega) : 0 < t < T\right\} < \infty, \tag{4.1}$$

then $T = +\infty$. If, in addition, $u_0, v_0, w_0 \in W_p^2(\Omega)$, then $u, v, w \in C([0,\infty), W_p^2(\Omega))$.

In this section, we consider the existence and the convergence of global solutions to the system (1.4).

Theorem 4.2. *Let $\alpha_{11}, \alpha_{22} > 0$ and the space dimension $n < 6$. Suppose that $u_0, v_0, w_0 \in C^{2+\lambda}(\overline{\Omega})$ $(0 < \lambda < 1)$ are nonnegative functions and satisfy zero Neumann boundary conditions. Then* (1.4) *has a unique nonnegative solution $u, v, w \in C^{2+\lambda, 1+\lambda/2}(\overline{\Omega} \times [0,\infty))$.*

In order to prove Theorem 4.2, some preparations are collected firstly.

Lemma 4.3. *Let (u, v, w) be a solution of* (1.4). *Then*

$$\begin{gathered} u, v \ge 0, \quad 0 \le w \le M_1, \quad \text{in } Q_T \equiv \Omega \times (0,T), \\ \sup_{0<t<T} \|u(\cdot,t)\|_{L^1(\Omega)}, \sup_{0<t<T} \|v(\cdot,t)\|_{L^1(\Omega)} \le C_1(T), \\ \|u\|_{L^2(Q_T)}, \|v\|_{L^2(Q_T)} \le C_2(T), \end{gathered} \tag{4.2}$$

where $M_1 = \max\{\alpha/\gamma, \|w_0\|_{L^\infty(\Omega)}\}$.

Proof. From the maximum principle for parabolic equations, it is not hard to verify that $u, v, w \ge 0$ and w is bounded.

Multiplying the second equation of (1.4) by $(a + \beta)$, adding up the first equation of (1.4), and integrating the result over Ω, we obtain

$$\frac{d}{dt}\int_\Omega [u + (a+\beta)v]\, dx \le -a\int_\Omega v\, dx + \int_\Omega \left(\beta u - bu^2\right) dx. \tag{4.3}$$

Using Young inequality and Hölder inequality, we have

$$\int_\Omega \left(\beta u - bu^2\right) dx \le C_{2,1} - \frac{a}{a+\beta}\int_\Omega u\, dx, \tag{4.4}$$

where $C_{2,1} = (1/4b)(\beta + a/(a+\beta))^2|\Omega|$. It follows from (4.3) and (4.4) that

$$\frac{d}{dt}\int_{\Omega}[u + (a+\beta)v]\,dx \le C_{2,1} - \frac{a}{a+\beta}\int_{\Omega}[u + (a+\beta)v]\,dx. \tag{4.5}$$

Thus,

$$\|u(\cdot,t)\|_{L^1(\Omega)}, \|v(\cdot,t)\|_{L^1(\Omega)} \le C_{2,2}, \tag{4.6}$$

where $C_{2,2}$ depends on $\|v_0\|_{L^1(\Omega)}, \|u_0\|_{L^1(\Omega)}$ and coefficients of (1.4). In addition, there exists a positive constant $C_1(T)$, such that

$$\sup_{0<t<T}\|u(\cdot,t)\|_{L^1(\Omega)}, \sup_{0<t<T}\|v(\cdot,t)\|_{L^1(\Omega)} \le C_1(T). \tag{4.7}$$

Integrating the first equation of (1.4) over Ω, we have

$$\frac{d}{dt}\int_{\Omega}u\,dx \le \beta\int_{\Omega}v\,dx - b\int_{\Omega}u^2dx. \tag{4.8}$$

Integrating (4.8) from 0 to T, we have

$$\int_{\Omega}u(x,T)dx - \int_{\Omega}u(x,0)dx \le \beta\int_0^T\int_{\Omega}v\,dx\,dt - b\int_0^T\int_{\Omega}u^2dx\,dt. \tag{4.9}$$

