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Research Article

First-Order Singular and Discontinuous Differential Equations

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We use subfunctions and superfunctions to derive sufficient conditions for the existence of extremal solutions to initial value problems for ordinary differential equations with discontinuous and singular nonlinearities.

1. Introduction

Let $t_0, x_0 \in \mathbb{R}$ and $L > 0$ be fixed and let $f : [t_0, t_0 + L] \times \mathbb{R} \to \mathbb{R}$ be a given mapping. We are concerned with the existence of solutions of the initial value problem

$$x' = f(t, x), \quad t \in I := [t_0, t_0 + L], \quad x(t_0) = x_0. \tag{1.1}$$

It is well-known that Peano's theorem ensures the existence of local continuously differentiable solutions of (1.1) in case f is continuous. Despite its fundamental importance, it is probably true that Peano's proof of his theorem is even more important than the result itself, which nowadays we know can be deduced quickly from standard fixed point theorems (see [1, Theorem 6.2.2] for a proof based on the Schauder's theorem). The reason for believing this is that Peano's original approach to the problem in [2] consisted in obtaining the greatest solution as the pointwise infimum of strict upper solutions. Subsequently this idea was improved by Perron in [3], who also adapted it to study the Laplace equation by means of what we call today Perron's method. For a more recent and important revisitation of the method we mention the work by Goodman [4] on (1.1) in case f is a Carathéodory function. For our purposes in this paper, the importance of Peano's original ideas is that they can be adapted to prove existence results for (1.1) under such weak conditions that standard functional analysis arguments are no longer valid. We refer to differential equations which depend discontinuously on the unknown and several results obtained in papers as [5–9], see also the monographs [10, 11].

On the other hand, singular differential equations have been receiving a lot of attention in the last years, and we can quote [7, 12–19]. The main objective in this paper is to establish an existence result for (1.1) with discontinuous and singular nonlinearities which generalizes in some aspects some of the previously mentioned works.

This paper is organized as follows. In Section 2 we introduce the relevant definitions together with some previously published material which will serve as a basis for proving our main results. In Section 3 we prove the existence of the greatest and the smallest Carathéodory solutions for (1.1) between given lower and upper solutions and assuming the existence of a L^1-bound for f on the sector delimited by the graphs of the lower and upper solutions (regular problems), and we give some examples. In Section 4 we show that looking for piecewise continuous lower and upper solutions is good in practice, but once we have found them we can immediately construct a pair of continuous lower and upper solutions which provide better information on the location of the solutions. In Section 5 we prove two existence results in case f does not have such a strong bound as in Section 3 (singular problems), which requires the addition of some assumptions over the lower and upper solutions. Finally, we prove a result for singular quasimonotone systems in Section 6 and we give some examples in Section 7. Comparison with the literature is provided throughout the paper.

2. Preliminaries

In the following definition $\mathrm{AC}(I)$ stands for the set of absolutely continuous functions on I.

Definition 2.1. A lower solution of (1.1) is a function $l \in \mathrm{AC}(I)$ such that $l(t_0) \leq x_0$ and $l'(t) \leq f(t, l(t))$ for almost all (a.a.) $t \in I$; an upper solution is defined analogously reversing the inequalities. One says that x is a (Carathéodory) solution of (1.1) if it is both a lower and an upper solution. On the other hand, one says that a solution x_* is the least one if $x_* \leq x$ on I for any other solution x, and one defines the greatest solution in a similar way. When both the least and the greatest solutions exist, one calls them the extremal solutions.

It is proven in [8] that (1.1) has extremal solutions if f is L^1-bounded for all $x \in \mathbb{R}$, $f(\cdot, x)$ is measurable, and for a.a. $t \in I$ $f(t, \cdot)$ is *quasi-semicontinuous*, namely, for all $x \in \mathbb{R}$ we have

$$\limsup_{y \to x^-} f(t, y) \leq f(t, x) \leq \liminf_{y \to x^+} f(t, y). \tag{2.1}$$

A similar result was established in [20] assuming moreover that f is superpositionally measurable, and the systems case was considered in [5, 8]. The term "quasi-semicontinuous" in connection with (2.1) was introduced in [5] for the first time and it appears to be conveniently short and descriptive. We note however that, rigorously speaking, we should say that $f(t, \cdot)$ is left upper and right lower semicontinuous.

