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Research Article

The Existence and Behavior of Solutions for Nonlocal Boundary Problems

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The purpose of this work is to investigate the uniqueness and existence of local solutions for the boundary value problem of a quasilinear parabolic equation. The result is obtained via the abstract theory of maximal regularity. Applications are given to some model problems in nonstationary radiative heat transfer and reaction-diffusion equation with nonlocal boundary flux conditions.

1. Introduction

The existence of solutions for quasilinear parabolic equation with boundary conditions and initial conditions can be discussed by maximal regularity, and more and more works on this field show that the maximal regularity method is efficient. Here we will use some of recently results developed by H. Amann to investigate a specific boundary value problems and then apply the existence theorem to two nonlocal problems.

This paper consists of three parts. In the next section we present and prove the existence and unique theorem of an abstract boundary problem. Then we give some applications of the results in Sections 3 and 4 to two reaction-diffusion model problems that arise from nonstationary radiative heat transfer in a system of moving absolutely black bodies and a reaction-diffusion equation with nonlocal boundary flux conditions.

2. Notations and Abstract Result

We consider the following quasilinear parabolic initial boundary value problem (IBVP for short):

$$u_t + \mathcal{A}(t, x, u)u = f(t, x, u, \nabla u), \quad \text{in } Q_T,$$

$$\mathcal{B}(t,x,u)u = \delta g(t,x,u,\nabla u), \quad \text{on } \partial\Omega,$$
$$u(x,0) = u_0(x), \quad \text{on } \Omega, \tag{2.1}$$

where Ω is a bounded strictly Lipschitz domain with its boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$, $Q_T = [0,T) \times \Omega$,

$$\mathcal{A}(t,x,u)u = -\nabla(\mathbf{a}(t,x,u)\nabla u), \tag{2.2}$$

and $-\mathcal{A}$ is a second-order strongly elliptic differential operator with the boundary operator given by

$$\mathcal{B}(t,x,u)u := \delta\partial_{\nu_{\mathbf{a}}} u + (1-\delta)\gamma u. \tag{2.3}$$

The coefficient matrix $\mathbf{a} = (a_{ij})_{n\times n}$ satisfies regularity conditions on $\overline{Q}_T \times \mathbb{R}$, respectively. The directional derivative $\partial_{\nu_{\mathbf{a}}} u := \gamma \mathbf{a}\nabla u \cdot \nu$, ν is the outer unit-normal vector on Γ; the function $\delta : \Gamma \to \{0, 1\}$ is defined as $\delta^{-1}(j) := \Gamma_j$ for $j = 0,1$; γ denotes the trace operator.

We introduce precise assumptions:

$$f : \overline{Q}_T \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R},$$
$$g : \mathbb{R}^+ \times \Gamma \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}, \tag{2.4}$$

where ($\mathbb{R}^+ := [0,+\infty)$) are Carathéodory functions; that is, f (resp., g) is measurable in $(t,x) \in Q_T$ (resp., in $(t,x,y) \in \mathbb{R}^+ \times \Gamma \times \Omega$) for each $u \in \mathbb{R}$ and continuous in u for a.e. $(t,x) \in Q_T$ (resp., $(t,x) \in \mathbb{R}^+ \times \Gamma$). More general, the function g can be a nonlocal function, for example, $g(t,x,u) = \int_\Omega k(t,x,y,u)dy$ or $g(t,x,u) = \int_\Gamma k(t,x,u)d\sigma$.

Let X and Y be Banach spaces, we introduce some notations as follows:

(i) $J_T := [0,T)$, $\mathring{J}_T := J_T \setminus \{0\}$. $\alpha \wedge \beta := \min\{\alpha,\beta\}$, $\alpha \vee \beta := \max\{\alpha,\beta\}$.

(ii) $\mathfrak{D}(D,Y) := \{\phi \in C^\infty(D), \phi : D \mapsto Y, \ \operatorname{supp}\phi \subset D\}$ for $D \subset \mathbb{R}^l$, $\mathfrak{D}_1 := \{v \in \mathfrak{D}(J_T \times \overline{\Omega})\} \cap \{v|_{\Gamma_0} = 0\}$.

(iii) $\mathcal{L}(X,Y) := \{$all continuous linear operators from X into $Y\}$, and $\mathcal{L}(X) := \mathcal{L}(X,X)$.

(iv) $f(t,u)$ denotes the Nemytskii operator induced by $f(t,x,u(t,x),\nabla u(t,x))$.

(v) $C^{1-}(X,Y)$ denotes the set of all locally Lipschitz-continuous functions from X into Y.

(vi) $\mathrm{Car}_{0,\lambda,\overline{\lambda}}(M \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, λ, and $\overline{\lambda} \geq 1$, denotes the set of all Carathéodory functions f on $m \in M$ such that $f(m, 0) = 0$, and there exists a nondecreasing function $\psi \geq 0$ with

$$\begin{aligned} \left|f(m,u,\xi) - f(m,v,\xi)\right| &\leq \psi(\rho)\left(1 + |\xi|^{\lambda-1}\right)|u - v|, \\ \left|f(m,u,\xi) - f(m,u,\eta)\right| &\leq \psi(\rho)\left(1 + |\xi|^{\overline{\lambda}-1} + |\eta|^{\overline{\lambda}-1}\right)|\xi - \eta| \end{aligned} \quad \text{for } |u|,\ |v| \leq \rho. \tag{2.5}$$

Particularly, f is independent of ξ if $\lambda \wedge \overline{\lambda} \leq 1$.

(vii) $W_p^s(\Omega)$ denotes the Sobolev-Slobodeckii space for $s \in \mathbb{R}$ and $p \geq 1$ with the norm $\|\cdot\|_{W_p^s}$, especially, $W_p^0(\Omega) = L_p(\Omega)$; and

$$\partial W_p^s := \partial W_p^s(\Gamma) := W_p^{s-1/p}(\Gamma_0) \times W_p^{s-1-1/p}(\Gamma_1) \quad (p > 1). \tag{2.6}$$

(viii) $W_{p,\mathcal{B}}^s(\Omega)$, $s \in [-2,\ 2] \setminus \{\mathbb{Z} + 1/p\}$ ($\mathbb{Z}$ is the set of integral numbers), is defined as

$$W_{q,\mathcal{B}}^s \begin{cases} \left\{u \in W_q^s; \mathcal{B}u = 0\right\}, & 1 + \dfrac{1}{q} < s \leq 2, \\ \left\{u \in W_q^s; \gamma u = 0 \text{ on } \Gamma_0\right\}, & \dfrac{1}{q} < s < 1 + \dfrac{1}{q}, \\ W_q^s, & 0 \leq s < \dfrac{1}{q}, \\ \left(W_{q',\mathcal{B}}^{-s}\right)', & -2 \leq s < 0,\ s \notin \mathbb{Z} + \dfrac{1}{q}, \end{cases} \tag{2.7}$$

where $q' = q/(q-1)$, X' is the dual space of X, and $\mathcal{B}$ is the formally adjoint operator.