According to (4.7), there exists a positive constant $C_2(T)$, such that

$$\|u\|_{L^2(Q_T)} \le C_2(T). \tag{4.10}$$

Multiplying the second equation of (1.4) by v and integrating it over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega}v^2dx &= -\int_{\Omega}(d_2 + 2\alpha_{22}v)|\nabla v|^2dx + \int_{\Omega}\left(uv - v^2\right)dx\\ &\le \frac{1}{2}\int_{\Omega}u^2dx - \frac{1}{2}\int_{\Omega}v^2dx.\end{aligned} \tag{4.11}$$

Integrating the previous inequation from 0 to T, we have

$$\|v\|_{L^2(Q_T)} \le C_2(T). \tag{4.12}$$

□

Lemma 4.4. *Let (u,v,w) be a solution of (1.4), $w_1 = (d_3 + \alpha_{33}w)w$, and $\tau < T$. Then there exists a positive constant $C_3(\tau)$ depending on $\|w_0\|_{W_2^1(\Omega)}$ and $\|w_0\|_{L^\infty(\Omega)}$, such that*

$$\|w_1\|_{W_2^{2,1}(Q_\tau)} \le C_3(\tau). \tag{4.13}$$

Furthermore $\nabla w_1 \in V_2(Q_\tau)$ and $\nabla w_1 \in L^{2(n+2)/n}(Q_\tau)$.

Proof. w_1 satisfies the equation

$$w_{1t} = (d_3 + 2\alpha_{33}w)\Delta w_1 + c_1 + c_2\frac{u^2}{1+u^2}, \tag{4.14}$$

where c_1, c_2 are functions of w and so are bounded because of Lemma 4.3.

Multiply the second equation of (1.4) by $-\Delta w_1$ and integrate it over Q_τ to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla w_1|^2(x,\tau)dx - \frac{1}{2}\int_\Omega |\nabla w_1|^2(x,0)dx + d_3\int_{Q_\tau}|\nabla w_1|^2 dx\,ds \\
&\quad\le \int_{Q_\tau}|\Delta w_1|\left|c_1 + c_2\frac{u^2}{1+u^2}\right|dx\,ds \\
&\quad\le m_1\|\Delta w_1\|_{L^2(Q_\tau)} \\
&\quad\le \frac{d_3}{2}\|\Delta w_1\|^2_{L^2(Q_\tau)} + \frac{m_1}{2d_3}.
\end{aligned} \tag{4.15}$$

Then

$$\begin{aligned}
&\int_\Omega |\nabla w_1|^2(x,\tau)dx + d_3\int_{Q_\tau}|\nabla w_1|^2 dx\,ds \\
&\quad\le \int_\Omega |\nabla w_1|^2(x,0)dx + \frac{m_1}{2d_3},
\end{aligned} \tag{4.16}$$

and $w_1 \in W_2^{2,1}(Q_T)$. From a disposal similar to the proof of Lemma 2.2 in [23], we have $\nabla w_1 \in V_2(Q_\tau)$. Using a standard embedding result, we obtain $\nabla w_1 \in L^{2(n+2)/n}(Q_\tau)$. □

Lemma 4.5 (see [23, Lemmas 2.3 and 2.4]). *Let $q > 1, \tilde{q} = 2 + 4q/n(q+1)$, $\tilde{\beta} \in (0,1)$, and let $C_T > 0$ be any number which may depend on T. Then there is a constant M_2 depending on $n, q, \Omega, \tilde{\beta}$, and C_T such that*

$$\|g\|_{L^{\tilde{q}}_{(Q_T)}} \le M_2\left\{1 + \left(\sup_{0\le t\le T}\|g(\cdot,t)\|_{L^{2q/(q+1)}(\Omega)}\right)^{4q/n(q+1)\tilde{q}}\|\nabla g\|^{2/\tilde{q}}_{L^2(Q_T)}\right\}, \tag{4.17}$$

for any $g \in C([0,T), W_2^1(\Omega))$ with $\left(\int_\Omega |g(\cdot,t)|^{\tilde{\beta}}dx\right)^{1/\tilde{\beta}} \le C_T$ for all $t \in [0,T]$.