On the other hand, the above assumptions imply that the extremal solutions of (1.1) are given as the infimum of all upper solutions and the supremum of all lower solutions, that is, the least solution of (1.1) is given by

$$u_{\inf}(t) = \inf\{u(t) : u \text{ upper solution of } (1.1)\}, \quad t \in I, \tag{2.2}$$

and the greatest solution is

$$l_{\sup}(t) = \sup\{l(t) : l \text{ lower solution of } (1.1)\}, \quad t \in I. \tag{2.3}$$

The mappings $u_{\inf}$ and $l_{\sup}$ turn out to be the extremal solutions even under more general conditions. It is proven in [9] that solutions exist even if (2.1) fails on the points of a countable family of curves in the conditions of the following definition.

Definition 2.2. An admissible non-quasi-semicontinuity (nqsc) curve for the differential equation $x' = f(t,x)$ is the graph of an absolutely continuous function $\gamma : [a,b] \subset [t_0, t_0+L] \to \mathbb{R}$ such that for a.a. $t \in [a,b]$ one has either $\gamma'(t) = f(t,\gamma(t))$, or

$$\gamma'(t) \geq f(t,\gamma(t)) \quad \text{whenever } \gamma'(t) \geq \liminf_{y \to (\gamma(t))^+} f(t,y), \tag{2.4}$$

$$\gamma'(t) \leq f(t,\gamma(t)) \quad \text{whenever } \gamma'(t) \leq \limsup_{y \to (\gamma(t))^-} f(t,y). \tag{2.5}$$

Remark 2.3. The condition (2.1) cannot fail over arbitrary curves. As an example note that (1.1) has no solution for $t_0 = 0 = x_0$ and

$$f(t,x) = \begin{cases} 1, & \text{if } x < 0, \\ -1, & \text{if } x \geq 0. \end{cases} \tag{2.6}$$

In this case (2.1) only fails over the line $x = 0$, but solutions coming from above that line collide with solutions coming from below and there is no way of continuing them to the right once they reach the level $x = 0$. Following Binding [21] we can say that the equation "jams" at $x = 0$.

An easily applicable sufficient condition for an absolutely continuous function $\gamma : [a,b] \subset I \to \mathbb{R}$ to be an admissible nqsc curve is that either it is a solution or there exist $\varepsilon > 0$ and $\delta > 0$ such that one of the following conditions hold:

(1) $\gamma'(t) \geq f(t,y) + \varepsilon$ for a.a. $t \in [a,b]$ and all $y \in [\gamma(t) - \delta, \gamma(t) + \delta]$,

(2) $\gamma'(t) \leq f(t,y) - \varepsilon$ for a.a. $t \in [a,b]$ and all $y \in [\gamma(t) - \delta, \gamma(t) + \delta]$.

These conditions prevent the differential equation from exhibiting the behavior of the previous example over the line $x = 0$ in several ways. First, if γ is a solution of $x' = f(t,x)$ then any other solution can be continued over γ once they contact each other and independently of the definition of f around the graph of γ. On the other hand, if (1) holds then solutions of $x' = f(t,x)$ can cross γ from above to below (hence at most once), and if (2) holds then solutions can cross γ from below to above, so in both cases the equation does not jam over the graph of γ.

For the convenience of the reader we state the main results in [9]. The next result establishes the fact that we can have "weak" solutions in a sense just by assuming very general conditions over f.

Theorem 2.4. *Suppose that there exists a null-measure set $N \subset I$ such that the following conditions hold:*

(1) *condition* (2.1) *holds for all $(t,x) \in (I \setminus N) \times \mathbb{R}$ except, at most, over a countable family of admissible non-quasi-semicontinuity curves;*

(2) *there exists an integrable function $g = g(t)$, $t \in I$, such that*

$$|f(t,x)| \leq g(t) \quad \forall (t,x) \in (I \setminus N) \times \mathbb{R}. \tag{2.7}$$