(x) $\mathbb{W}_p^1(J) := \mathbb{W}_p^1(J, (E_1, E_0)) := W_p^1(J, E_0) \cap L_p(J, E_1)$ if $E_1 \overset{d}{\hookrightarrow} E_0$ and J is an interval in $\mathbb{R}$.

(xi) $\mathcal{M}R_p(J_T) := \mathcal{M}R_p(J_T, (E_1, E_0))$ denotes all maps B possessing the property of *maximal L_p regularity* on J_T with respect to (E_1, E_0), that is, given $h \in L_p(J_T, E_0)$, the initial problem

$$\dot{v} + Bv = h, \quad \text{in } \overset{\circ}{J}_T, \quad v(0) = 0 \tag{2.8}$$

has a unique solution $v \in \mathbb{W}_p^1(J_T, (E_1, E_0))$.

Now we turn to discuss the local existence result. We write

$$E_1 = W_{p,B}^s(\Omega), \qquad E_0 = W_{p,B}^{s-2}(\Omega), \qquad E = E_{1/p',p} = W_{p,B}^{s-2/p}, \tag{2.9}$$

then,

$$\mathbb{W}_p^1(J_T,(E_1,E_0)) \hookrightarrow \begin{cases} C(J_T,E), \\ C\left(\overline{Q}_T\right), & \text{if } p > n+2. \end{cases} \tag{2.10}$$

Exactly, $\mathbb{W}_p^1(J_T,(E_1,E_0)) \hookrightarrow \mathrm{BUC}(Q_T)$ as $p > n+2$, where $\mathrm{BUC}(Q_T)$ denotes the Banach space of all functions being bounded and uniformly continuous in Q_T. So, we will not emphasize it in the following.

A (weak) solution u of IBVP (2.1) is defined as a $\mathbb{W}_p^1(J_T,(E_1,E_0))$ function u, $s \in [1, 1+1/p)$, satisfying

$$\begin{aligned} &\int_0^T (\langle -\dot{v}, u\rangle + \langle \nabla v, \mathbf{a}(t,u)\nabla u\rangle)dt - \langle v(0), u_0\rangle \\ &\quad = \int_0^T \{\langle v, f(t,u)\rangle + \langle \gamma v, g(t,u)\rangle_{\partial_1}\}dt \quad \forall v \in \mathfrak{D}_1, \end{aligned} \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\partial_j}$ denote the obvious duality pairings on Ω and Γ_j, respectively.

Set

$$f(t,u) := f_0(t) + f_1(t,u), g(t,u) := g_0(t) + g_1(t,u) \quad \text{with} \quad f_1(t,0) = g_1(t,0) = 0. \tag{2.12}$$

After these preparations we introduce the following hypotheses:

(H1) $p > n+2$ and $s \in [1, 1+1/p)$.

(H2) $\mathbf{a}(t,x,u) \in C^{0,\alpha,1-}(\overline{J}_T \times \overline{\Omega} \times \mathbb{R}, \mathbb{R}^{n\times n})$ *with* $\alpha + 1 > s$, *and there exists a* $\delta_0 \in (0,1)$ *such that*

$$\delta_0|\xi|^2 \le \xi \cdot \mathbf{a}(t,x,u)\xi \le |\xi|^2/\delta_0, \quad \forall (t,x,u) \in J_T \times \overline{\Omega} \times \mathbb{R}. \tag{2.13}$$

$(f_0, g_0) \in L_p(J, E_0 \times \partial W_p^s(\Gamma))$, $f_1 \in \mathrm{Car}_{0,\lambda_0,\lambda_1}(\overline{Q}_T \times \mathbb{R} \times \mathbb{R}^n)$ with $\lambda_0 \in [1, 2)$, and $\lambda_1 < 1 + p(s-1)/(2+p(1-s))$.

(H3) $g_1(t,\cdot) \in C^{1-}(\mathbb{W}_p^1(J_T), L_r(J_T, \partial W_p^s(\Gamma)))$ for some $r > p$.

Theorem 2.1. *Let assumptions (H1)–(H3) be satisfied. Then for each* $u_0 \in E$ *the quasilinear problem* (2.1) *possesses a unique weak solution* $u(t,x) \in \mathbb{W}_p^1(J_{T^*},(E_1,E_0))$ *for some* $T^* > 0$.

Proof. Recall that

$$E_1 \hookrightarrow C^{s-n/p}\left(\overline{\Omega}\right), \qquad E \hookrightarrow C^{s-(n+2)/p}\left(\overline{\Omega}\right). \tag{2.14}$$

The Nemytskii operator $\mathbf{a}(\cdot,u)$ is defined as $\mathbf{a}(\cdot,u)(t,x) := \mathbf{a}(t,x,u(t,x))$. The fact

$$\mathbf{a}(\cdot,u)(t,x) \in C^{0,\alpha\wedge(s-(n+2)/p)}\left(\overline{Q}_T\right) \tag{2.15}$$

shows the maximal regularity of the operator $\mathcal{A}$. By [1, Theorem 2.1], if, for $t \leq T$, $f_1 \in C^{1-}(\mathbb{W}_p^1(J_t), L_r(J_t, E_0))$ for some $r > p$, then the existence and the uniqueness of a local solution will be proved.

The remain work is to check the Lipschitz-continuity. Set

$$q := \frac{np}{n + (2-s)p},$$
$$\theta := 1 + \frac{p}{2}(1-s). \tag{2.16}$$

Then $L_q(\Omega) \hookrightarrow E_0$. So, for $u, v \in \mathbb{W}_p^1(J_t)$ with $|u|,\ |v| \leq \rho$ we have

$$\begin{aligned}
&\left\| f_1(t,x,u,\nabla u) - f_1(t,x,u,\nabla v) \right\|_{L_r(J_t,E_0)} \\
&\quad \leq \left\| f_1(t,x,u,\nabla u) - f_1(t,x,u,\nabla v) \right\|_{L_r(J_t,L_q(\Omega))} \\
&\quad \leq c(\rho) \left\| |\nabla u|^{\lambda_1 - 1} |\nabla (u-v)| \right\|_{L_r(J_t,L_q(\Omega))} \\
&\quad \leq c(\rho) \left\| \|\nabla u\|^{\lambda_1-1}_{L_{(\lambda_1-1)pq/(p-q)}(\Omega)} \|\nabla(u-v)\|_{L_p(\Omega)} \right\|_{L_r(J_t)}.
\end{aligned} \tag{2.17}$$