To obtain L^∞-estimates of u, v, we establish L^q-estimates of u, v in the following lemma.

Lemma 4.6. *Let $\alpha_{11}, \alpha_{22} > 0$, $1 < q < 2(n+1)/(n-2)$, then there exist positive constants $C(q,T)$ and $C(T)$, such that*

$$\|u\|_{L^q(Q_T)}, \|v\|_{L^q(Q_T)} \le C(q,T), \qquad \|u\|_{V_2(Q_T)}, \|v\|_{V_2(Q_T)} \le C(T). \tag{4.18}$$

Proof. Multiply the first equation of (1.4) by qu^{q-1} for $q > 1$ and integrate by parts over Ω to obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} u^q dx \leq -q(q-1)\int_{\Omega} u^{q-2}(d_1 + 2\alpha_{11}u)|\nabla u|^2 dx \\ - \alpha_{13}(q-1)\int_{\Omega} \nabla(u^q)\cdot\nabla w\, dx + q\beta\int_{\Omega} u^{q-1}v\, dx.\end{aligned} \tag{4.19}$$

Integrating (4.19) from 0 to t, we have

$$\begin{aligned}\int_{\Omega} u^q(x,t)dx - \int_{\Omega} u_0^q(x)dx + q(q-1)\int_{Q_t} u^{q-2}(d_1 + 2\alpha_{11}u)|\nabla u|^2 dx\, ds \\ \leq -\alpha_{13}(q-1)\int_{Q_t} \nabla(u^q)\cdot\nabla w\, dx\, ds + q\beta\int_{Q_t} u^{q-1}v\, dx\, ds.\end{aligned} \tag{4.20}$$

Then substitution of $u^{q-2}|\nabla u|^2 = (4/q^2)|\nabla(u^{q/2})|^2$, $u^{q-1}|\nabla u|^2 = (4/(q+1)^2)|\nabla(u^{(q+1)/2})|^2$ into (4.20) leads to

$$\begin{aligned}\int_{\Omega} u^q(x,t)dx + \frac{4(q-1)d_1}{q}\int_{Q_t}\left|\nabla\left(u^{q/2}\right)\right|^2 dx\, ds + \frac{8\alpha_{11}q(q-1)}{(q+1)^2}\int_{Q_t}\left|\nabla\left(u^{(q+1)/2}\right)\right|^2 dx\, ds \\ \leq \int_{\Omega} u_0^q(x)dx - \alpha_{13}(q-1)\int_{Q_t}\nabla(u^q)\cdot\nabla w\, dx\, ds + q\beta\int_{Q_t} u^{q-1}v.\end{aligned} \tag{4.21}$$

It follows from Hölder inequality and Lemma 4.3 that

$$\begin{aligned}q\beta\int_{Q_t} u^{q-1}v &\leq q\beta\|u^{(q-1)/2}\|_{L^{n+2}(Q_T)}\|u^{(q-1)/2}\|_{L^{2(n+2)/n}(Q_T)}\|v\|_{L^2(Q_T)} \\ &\leq C_{3,1}\|u\|^{q-1}_{L^{(q-1)(n+2)/2}(Q_T)}.\end{aligned} \tag{4.22}$$

Note that $1/2 + 1/(n+2) + n/2(n+2) = 1$, and $n+2 \geq 2(n+2)/n$ for $n \geq 2$. From Hölder inequality, Young inequality, and Lemma 4.4, we have