Then the mapping

$$u^*_{\inf}(t) = \inf\{u(t) : u \text{ upper solution of } (1.1), |u'| \leq g + 1 \text{ a.e.}\}, \quad t \in I \tag{2.8}$$

is absolutely continuous on I and satisfies $u^*_{\inf}(t_0) = x_0$ and ${u^*_{\inf}}'(t) = f(t, u^*_{\inf}(t))$ for a.a. $t \in I \setminus J$, where $J = \cup_{n,m \in \mathbb{N}} J_{n,m}$ and for all $n, m \in \mathbb{N}$ the set

$$J_{n,m} := \left\{ t \in I : {u^*_{\inf}}'(t) - \frac{1}{n} > \sup\left\{ f(t,y) : u^*_{\inf}(t) - \frac{1}{m} < y < u^*_{\inf}(t) \right\} \right\} \tag{2.9}$$

contains no positive measure set.

Analogously, the mapping

$$l^*_{\sup}(t) = \sup\{l(t) : l \text{ lower solution of } (1.1), |l'| \leq g + 1 \text{ a.e.}\}, \quad t \in I, \tag{2.10}$$

is absolutely continuous on I and satisfies $l^*_{\sup}(t_0) = x_0$ and ${l^*_{\sup}}'(t) = f(t, l^*_{\sup}(t))$ for a.a. $t \in I \setminus K$, where $K = \cup_{n,m \in \mathbb{N}} K_{n,m}$ and for all $n, m \in \mathbb{N}$ the set

$$K_{n,m} := \left\{ t \in I : {l^*_{\sup}}'(t) + \frac{1}{n} < \inf\left\{ f(t,y) : l^*_{\sup}(t) < y < l^*_{\sup}(t) + \frac{1}{m} \right\} \right\} \tag{2.11}$$

contains no positive measure set.

Note that if the sets $J_{n,m}$ and $K_{n,m}$ are measurable then $u^*_{\inf}$ and $l^*_{\sup}$ immediately become the extremal Carathéodory solutions of (1.1). In turn, measurability of those sets can be deduced from some measurability assumptions on f. The next lemma is a slight generalization of some results in [8] and the reader can find its proof in [9].

Lemma 2.5. *Assume that for a null-measure set $N \subset I$ the mapping f satisfies the following condition.*

For each $q \in \mathbb{Q}$, $f(\cdot, q)$ is measurable, and for $(t,x) \in (I \setminus N) \times \mathbb{R}$ one has

$$\min\left\{\limsup_{y \to x^-} f(t,y), \limsup_{y \to x^+} f(t,y)\right\} \le f(t,x) \le \max\left\{\liminf_{y \to x^-} f(t,y), \liminf_{y \to x^+} f(t,y)\right\}. \tag{2.12}$$

Then the mappings $t \in I \mapsto \sup\{f(t,y) : x_1(t) < y < x_2(t)\}$ and $t \in I \mapsto \inf\{f(t,y) : x_1(t) < y < x_2(t)\}$ are measurable for each pair $x_1, x_2 \in C(I)$ such that $x_1(t) < x_2(t)$ for all $t \in I$.

Remark 2.6. A revision of the proof of [9, Lemma 2] shows that it suffices to impose (2.12) for all $(t,x) \in (I \setminus N) \times \mathbb{R}$ such that $x_1(t) < x < x_2(t)$. This fact will be taken into account in this paper.

As a consequence of Theorem 2.4 and Lemma 2.5 we have a result about existence of extremal Carathéodory solutions for (1.1) and L^1-bounded nonlinearities. Note that the assumptions in Lemma 2.5 include a restriction over the type of discontinuities that can occur over the admissible nonqsc curves, but remember that such a restriction only plays the role of implying that the sets $J_{n,m}$ and $K_{n,m}$ in Theorem 2.4 are measurable. Therefore, only using the axiom of choice one can find a mapping f in the conditions of Theorem 2.4 which does not satisfy the assumptions in Lemma 2.5 and for which the corresponding problem (1.1) lacks the greatest (or the least) Carathéodory solution.