From $(\lambda_1 - 1)q < p - q$, we infer that

$$\begin{aligned}
&\left\| f_1(t,x,u,\nabla u) - f_1(t,x,u,\nabla v) \right\|_{L_r(J_t,E_0)} \\
&\quad \leq c(\rho) \left\| \|u\|^{\lambda_1-1}_{W_p^1(\Omega)} \|u-v\|_{W_p^1(\Omega)} \right\|_{L_r(J_t)} \\
&\quad \leq c(\rho)\ \|u\|^{(\lambda_1-1)(1-\theta)}_{C(I_t,E)} \cdot \|u-v\|^{1-\theta}_{C(I_t,E)} \cdot \left\| \|u\|^{(\lambda_1-1)\theta}_{E_1} \|u-v\|^{\theta}_{E_1} \right\|_{L_r(J_t)} \\
&\quad \leq c(\rho)\ \|u\|^{(\lambda_1-1)(1-\theta)}_{C(I_t,E)} \cdot \|u-v\|^{1-\theta}_{C(I_t,E)} \cdot \|u\|^{\theta(\lambda_1-1)}_{L_{r^*}(J_t,E_1)} \cdot \|u-v\|^{\theta}_{L_p(J_t,E_1)},
\end{aligned} \tag{2.18}$$

where $r^* := (\lambda_1 - 1)pr\theta/(p - \theta r)$. Note that $\lambda_1 \theta < 1$, we can choose $r > p$ such that

$$\left\| f_1(t,x,u,\nabla u) - f_1(t,x,u,\nabla v) \right\|_{L_r(J_t,E_0)} \leq c\Big(\|u\|_{\mathbb{W}_p^1(J_t)}\Big) \cdot \|u-v\|_{\mathbb{W}_p^1(J_t)}. \tag{2.19}$$

On the other hand, the hypotheses guarantee that

$$\begin{aligned}
&\left\| f(t,x,u,\nabla v) - f(t,x,v,\nabla v) \right\|_{L_r(J_t,E_0)} \\
&\quad \leq \left\| f(t,x,u,\nabla v) - f(t,x,v,\nabla v) \right\|_{L_r(J_t,L_q(\Omega))} \\
&\quad \leq c(\rho) \left\| |\nabla v|^{\lambda_0-1} |u-v| \right\|_{L_r(J_t,L_q(\Omega))} \\
&\quad \leq c(\rho) \|u-v\|_{C(J_t,E)} \cdot \left\| |\nabla v|^{\lambda_0-1} \right\|_{L_r\left(J_t,L_q(\Omega)\right)}.
\end{aligned} \tag{2.20}$$

Due to $q < p$ and $\lambda_0 \in [1,\ 2)$, Hölder inequality follows that

$$\left\| f(t,x,u,\nabla v) - f(t,x,v,\nabla v) \right\|_{L_r(J_t,E_0)} \le c(\rho)\ \|u - v\|_{C(J_t,E)} \cdot \left(\int_0^t \|\nabla v\|_{L_p(\Omega)}^{(\lambda_0 - 1)r} d\tau \right)^{1/r}. \quad (2.21)$$

The hypothesis of λ_0 means that one can find an $r > p$ such that

$$\left\| f(t,x,u,\nabla v) - f(t,x,v,\nabla v) \right\|_{L_r(J_t,E_0)} \le c\left(\rho, \|v\|_{\mathbb{W}_p^1(J_t)}\right) \cdot \|u - v\|_{\mathbb{W}_p^1(J_t)}. \quad (2.22)$$

Obviously, if $\lambda_0 = 1$, the above inequality is followed from (2.20) immediately. Hence it follows from (2.19) and (2.22) that

$$\left\| f(t,x,u,\nabla u) - f(t,x,v,\nabla v) \right\|_{L_r(J_t,E_0)} \le c\left(\|u, v\|_{\mathbb{W}_p^1(J_t)}\right) \cdot \|u - v\|_{\mathbb{W}_p^1(J_t)}. \quad (2.23)$$

This ends the proof. □

We apply the above theorem to the following two examples in next sections. For this, in the remainder we suppose that hypotheses (H1)-(H2) hold and that

$$\Gamma = \Gamma_1 (\Gamma_0 = \emptyset), \qquad \dot{p} := \frac{(n-1)p}{n + (1-s)p}. \quad (2.24)$$

3. A Radiative Heat Transfer Problem

We see a nonlinear initial-boundary value problem, which, in particular, describes a nonstationary radiative heat transfer in a system of absolutely black bodies (e.g., refer to [2]). A problem is

$$\begin{gathered} u_t - \nabla(\mathbf{a}(t,x,u)\nabla u) = f_0(t,x) \quad \text{in } Q_T, \\ \partial_{\nu_{\mathbf{a}(u)}} u = \int_\Gamma h(u(t,y))\varphi(t,x,y)\,d\sigma(y) - h(u(t,x)) + g_0(t,x) \quad \text{on } [0,T] \times \Gamma, \\ u(0,x) = u_0(x), \quad \text{on } \overline{\Omega}. \end{gathered} \quad (3.1)$$

3.1. Local Solvability

We assume that (Hr)

(Hr1) $\varphi \in L_r(J_T, L_{\dot{p}}(\Gamma \times \Gamma))$;

(Hr2) *h is locally Lipschitz continuous and* $h(0) = 0$.

Theorem 3.1. *Let assumptions (H1)-(H2) and (Hr) be satisfied. Then problem* (3.1)*, for all $u_0 \in E$, has a unique $u \in \mathbb{W}_p^1(J_{T^*})$ for some $T^* > 0$.*

Proof. Note that the embedding (2.14) holds:

$$L_{\dot{p}}(\Gamma) \hookrightarrow \partial W_p^s, \qquad L_{\dot{p}}(\Gamma) \hookrightarrow W_{p'}^{1/p-s}(\Gamma) = \left(W_p^{s-1/p}(\Gamma)\right)'. \tag{3.2}$$

Hence Theorem 2.1 implies the result immediately. □

In fact, Amosov proved in 2005 the uniqueness of the solution for a simple case, that is, problem in which the matrix $\mathbf{a}$ is independent of u (see [2, Theorem 1.4]). In this paper, we also can get the positivity of the solution and the estimates of the solution in $W_2^1(\Omega)$ and $L_\infty(\Omega)$ in this part. We have tried to achieve the global existence, but it is still an open problem.

In the rest of this section, we always assume that (H1)-(H2) and (Hr) hold.