$$\begin{aligned}\left|\int_{Q_T}\nabla(u^q)\cdot\nabla w\, dx\, dt\right| &= \frac{2q}{q+1}\left|\int_{Q_T} u^{(q-1)/2}\nabla w\cdot\nabla\left(u^{(q+1)/2}\right)dx\, dt\right| \\ &\leq \frac{2q}{q+1}\|u\|^{(q-1)/2}_{L^{(q-1)(n+2)/2}(Q_T)}\|\nabla w\|_{L^{2(n+2)/n}(Q_T)}\left\|\nabla\left(u^{(q+1)/2}\right)\right\|_{L^2(Q_T)} \\ &\leq C_{3,2}\|u\|^{(q-1)/2}_{L^{(q-1)(n+2)/2}(Q_T)}\left\|\nabla\left(u^{(q+1)/2}\right)\right\|_{L^2(Q_T)} \\ &\leq \frac{C_{3,2}\epsilon_1}{2}\left\|\nabla\left(u^{(q+1)/2}\right)\right\|^2_{L^2(Q_T)} + \frac{C_{3,2}}{2\epsilon_1}\|u\|^{q-1}_{L^{(q-1)(n+2)/2}(Q_T)}.\end{aligned} \tag{4.23}$$

Substitution of (4.22) and (4.23) into (4.21) leads to

$$\begin{aligned}&\int_\Omega u_1^{2q/(q+1)}(x,t)dx+\frac{4(q-1)d_1}{q}\int_{Q_t}\left|\nabla\left(u^{q/2}\right)\right|^2dx\,dt+\frac{8\alpha_{11}q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla u_1|^2dx\,dt\\&\quad\le\int_\Omega u_0^q(x)dx+\epsilon C_{3,3}\|\nabla u_1\|^2_{L^2(Q_T)}+\frac{C_{3,4}}{\epsilon}\|u_1\|^{2(q-1)/(q+1)}_{L^{(q-1)(n+2)/(q+1)}(Q_T)},\end{aligned}\tag{4.24}$$

where $\epsilon>0$ is arbitrary and $u_1=u^{(q+1)/2}$.

Choose ϵ such that

$$\epsilon\alpha_{13}C_{3,3}<\frac{4\alpha_{11}q}{(q+1)^2},\tag{4.25}$$

then it follows from (4.24) that

$$\sup_{0<t<T}\int_\Omega u_1^{2q/(q+1)}(x,t)dx+\int_{Q_T}|\nabla u_1|^2dx\,dt\le C_{3,5}\left(1+\|u_1\|^{2(q-1)/(q+1)}_{L^{(q-1)(n+2)/(q+1)}(Q_T)}\right).\tag{4.26}$$

Let

$$E\equiv\sup_{0<t<T}\int_\Omega u_1^{2q/(q+1)}(x,t)dx+\int_{Q_T}|\nabla u_1|^2dx\,dt.\tag{4.27}$$

Then $(q-1)(n+2)/(q+1)<\tilde{q}$ for

$$1<q<\frac{n(n+4)}{n^2-4}.\tag{4.28}$$

According to Lemma 4.5 and the definition of E, we can see

$$\|u_1\|_{L^{\tilde{q}}(Q_T)}\le M_3\left(1+E^{2/n\tilde{q}}E^{1/\tilde{q}}\right).\tag{4.29}$$

Combining (4.26) and (4.29), we have

$$\begin{aligned}E&\le C_{3,5}\left(1+\|u_1\|^{2(q-1)/(q+1)}_{L^{\tilde{q}}(Q_T)}\right)\\&\le C_{3,5}\left\{1+\left[M_3\left(1+E^{2/n\tilde{q}}E^{1/\tilde{q}}\right)\right]^{2(q-1)/(q+1)}\right\}\\&\le C_{3,6}(1+E^\mu),\end{aligned}\tag{4.30}$$

where $\mu=(2(q-1)/\tilde{q}(q+1))(2/n+1)<1/\tilde{q}[4q/n(q+1)+2]=1$. Therefore E is bounded from (4.30).

From (4.29), we have $u_1 \in L^{\tilde{q}}(Q_T)$. Namely, $u \in L^r(Q_T)$, $r = \tilde{q}(q+1)/2 = q+1+2q/n$. Combining (4.28), we have $u \in L^r(Q_T)$, where $r < 2(n+1)/(n-2)$.

Setting $q = 2$ in (4.20) (it is easily checked that $q = 2 < n(n+4)/(n^2-4)$, i.e., $n = 2,3,4,5$), we have $\|u\|_{V_2(Q_T)} \le C_T$.

Multiplying the second equation of (1.4) by qv^{q-1} and integrating it over Ω, we have

$$\frac{d}{dt}\int_\Omega v^q dx = -q(q-1)\int_\Omega v^{q-2}(d_2 + 2\alpha_{22}v)|\nabla v|^2 dx + q\int_\Omega v^{q-1}(u-v)dx. \tag{4.31}$$

The result of v can be obtained from an analogue of the previous proof of u's. □

Lemma 4.7. *Let $n = 2,3,4,5$, then there exists a positive constant M_4 such that*

$$\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)} \le M_4. \tag{4.32}$$

Proof. We will prove this lemma by [37, Theorem 7.1, page 181]. At first, we rewrite the first two equations of (1.4) as

$$\begin{gathered}\frac{\partial u}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(a_i u) + u\left(a + bu + cu^2 + \frac{uw}{1+u^2}\right) = \beta v,\\ \frac{\partial v}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial v}{\partial x_j}\right) + v = u,\end{gathered} \tag{4.33}$$

where $a_{ij}(x,t) = (d_1 + 2\alpha_{11}u + \alpha_{13}w)\delta_{ij}$, $a_i(x,t) = \alpha_{13}(\partial w/\partial x_i)$, $b_{ij}(x,t) = (d_2 + 2\alpha_{22}v)\delta_{ij}$, δ_{ij} is *Kronecker* symbol. It follows from Lemma 4.6 that $u \in L^q(Q_T)$, $1 < q < 2(n+1)/(n-2)$.

By the third equation of (1.4), we have

$$w_t = \nabla[(d_3 + 2\alpha_{33}w)\nabla w] - kw - \gamma w^2 + \frac{\alpha u^2 w}{1+u^2}. \tag{4.34}$$

It follows from Lemma 4.3 that $d_3 + 2\alpha_{33}w, -kw - \gamma w^2 + \alpha u^2 w/(1+u^2)$ is bounded in Q_T. Applying Theorem 10.1 [37, Page 204] to (4.34), we have

$$w \in C^{\lambda_1,\lambda_1/2}\left(\overline{Q}_T\right), \quad \lambda_1 > 0. \tag{4.35}$$

Recall that $w_1 = (d_3 + \alpha_{33}w)w$ satisfy (4.14) in Lemma 4.4, that is,

$$w_{1t} = (d_3 + \alpha_{33}w)\Delta w_1 + c_1 + c_2\frac{u^2}{1+u^2}, \tag{4.36}$$

where $c_1 + c_2(u^2/(1+u^2))$ is bounded. Since $d_3 + 2\alpha_{33}w \in C^{\lambda_1,\lambda_1/2}(\overline{Q}_T)$ by (4.35), applying Theorem 9.1 [37, page 341-342] to (4.36), we have

$$w_1 \in W_q^{2,1}(Q_T). \tag{4.37}$$

It follows from [37, Lemma 3.3, page 80] that $\nabla w_1 \in L^{(n+2)q/(n+2-q)}(Q_T)$ and so $\nabla w = \nabla w_1/(d_3 + 2\alpha_{33}w) \in L^{(n+2)q/(n+2-q)}(Q_T)$. Recall from Lemma 4.6 that $u, v \in V_2(Q_T)$, so that $u, v \in L^\infty(Q_T)$ by applying Theorem 7.1 [37, Page 181] to (4.33). □

Proof of Theorem 4.2. Firstly, Theorem 4.2 can be proved in a similar way as Theorem 2 in [21, 25] when the space dimension $n = 1$.