3.2. Positivity

Assume that

(H^+) $h(u)$ *is nondecreasing with* $h(0) = 0$, *and*

$$\begin{gathered} \varphi(t,x,y) = \varphi(t,y,x) \geq 0, \\ \tilde{\varphi}(t,x) := \int_\Gamma \varphi(t,x,y)\,d\sigma(y) < 1. \end{gathered} \tag{3.3}$$

Theorem 3.2. *Let assumption (H^+) be satisfied. If (f_0, g_0, u_0) is nonnegative, then the solution u of problem* (3.1) *is also nonnegative.*

Proof. Put $u_- := 0 \wedge u$. Multiplying the equation with u_- and integrating over Ω, we have

$$\begin{aligned} \int_\Omega f_0 u_- dx &= \frac{1}{2}\int_\Omega \left(u_-^2\right)_t dx - \int_\Omega \nabla(\mathbf{a}(u)\nabla u)\; u_- dx \\ &= \frac{1}{2}\int_\Omega \left(u_-^2\right)_t dx + \int_\Omega \mathbf{a}(u)\nabla u \cdot \nabla u_- dx - \int_\Gamma \frac{\partial u}{\partial \nu_{\mathbf{a}}} u_- d\sigma \\ &= \frac{1}{2}\int_\Omega \left(u_-^2\right)_t dx + \int_\Omega \mathbf{a}(u)\nabla u_- \cdot \nabla u_- dx \\ &\quad + \int_\Gamma \left[h(u(t,x)) - \int_\Gamma h(u(t,y))\varphi(t,x,y)\,d\sigma(y)\right] u_- d\sigma - \int_\Gamma g_0 u_- d\sigma. \end{aligned} \tag{3.4}$$

By using the assumption of (H^+), we can get following equality:

$$\begin{aligned} &\int_{\Gamma\times\Gamma} h(u(t,y))\varphi(t,x,y)(u(t,y) - u(t,x))\,d\sigma(y)\,d\sigma \\ &\quad = \frac{1}{2}\int_{\Gamma\times\Gamma} [h(u(t,x)) - h(u(t,y))]\varphi(t,y,x)(u(t,x) - u(t,y))\,d\sigma(y)\,d\sigma. \end{aligned} \tag{3.5}$$

So,

$$\begin{aligned}
&\int_\Gamma \left[h(u(t,x)) - \int_\Gamma h(u(t,y))\varphi(t,x,y)d\sigma(y)\right]u_-(t,x)d\sigma \\
&\quad = \int_\Gamma h(u(t,x))u_-(t,x)(1-\tilde{\varphi}(t,x))d\sigma \\
&\qquad + \int_{\Gamma\times\Gamma} h(u(t,y))\varphi(t,x,y)(u_-(t,y)-u_-(t,x))d\sigma(y)d\sigma \\
&\quad = \int_\Gamma h(u(t,x))u_-(t,x)(1-\tilde{\varphi}(t,x))d\sigma \\
&\qquad + \frac{1}{2}\int_{\Gamma\times\Gamma} [h(u(t,x)) - h(u(t,y))](u_-(t,x)-u_-(t,y))\ \varphi(t,y,x)d\sigma(y)d\sigma \ge 0.
\end{aligned} \tag{3.6}$$

At the last inequality, the monotonity of h on u and the restriction $\tilde{\varphi} < 1$ are used. Therefore,

$$\frac{d}{dt}\|u_-\|^2_{L_2(\Omega)} \le \int_\Omega f_0 u_- dx + \int_\Gamma g_0 u_- d\sigma \le 0. \tag{3.7}$$

If $u_0 \ge 0$, then $u_-(t,x) \equiv 0$. The assertion follows. □

3.3. $W_2^1(\Omega)$*-norm*

We denote by $J_{\max}$ the maximal interval of the solution of problem (3.1).

Lemma 3.3. *There exists a constant* $C = C(T; f_0, g_0, u_0)$ *such that the solution* u *of problem* (3.1) *satisfies*

$$\|u(t,\cdot)\|_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(Q_t)} \le C \quad \text{for } t \le J_{\max}. \tag{3.8}$$

Proof. Multiplying by u and integrating over Ω, we have

$$\int_\Omega [u_t - \nabla(\mathbf{a}(u)\nabla u)u]dx = \int_\Omega f_0\ u\,dx. \tag{3.9}$$

That is,

$$\frac{1}{2}\ \frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega a(u)\nabla u\cdot\nabla u\,dx - \int_\Gamma u\ \partial_{\nu_a} u\,d\sigma = \int_\Omega f_0\ u\,dx. \tag{3.10}$$

As similar as the inequality (3.6), we have

$$
\begin{aligned}
-\int_{\Gamma} u\, \partial_{\nu_a} u\, d\sigma &= -\int_{\Gamma} u\left[\int_{\Gamma} h(u(t,y))\ \varphi(t,x,y)d\sigma(y) - h(t,x)) + g_0\right] d\sigma \\
&= \int_{\Gamma} (1-\tilde{\varphi}(t,x))\ h(u(t,x))\ u\, d\sigma - \int_{\Gamma} g_0\ u\, d\sigma \\
&\quad + \frac{1}{2}\int_{\Gamma\times\Gamma} \{h(u(t,x)) - h(u(t,y))\}[u(t,x)-u(t,y)]\varphi(t,x,y)d\sigma\, d\sigma \\
&\geq -\int_{\Gamma} g_0\ u\, d\sigma.
\end{aligned}
\tag{3.11}
$$

Hence,

$$
\begin{aligned}
\frac{1}{2}\ \frac{d}{dt}\int_{\Omega} |u|^2 dx + \delta_0 \int_{\Omega} |\nabla u|^2 dx &\leq \int_{\Omega} f_0\ u\, dx + \int_{\Gamma} g_0\ u\, d\sigma \\
&\leq \epsilon\left\{\int_{\Omega} u^2 dx + \int_{\Gamma} u^2 d\sigma\right\} + \frac{1}{\epsilon}\left\{\int_{\Omega} f_0^2 dx + \int_{\Gamma} g_0^2 d\sigma\right\}.
\end{aligned}
\tag{3.12}
$$

By using the embedding $W^1_{2,\mathcal{B}}(\Omega) \hookrightarrow L_2(\Gamma)$ and letting ϵ small enough, it is easy to get that

$$
\|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(Q_t)} \leq C(T; f_0, g_0, u_0), \qquad \text{for } t \leq J_{\max}. \tag{3.13}
$$

□

3.4. $L_\infty(\Omega)$-norm

Theorem 3.4. *If $f_0 \in L_\infty(\overline{Q}_T)$ and $g_0 \in L_\infty([0,T]\times\Gamma)$, then the solution $u(t,x)$ of problem* (3.1) *is bounded with its L_∞-norm for all $t \in J_{\max}$.*

Proof. From the hypothesis (H1) and embedding (2.10), one has that $u \in C(\overline{Q}_T)$ and $u^n \in W^1_2(\Omega)$, $n = 1,2,\ldots$. By multiplying with u^{2^k-1} ($k \in \mathbb{Z}^+$ and $k \geq 2$) and integrating over Ω, we have

$$
\int_{\Omega} [u_t - \nabla(\mathbf{a}(u)\nabla u)]u^{2^k-1} dx = \int_{\Omega} f_0\ u^{2^k-1} dx. \tag{3.14}
$$