Secondly, for $2 \le n < 6$, applying Lemma 3.3 [37, Page 80] to (4.36), we have

$$w_1 \in C^{1+\lambda_2,(1+\lambda_2)/2}\left(\overline{Q}_T\right), \quad 0 < \lambda_2 < 1. \tag{4.38}$$

Since $w = (-d_3 + \sqrt{d_3^2 + 4\alpha_{33}w_1})/2\alpha_{33}$, we obtain

$$w \in C^{1+\lambda_2,(1+\lambda_2)/2}\left(\overline{Q}_T\right), \quad 0 < \lambda_2 < 1. \tag{4.39}$$

The first two equations can be written in the divergence form as

$$\begin{aligned} u_t &= \nabla[(d_1 + 2\alpha_{11}u + \alpha_{13}w)\nabla u + \alpha_{13}u\nabla w] + g_1(x,t), \\ v_t &= \nabla[(d_2 + 2\alpha_{22}v)\nabla v] + g_2(x,t), \end{aligned} \tag{4.40}$$

where $g_1 = \beta v - au - bu^2 - cu^3 - u^2 w/(1+u^2) \in L^\infty(Q_T)$, $g_2 = u - v \in L^\infty(Q_T)$. It follows from Lemmas 4.1, 4.5, and (4.39) that $u, v, w, \nabla w$ are bounded. Thus applying Theorem 10.1 [37, Page 204] to (4.40) leads to

$$u, v \in C^{\lambda_3,\lambda_3/2}\left(\overline{Q}_T\right), \quad 0 < \lambda_3 < 1. \tag{4.41}$$

We rewrite the third equation of (1.4) as

$$w_t = (d_3 + 2\alpha_{33}w)\Delta w + g_3(x,t), \tag{4.42}$$

where $g_3 = 2\alpha_{33}|\nabla w|^2 - kw - \gamma w^2 + \alpha u^2 w/(1+u^2) \in C^{\lambda_3,\lambda_3/2}(\overline{Q}_T)$. Applying Schauder estimate [29, Theorem 3.2.6, page 114] to (4.42) gives

$$w \in C^{2+\lambda_4,(2+\lambda_4)/2}\left(\overline{Q}_T\right), \quad \text{where } \lambda_4 = \min\{\lambda, \lambda_3\}. \tag{4.43}$$

Let

$$u_2 = (d_1 + \alpha_{11}u + \alpha_{13}w)u, \qquad v_2 = (d_2 + \alpha_{22}v)v, \tag{4.44}$$

then

$$\begin{aligned} u_{2t} &= (d_1 + 2\alpha_{11}u + \alpha_{13}w)\Delta u_2 + g_4(x,t), \\ v_{2t} &= (d_2 + 2\alpha_{22}v)\Delta v_2 + g_5(x,t), \end{aligned} \tag{4.45}$$

where $g_4 = (d_1+2\alpha_{11}u+\alpha_{13}w)(\beta v-au-bu^2-cu^3-u^2w/(1+u^2))+\alpha_{13}uw_t$, $g_5 = (d_2+2\alpha_{22}v)(u-v)$. From (4.41), we have $d_1 + 2\alpha_{11}u + \alpha_{13}w, d_2 + 2\alpha_{22}v \in C^{\lambda_3,\lambda_3/2}(\overline{Q}_T)$. It follows from (4.41) and (4.43) that $g_4(x,t), g_5(x,t) \in C^{\lambda_4,\lambda_4/2}(\overline{Q}_T)$. Applying Schauder estimate to (4.45) gives

$$u_2, v_2 \in C^{2+\lambda_4,(2+\lambda_4)/2}\left(\overline{Q}_T\right). \tag{4.46}$$

Solving equations (4.44) for u, v, respectively, we have

$$u, v \in C^{2+\lambda_4,(2+\lambda_4)/2}\left(\overline{Q}_T\right). \tag{4.47}$$

In particular, to conclude $u, v, w \in C^{2+\lambda,(2+\lambda)/2}(\overline{Q}_T)$, we need to repeat the above bootstrap technique. Since T is arbitrary, so the classical solution (u, v, w) of (1.4) exists globally in time.