That is,

$$
2^{-k}\ \frac{d}{dt}\int_{\Omega} |u|^{2^k} dx + \int_{\Omega} \mathbf{a}(u)\nabla u\cdot\nabla u^{2^k-1} dx - \int_{\Gamma} u^{2^k-1}\ \partial_{\nu_a} u\, d\sigma = \int_{\Omega} f_0\ u^{2^k-1} dx. \tag{3.15}
$$

But,

$$\begin{aligned}
\int_\Omega \mathbf{a}(u)\nabla u\cdot\nabla u^{2^k-1}dx &= \left(2^k-1\right)\int_\Omega u^{2^k-2}\ \mathbf{a}(u)\nabla u\cdot\nabla u\,dx \\
&= \left(2^k-1\right)2^{2-2k}\int_\Omega \mathbf{a}(u)\nabla u^{2^{k-1}}\cdot\nabla u^{2^{k-1}}dx, \\
-\int_\Gamma u^{2^k-1}\partial_{\nu_{\mathbf{a}}}u\,d\sigma &= -\int_\Gamma u^{2^k-1}\left[\int_\Gamma h(u(t,y))\varphi(t,x,y)d\sigma(y)-h((t,x))+g_0\right]d\sigma \\
&= \int_\Gamma (1-\tilde{\varphi}(t,x))u^{2^k-1}\ h(u(t,x))u\,d\sigma-\int_\Gamma g_0u^{2^k-1}d\sigma \\
&\quad+\frac{1}{2}\int_{\Gamma\times\Gamma}\{h(u(t,x))-h(u(t,y))\}\left[u^{2^k-1}(t,x)-u^{2^k-1}(t,y)\right]\varphi(t,x,y)d\sigma^2, \\
&\geq -\int_\Gamma g_0u^{2^k-1}d\sigma.
\end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2^k}\ \frac{d}{dt}\int_\Omega |u|^{2^k}dx+\delta_0\left(2^k-1\right)\int_\Omega |u|^{2^k-2}|\nabla u|^2dx \\
&\quad= \frac{1}{2^k}\ \frac{d}{dt}\int_\Omega |u|^{2^k}dx+\delta_0\frac{2^k-1}{4^{k-1}}\int_\Omega\left|\nabla u^{2^{k-1}}\right|^2dx \\
&\quad\leq \int_\Omega f\ u^{2^k-1}dx+\int_\Gamma g_0\ u^{2^k-1}d\sigma \\
&\quad\leq \epsilon\left(1-2^{-k}\right)\left\{\int_\Omega u^{2^k}dx+\int_\Gamma u^{2^k}d\sigma\right\} \\
&\qquad+\epsilon^{1-2^k}2^{-k}\left\{\int_\Omega f_0^{2^k}dx+\int_\Gamma g_0^{2^k}d\sigma\right\},
\end{aligned} \tag{3.17}$$

where Young's inequality, $\alpha\beta\leq(\epsilon/r)\alpha^r+\epsilon^{-r'/r}/r'\beta^{r'}\,(\alpha,\beta\geq 0, 0<\varepsilon<1)$, has been used at the last inequality. We apply the embedding $W^1_{2,\mathcal{B}}(\Omega)\hookrightarrow L_2(\Gamma)$ again with $v=u^{2^{k-1}}$ and choose ϵ small enough, then we attain the following inequality:

$$\frac{d}{dt}\int_\Omega u^{2^k}dx-\left\{(\epsilon+C)2^k-\epsilon\right\}\int_\Omega u^{2^k}dx\leq\epsilon^{1-2^k}\left\{\int_\Omega f_0^{2^k}dx+\int_\Gamma g_0^{2^k}d\sigma\right\}. \tag{3.18}$$

By Gronwall's inequality, the inequality (3.18) becomes

$$\int_\Omega u^{2^k}dx\leq e^{[(\epsilon+C)2^k-\varepsilon]t}\cdot\int_\Omega u_0^{2^k}dx+\epsilon^{1-2^k}\int_0^t e^{[(\epsilon+C)2^k-\varepsilon](t-\tau)}\left\{\int_\Omega f_0^{2^k}dx+\int_\Gamma g_0^{2^k}d\sigma\right\}d\tau. \tag{3.19}$$

Set $B_\infty := 1 + \|u_0\|_{L_\infty(\Omega)} + \|f_0\|_{L_\infty(\overline{Q}_T)} + \|g_0\|_{L_\infty([0,T]\times\Gamma)}$, then we deduce that

$$\begin{aligned}\|u\|_{L_{2^k}(\Omega)} &\leq \frac{B_\infty e^{(\varepsilon+C)t}}{\varepsilon}\left[e^{-\epsilon t}\int_\Omega \frac{(\varepsilon u_0)^{2^k}}{B_\infty^{2^k}}\,dx + \epsilon\int_0^t\left\{\int_\Omega \frac{f_0^{2^k}}{B_\infty^{2^k}}dx + \int_\Gamma \frac{g_0^{2^k}}{B_\infty^{2^k}}d\sigma\right\}d\tau\right]^{1/2^k} \\ &\leq \varepsilon^{-1}B_\infty e^{(\varepsilon+C)t}\left[|\Omega| + t(|\Omega|+|\Gamma|)\right]^{1/2^k}.\end{aligned} \tag{3.20}$$

Let $k \to +\infty$, the inequality (3.20) implies

$$\|u(t,\cdot)\|_{L_\infty(\Omega)} \leq e^{(\epsilon+C)t}\|u_0\|_{L_\infty(\Omega)} + C\Big(T; \|f_0\|_{L_\infty(\overline{Q}_T)}, \|g_0\|_{L_\infty([0,T]\times\Gamma)}\Big) \quad \text{for } t \in J_{\max}. \tag{3.21}$$

The claim follows. □

One immediate consequence of the above theorem is.

Corollary 3.5. *The L_∞-norm of the solution u, that is, $\|u(t,\cdot)\|_{L_\infty(\Omega)}$, of problem* (3.1) *is nonincreasing if $f_0 = g_0 = 0$.*

4. A Nonlocal Boundary Value Problem

We now consider the problem (2.1) with the following boundary value condition:

$$\mathcal{B}(t,x,u) = g(t,x,u,\nabla u) = \Phi(u)(t,x) + g_1(t,x,u) + g_0. \tag{4.1}$$

The function Φ in (4.1) can be in nonlocal form.

IBVP (2.1) with a nonlocal term stands, for example, for a model problem arising from quasistatic thermoelasticity. Results on linear problems can be found in [3–5]. As far as we know, this kind of nonlocal boundary condition appeared first in 1952 in a paper [6] by W. Feller who discussed the existence of semigroups. There are other problems leading to this boundary condition, for example, control theory (see [7–12] etc.). Some other fields such as environmental science [13] and chemical diffusion [14] also give rise to such kinds of problems. We do not give further comments here.

Carl and Heikkilä [15] proved the existence of local solutions of the semilinear problem by using upper and lower solutions and pseudomonotone operators. But their results based on the monotonicity hypotheses of f, g, and Φ with respect to u.

In this section, we assume that (H1) and (H2) always hold and assume that

(Hn1) $\Phi(0) = 0$ and $\Phi \in C^{1-}(\mathbb{W}_p^1(J_T), L_r(J_T, L_{\dot{p}}(\Gamma)))$ for some $r > p$;

(Hn2) $g_1(t,x,0) = 0$, g_1 satisfies the Carathéodory condition on (t,x) and $g_1(t,x,\cdot) \in C^{1-}(\mathbb{R})$.

By the embedding theorem and Theorem 2.1, we get immediately.

Theorem 4.1. *Suppose hypotheses of (Hn) satisfy. Then problem* (2.1), *for all $u_0 \in E$, with g defined in* (4.1) *has a unique $u \in \mathbb{W}_p^1(J_{T^*})$ for some $T^* > 0$.*

For the simplicity in expression, we turn to consider a problem with nonlocal boundary value

$$\begin{gathered} u_t + \mathcal{A}(t,x,u)u = f(t,x,u,\nabla u), \quad \text{in } Q_T, \\ \mathcal{B}(t,x,u)u = \kappa(u)(t,x) + g_1(t,x,u) + g_0, \quad \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad \text{on } \Omega, \end{gathered} \tag{4.2}$$

where

$$\kappa(u)(t,x) := \int_\Omega k(t,x,y,u(t,y),\nabla u(t,y))dy, \tag{4.3}$$

and

(Hk) *The function* k *satisfies the Carathéodory condition on* $(t,x,y) \in Q_T := [0,T] \times \Gamma \times \overline{\Omega}, k|_{u=0} = 0$ *and* f $k \in \mathrm{Car}_{0,\lambda_2,\widetilde{\lambda}_2}(Q_T \times \mathbb{R} \times \mathbb{R}^n)$ with

$$\lambda_2 < 1 + \frac{p(s-1)}{2+p(1-s)}, \qquad \widetilde{\lambda}_2 < p+1. \tag{4.4}$$

Theorem 4.2. *Let assumption (Hk) be satisfied. Then Problem* (4.2), *for any* $u_0 \in E$, *has a unique solution* $u \in \mathbb{W}_p^1(J_{T^*})$ *for some* $T^* > 0$.

Proof. First, we see that

$$\begin{aligned} \left\| \int_\Omega |\nabla u|^{\lambda_2-1}|\nabla(u-v)|dy \right\|_{L_r(J_t)} &\le \left\| \|\nabla u\|^{\lambda_2-1}_{L_{(\lambda_2-1)p'}(\Omega)} \cdot \|\nabla(u-v)\|_{L_p(\Omega)} \right\|_{L_r(J_t)} \\ &\le c \left\| \|u\|^{\lambda_2-1}_{W_p^1} \cdot \|u-v\|_{W_p^1} \right\|_{L_r(J_t)}. \end{aligned} \tag{4.5}$$

Choose $\theta \in (0,\ 1)$ such that $\lambda_1\theta < 1$, then $(\lambda_2-1)\theta/(1-\theta) < 1$. Consequently, there exists $r > p$ such that

$$\begin{aligned} &\left\| \int_\Omega |\nabla u|^{\lambda_2-1}|\nabla(u-v)|dy \right\|_{L_r(J_t)} \\ &\quad \le c\ \|u\|^{(\lambda_2-1)(1-\theta)}_{C(J_t,E)} \|u-v\|^{1-\theta}_{C(J_t,E)} \cdot \left\| \|u\|^{(\lambda_2-1)p\theta/(p-\theta r)}_{E_1} \right\|_{L_r(J_t)} \left\| \|u-v\|^{\theta}_{E_1} \right\|_{L_r(J_t)} \\ &\quad \le c\Big(\|u\|_{W_p^1(J_t)}\Big) \cdot \|u-v\|_{W_p^1(J_t)}. \end{aligned} \tag{4.6}$$

Similarly, from $\tilde{\lambda}_2 \le p+1$ we have

$$\left\| \int_\Omega |\nabla u|^{\tilde{\lambda}_2 - 1} |u-v| dy \right\|_{L_r(J_t)} \le c \ \ \|u-v\|_{C(J_t,E)} \cdot \left(\int_0^t \|u\|^r_{W^1_p} d\tau \right)^{1/r} \le c\Big(\|u\|_{W^1_p(J_t)}\Big) \cdot \|u-v\|_{W^1_p(J_t)}. \tag{4.7}$$

Combining two inequalities (4.6) and (4.7), we obtain that

$$\begin{aligned} &\|\kappa(u) - \kappa(v)\|_{L_r(J_t,\partial W^s_p)} \\ &\quad \le \left\| \int_\Omega k(t,x,y,u,\nabla u) - k(t,x,y,v,\nabla v) dy \right\|_{L_r(J_t,L_p(\Gamma))} \\ &\quad \le c \left\| \int_\Omega \psi(\rho)\Big[\Big(1 + |\nabla u|^{\lambda_2 - 1} + |\nabla v|^{\lambda_2 - 1}\Big) \mid \nabla(u-v) \mid + \Big(1 + |\nabla v|^{\tilde{\lambda}_2 - 1}\Big)|u-v|\Big] dy \right\|_{L_r(J_t)} \\ &\quad \le c\Big(\|u\|_{W^1_p(J_t)}\Big) \cdot \|u-v\|_{W^1_p(J_t)}. \end{aligned} \tag{4.8}$$

The claim follows immediately from Theorem 4.1. □

A special case of problem (4.2) is

$$\begin{aligned} u_t - \nabla(\mathbf{a}(u)\nabla u) &= f(t,x,u), \quad \text{in } Q_T, \\ \partial_{\nu_{\mathbf{a}(u)}} u &= \int_\Omega k(t,x,y,u) dy + g(t,x,u), \quad \text{on } [0,T) \times \Gamma, \\ u(0,x) &= u_0(x), \quad \text{on } \overline{\Omega}. \end{aligned} \tag{4.9}$$

That is, f and k in (4.9) are independent of gradient ∇u.

4.1. $W^1_2(\Omega)$*-norm*

In order to discuss the global existence of solution, in the rest of this section we assume the following.