Now we discuss the global stability of the positive equilibrium $E^*(u^*, v^*, w^*)$ (see Section 2) for (1.4). □

Theorem 4.8. *Assume that the all conditions in Theorem 4.2,* (2.1)*, and*

$$\frac{1}{\beta}\left(a + bu^* + cu^{*2}\right) > 2 + \frac{\left(u^* + \sqrt{1 + u^{*2}}\right)^2}{8} + \frac{u^{*4}}{2\beta^2}, \qquad \frac{\gamma}{\alpha} > \frac{1}{2(1 + u^{*2})^2} \tag{4.48}$$

hold. Let (u^*, v^*, w^*) *be the unique positive equilibrium point of* (1.4)*, and let* (u, v, w) *be a positive solution for* (1.4)*. Then*

$$\|u(\cdot,t) - u^*\|_{L^2(\Omega)} \longrightarrow 0, \quad \|v(\cdot,t) - v^*\|_{L^2(\Omega)} \longrightarrow 0, \quad \|w(\cdot,t) - w^*\|_{L^2(\Omega)} \longrightarrow 0 \quad (t \longrightarrow \infty), \tag{4.49}$$

provided that $d_1 \cdot d_2 \cdot d_3$ *is large enough.*

Proof. Define the Lyapunov function

$$H(u, v, w) = \frac{1}{2\beta}\int_\Omega (u - u^*)^2 dx + \frac{1}{2}\int_\Omega (v - v^*)^2 dx + \frac{1}{\alpha}\int_\Omega \left(w - w^* - w^* \ln \frac{w}{w^*}\right)dx. \tag{4.50}$$

Let (u, v, w) be a positive solution of (1.4), Then

$$\begin{aligned}
\frac{dH}{dt} \leq & -\int_{\Omega}\left[\frac{1}{\beta}(d_1 + 2\alpha_{11}u + \alpha_{13}w)|\nabla u|^2 + (d_2 + 2\alpha_{22}v)|\nabla v|^2 \right.\\
& \left. + \frac{1}{\alpha}(d_3 + 2\alpha_{33}w)\frac{w^*}{w^2}|\nabla w|^2 + \frac{1}{\beta}\alpha_{13}u\nabla u\nabla w\right]dx \\
& -\int_{\Omega}\left\{(u-u^*)^2\frac{1}{\beta}\left[a + b(u+u^*) + c\left(u^2 + uu^* + u^{*2}\right) + \frac{w(u+u^*)}{(1+u^2)(1+u^{*2})}\right] - 2\right. \\
& \left. - \frac{1}{2}\left(\frac{u+u^*}{1+u^2} - \frac{u^{*2}}{\beta}\right)^2\right\}dx - \frac{1}{2}\int_{\Omega}(v-v^*)^2dx \\
& -\int_{\Omega}(w-w^*)^2\left[\frac{\gamma}{\alpha} - \frac{1}{2(1+u^{*2})^2}\right]dx.
\end{aligned} \tag{4.51}$$

The first integrand in the right hand of the previous inequality is positive definite if

$$\frac{4\beta}{\alpha}w^*(d_1 + 2\alpha_{11}u + \alpha_{13}w)(d_2 + 2\alpha_{22}v)(d_3 + 2\alpha_{33}w) > \alpha_{13}^2u^2w^2(d_2 + 2\alpha_{22}v). \tag{4.52}$$

Therefore, when the all conditions in Theorem 4.8 hold, there exists a positive constant δ such that

$$\frac{dH(u,v,w)}{dt} \leq -\delta\int_{\Omega}\left[(u-u^*)^2 + (v-v^*)^2 + (w-w^*)^2\right]dx. \tag{4.53}$$

This implies that $\|u(\cdot,t) - u^*\|_{L^2(\Omega)}$, $\|v(\cdot,t) - v^*\|_{L^2(\Omega)}$, $\|w(\cdot,t) - w^*\|_{L^2(\Omega)} \to 0$ as $t \to \infty$. So the proof of Theorem 4.8 is completed. □

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