(Hkl) *Suppose there exists a continuous function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\big|f_1(t,x,u)\big|, \quad \big|k(t,x,y,u) \ \big|, \quad \big|g_1 \ (t,x,u)\big| \le \phi(t)|u|. \tag{4.10}$$

Lemma 4.3. *There exists a constant* $C = C(T; f_0, g_0, u_0)$ *such that the solution of problem* (4.9) *satisfies*

$$\|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(Q_t)} \leq C, \quad \text{for } t \leq J_{\max}. \tag{4.11}$$

Proof. We multiply the first equation in (4.9) with u and then integrate over Ω, and we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \delta_0\|\nabla u\|^2_{L_2(\Omega)} \\
&\quad \leq \int_\Omega f(t,x,u)u\,dx + \int_\Gamma u\left[\int_\Omega k(t,x,y,u)dy + g(t,x,u)\right] d\sigma(x) \\
&\quad \leq \phi(t)\left\{\|u\|^2_{L_2(\Omega)} + \|u\|_{L_1(\Gamma)}\left(\|u\|_{L_1(\Omega)} + \|u\|_{L_1(\Gamma)}\right)\right\} + \varepsilon_1\left(\|u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Gamma)}\right) \\
&\qquad + \frac{1}{\varepsilon_1}\left(\|f_0\|^2_{L_2(\Omega)} + \|g_0\|^2_{L_2(\Gamma)}\right).
\end{aligned} \tag{4.12}$$

Since $W_2^\theta(\Omega) \hookrightarrow L_2(\Gamma)$ for $\theta \in (1/2, 1)$, by interpolation inequality and Young's inequality we have that

$$\begin{aligned}
\|u\|_{L_2(\Gamma)} &\leq C\ \|u\|_{W_2^\theta(\Omega)} \\
&\leq C\ \|u\|^\theta_{W_2^1(\Omega)}\|u\|^{1-\theta}_{L_2(\Omega)} \\
&\leq \varepsilon_2\|u\|_{W_2^1(\Omega)} + C(\varepsilon_2)\|u\|_{L_2(\Omega)}.
\end{aligned} \tag{4.13}$$

Apply Young's inequality again and then choose ε_j small enough ($j = 1, 2$); it is not difficult to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} - C(\varepsilon)\phi(t)\|u\|^2_{L_2(\Omega)} + \left(\delta_0 - \varepsilon\left(\phi(t)+1\right)\right)\|\nabla u\|^2_{L_2(\Omega)} \\
&\quad \leq \frac{1}{\varepsilon_1}\left(\|f_0\|^2_{L_2(\Omega)} + \|g_0\|^2_{L_2(\Gamma)}\right),
\end{aligned} \tag{4.14}$$

where $\delta_0 - \varepsilon(\phi(t) + 1) > 0$ for $t \in [0,\ T]$. Therefore, by multiplying with $e^{-2C(\varepsilon)\int_0^t \phi(\tau)d\tau}$ and integrating over $[0,\ t]$, the inequality (4.14) follows the claim. □

4.2. $L_\infty(\Omega)$*-norm*

Lemma 4.4. *Let assumptions of Lemma 4.3 be satisfied. If* $(f_0, g_0) \in L_\infty(\overline{Q}_T \times ([0,T] \times \Gamma))$*, then the solution* u *of problem* (4.9) *satisfies*

$$\|u\|_{L_\infty(\Omega)} \leq C(T; f_0, g_0, u_0), \quad \text{for } t \leq J_{\max}. \tag{4.15}$$

Proof. We multiply the first equation in (4.9) with u^{2^k-1} and integrate over Ω, then we reach that

$$\begin{aligned} J(t) &:= \frac{1}{2^k}\frac{d}{dt}\|u\|^{2^k}_{L_{2^k}(\Omega)} + \frac{\delta_0(2^k-1)}{4^{k-1}}\left\|\nabla u^{2^{k-1}}\right\|^2_{L_2(\Omega)} \\ &\le \int_\Omega f(t,x,u)u^{2^k-1}dx \\ &\quad + \int_\Gamma u^{2^k-1}\left[\int_\Omega k(t,x,y,u)dy + g(t,x,u)\right] d\sigma(x) \\ &\le \phi(t)\left\{\|u\|^{2^k}_{L_{2^k}(\Omega)} + \|u\|^{2^k-1}_{L_{2^k}(\Gamma)}\left(\|u\|_{L_1(\Omega)}|\Gamma|^{2^{-k}} + \|u\|_{L_{2^k}(\Gamma)}\right)\right\} \\ &\quad + \int_\Omega f_0 u^{2^k-1}dx + \int_\Gamma g_0 u^{2^k-1}d\sigma. \end{aligned} \tag{4.16}$$

As the same as the inequality (4.13), we have

$$\begin{aligned} \|u\|^{2^k}_{L_{2^k}(\Omega)} &\le \varepsilon_1\left\|\nabla u^{2^{k-1}}\right\|^2_{L_2(\Omega)} + C(\varepsilon_1)\|u\|^{2^k}_{L_{2^k}(\Omega)}, \\ \|u\|^{2^k-1}_{L_{2^k}(\Omega)}\cdot\|u\|_{L_1(\Omega)} &\le \left(\varepsilon_1\left\|\nabla u^{2^{k-1}}\right\|^2_{L_2(\Omega)} + C(\varepsilon_1)\|u\|^{2^k}_{L_{2^k}(\Omega)}\right)^{1-2^{-k}}\cdot\|u\|_{L_1(\Omega)} \\ &\le \varepsilon_1^{1-2^{-k}}\left\|\nabla u^{2^{k-1}}\right\|^{2(1-2^{-k})}_{L_2(\Omega)}\cdot\|u\|_{L_1(\Omega)} \\ &\quad + C(\varepsilon_1)\|u\|^{2^k}_{L_{2^k}(\Omega)}. \end{aligned} \tag{4.17}$$

Hence,

$$\begin{aligned} J(t) &\le \{\phi(t)(1+C(\varepsilon_1)) + C(\varepsilon_2)\}\|u\|^{2^k}_{L_{2^k}(\Omega)} \\ &\quad + (\varepsilon_1+\varepsilon_2)\left\|\nabla u^{2^{k-1}}\right\|^2_{L_2(\Omega)} + \varepsilon_1^{1-2^{-k}}|\Gamma|^{2^{-k}}\cdot\left\|\nabla u^{2^{k-1}}\right\|^{2(1-2^{-k})}_{L_2(\Omega)} \\ &\quad + C(\varepsilon_2)2^{-k}\left\{\|f_0\|^{2^k}_{L_{2^k}(\Omega)} + \|g_0\|^{2^k}_{L_{2^k}(\Gamma)}\right\}. \end{aligned} \tag{4.18}$$

We might as well assume that $\|u\|_{L_2(\Omega)} > 0$, so,

$$\left\|\nabla u^{2^{k-1}}\right\|^{-2^{1-k}}_{L_2(\Omega)} = 4^{(1-k)2^{-k}}\left[\int_\Omega u^{2^k-2}|\nabla u|^2dx\right]^{-2^{-k}} \longrightarrow \|u\|^{-1}_{L_\infty(\Omega)} \quad \text{as } k \longrightarrow +\infty. \tag{4.19}$$

The boundedness of solution $\|u\|_{L_2(\Omega)} \le C$ for $t \le J_{\max}$ is used in above deduction.

Let ε_j $(j = 1,2)$ small enough, then we have

$$2^{-k}\frac{d}{dt}\|u\|^{2^k}_{L_{2^k}(\Omega)} - C(\varepsilon)\phi(t)\|u\|^{2^k}_{L_{2^k}(\Omega)} \le C(\varepsilon)2^{-k}\left\{\|f_0\|^{2^k}_{L_{2^k}(\Omega)} + \|g_0\|^{2^k}_{L_{2^k}(\Gamma)}\right\}. \tag{4.20}$$

Multiplying with $2^k \cdot e^{-C(\varepsilon)2^k\int_0^t \phi(\tau)d\tau}$, then integrating over $[0,\ t]$, we obtain that

$$\left(\|u\|_{L_{2^k}(\Omega)} e^{-C(\varepsilon)\int_0^t \phi(\tau)d\tau}\right)^{2^k} \leq \|u_0\|_{L_{2^k}(\Omega)}^{2^k} + C(\varepsilon)\int_0^t e^{-C(\varepsilon)2^k\int_0^s \phi(\tau)d\tau}\left\{\|f_0\|_{L_{2^k}(\Omega)}^{2^k} + \|g_0\|_{L_{2^k}(\Gamma)}^{2^k}\right\}ds. \tag{4.21}$$

By a similar limitation process as in (3.21), we get

$$\|u\|_{L_\infty(\Omega)} \leq C(T; f_0, g_0, u_0) \quad \text{for } t \leq J_{\max}. \tag{4.22}$$

This closes the end of proof. □

4.3. *Decay Behavior*

In order to investigate the decay behavior of solution for problem (4.9), we assume that

(Hkd) there are two continuous function $\varphi(t) > 0$ and $\epsilon(t) \geq 0$ *($t \geq 0$) such that*

$$\begin{aligned} f_1(t,x,u)u &\leq -\varphi(t)u^2, \\ g_1(t,x,u)u &\leq 0, \\ |k(t,x,y,u)| &\leq \epsilon(t)|u| \end{aligned} \tag{4.23}$$

for all $(t,x,y) \in \mathbb{R}^+ \times \overline{\Omega} \times \Gamma$.

Theorem 4.5. *Let the assumption (Hkd) be satisfied and, u be the solution of problem* (4.9) *with* $(f_0, g_0) = 0$. *Then $\|u\|_{L_2(\Omega)}$ decay to zero as $t \to \infty$ for some small functions $\epsilon(t)$.*

Proof. We use u to multiply the first equation in the system (4.9) and then integrate over Ω. Thus, we get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|_{L_2(\Omega)}^2 + \delta_0\|\nabla u\|_{L_2(\Omega)}^2 \\ &\quad \leq \int_\Omega f(t,x,u)u\,dx + \int_\Gamma u\left[\int_\Omega k(t,x,y,u)dy + g(t,x,u)\right]d\sigma \\ &\quad \leq -\varphi(t)\|u\|_{L_2(\Omega)}^2 + \epsilon(t)\|u\|_{L_1(\Gamma)}\|u\|_{L(\Omega)} \\ &\quad \leq \left\{C\epsilon(t)\sqrt{|\Gamma||\Omega|} - \varphi(t)\right\} \cdot \|u\|_{L_2(\Omega)}^2 + C\epsilon(t)\sqrt{|\Gamma||\Omega|}\|\nabla u\|_{W_2^1(\Omega)}^2. \end{aligned} \tag{4.24}$$

In the above process the inequality (4.13) is used. If we choose $\epsilon(t)$ as

$$\epsilon(t) \leq \frac{1}{C\sqrt{|\Gamma||\Omega|}} \cdot \min\left\{\delta_0, \min_{t\in[0,\ T]} \varphi(t)\right\}, \quad t \in [0,\ T], \tag{4.25}$$

then

$$\|u\|_{L_2(\Omega)} \leq \|u_0\|_{L_2(\Omega)} \cdot e^{-\int_0^t \widetilde{\varphi}(\tau)d\tau}, \quad t \in [0,\ T]. \tag{4.26}$$

This ends the proof.

Moreover, one can verify that $\|u\|_{L_p(\Omega)}$ also decay to zero (as $t \to \infty$) if $p \geq 2$. □

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References

[1] H. Amann, "Quasilinear parabolic problems via maximal regularity," *Advances in Differential Equations*, vol. 10, no. 10, pp. 1081–1110, 2005.
[2] A. A. Amosov, "Global solvability of a nonlinear nonstationary problem with a nonlocal boundary condition of radiative heat transfer type," *Differential Equations*, vol. 41, no. 1, pp. 96–109, 2005.
[3] W. A. Day, "A decreasing property of solutions of parabolic equations with applications to thermoelasticity," *Quarterly of Applied Mathematics*, vol. 40, no. 4, pp. 468–475, 1983.
[4] A. Friedman, "Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions," *Quarterly of Applied Mathematics*, vol. 44, no. 3, pp. 401–407, 1986.
[5] L. A. Muraveĭ and A. V. Filinovskiĭ, "On a problem with nonlocal boundary condition for a parabolic equation," *Mathematics of the USSR-Sbornik*, vol. 74, no. 1, pp. 219–249, 1993.
[6] W. Feller, "The parabolic differential equations and the associated semi-groups of transformations," *Annals of Mathematics*, vol. 55, no. 3, pp. 468–519, 1952.
[7] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*, Springer, Berlin, Germany, 1972.
[8] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. II*, Springer, Berlin, Germany, 1972.
[9] I. Lasiecka, "Unified theory for abstract parabolic boundary problems—a semigroup approach," *Applied Mathematics and Optimization*, vol. 6, no. 1, pp. 287–333, 1980.
[10] H. Amann, "Feedback stabilization of linear and semilinear parabolic systems," in *Semigroup Theory and Applications (Trieste, 1987)*, P. Clement, S. Invernizzi, E. Mitidieri, and I. I. Vrabie, Eds., vol. 116 of *Lecture Notes in Pure and Applied Mathematics*, pp. 21–57, Dekker, New York, NY, USA, 1989.
[11] R. P. Agarwal, M. Bohner, and V. B. Shakhmurov, "Linear and nonlinear nonlocal boundary value problems for differential-operator equations," *Applicable Analysis*, vol. 85, no. 6-7, pp. 701–716, 2006.
[12] A. Ashyralyev, "Nonlocal boundary-value problems for abstract parabolic equations: well-posedness in Bochner spaces," *Journal of Evolution Equations*, vol. 6, no. 1, pp. 1–28, 2006.
[13] V. Capasso and K. Kunisch, "A reaction-diffusion system modelling man-environment epidemics," *Annals of Differential Equations*, vol. 1, no. 1, pp. 1–12, 1985.
[14] K. Taira, *Diffusion Processes and Partial Differential Equations*, Academic Press, Boston, Mass, USA, 1988.
[15] S. Carl and S. Heikkilä, "Discontinuous reaction-diffusion equations under discontinuous and nonlocal flux conditions," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1333–1344, 2000.