Theorem 2.7 ([9, Theorem 4]). *Suppose that there exists a null-measure set $N \subset I$ such that the following conditions hold:*

(i) *for every $q \in \mathbb{Q}$, $f(\cdot, q)$ is measurable;*

(ii) *for every $t \in I \setminus N$ and all $x \in \mathbb{R}$ one has either* (2.1) *or*

$$\liminf_{y \to x^-} f(t,y) \ge f(t,x) \ge \limsup_{y \to x^+} f(t,y), \tag{2.13}$$

and (2.1) *can fail, at most, over a countable family of admissible nonquasisemicontinuity curves;*

(iii) *there exists an integrable function $g = g(t)$, $t \in I$, such that*

$$|f(t,x)| \le g(t) \quad \forall (t,x) \in (I \setminus N) \times \mathbb{R}. \tag{2.14}$$

Then the mapping $u_{\inf}$ defined in (2.2) *is the least Carathéodory solution of* (1.1) *and the mapping $l_{\sup}$ defined in* (2.3) *is the greatest one.*

Remark 2.8. Theorem 4 in [9] actually asserts that $u^*_{\inf}$, as defined in (2.8), is the least Carathéodory solution, but it is easy to prove that in that case $u^*_{\inf} = u_{\inf}$, as defined in (2.2). Indeed, let U be an arbitrary upper solution of (1.1), let $\tilde{g} = \max\{|U'|, g\}$ and let

$$v^*_{\inf}(t) = \inf\{u(t) : u \text{ upper solution of (1.1)}, \left|u'\right| \leq \tilde{g} + 1 \text{ a.e.}\}, \quad t \in I. \tag{2.15}$$

Theorem 4 in [9] implies that also $v^*_{\inf}$ is the least Carathéodory solution of (1.1), thus $u^*_{\inf} = v^*_{\inf} \leq U$ on I. Hence $u^*_{\inf} = u_{\inf}$.

Analogously we can prove that $l^*_{\sup}$ can be replaced by $l_{\sup}$ in the statement of [21, Theorem 4].

3. Existence between Lower and Upper Solutions

Condition (iii) in Theorem 2.7 is rather restrictive and can be relaxed by assuming boundedness of f between a lower and an upper solution.

In this section we will prove the following result.

Theorem 3.1. *Suppose that* (1.1) *has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in I$ and let $E = \{(t,x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$.*

Suppose that there exists a null-measure set $N \subset I$ such that the following conditions hold:

($\mathrm{i}_{\alpha,\beta}$) *for every $q \in \mathbb{Q} \cap (\min_{t\in I}\alpha(t), \max_{t\in I}\beta(t))$, the mapping $f(\cdot,q)$ with domain $\{t \in I : \alpha(t) \leq q \leq \beta(t)\}$ is measurable;*

($\mathrm{ii}_{\alpha,\beta}$) *for every $(t,x) \in E$, $t \notin N$, one has either* (2.1) *or* (2.13), *and* (2.1) *can fail, at most, over a countable family of admissible non-quasisemicontinuity curves contained in E;*

($\mathrm{iii}_{\alpha,\beta}$) *there exists an integrable function $g = g(t)$, $t \in I$, such that*

$$\left|f(t,x)\right| \leq g(t) \quad \forall (t,x) \in E,\ t \notin N. \tag{3.1}$$

Then (1.1) *has extremal solutions in the set*

$$\left[\alpha,\beta\right] := \left\{z \in AC(I) : \alpha(t) \leq z(t) \leq \beta(t)\ \forall t \in I\right\}. \tag{3.2}$$

Moreover the least solution of (1.1) *in* $[\alpha,\beta]$ *is given by*

$$x_*(t) = \inf\left\{u(t) : u \text{ upper solution of (1.1)}, u \in \left[\alpha,\beta\right]\right\}, \quad t \in I, \tag{3.3}$$

and the greatest solution of (1.1) *in* $[\alpha,\beta]$ *is given by*

$$x^*(t) = \sup\left\{l(t) : l \text{ lower solution of (1.1)}, l \in \left[\alpha,\beta\right]\right\}, \quad t \in I. \tag{3.4}$$

Proof. Without loss of generality we suppose that α' and β' exist and satisfy $|\alpha'| \le g$, $|\beta'| \le g$, $\alpha' \le f(t,\alpha)$, and $\beta' \ge f(t,\beta)$ on $I \setminus N$. We also may (and we do) assume that every admissible nqsc curve in condition ($\text{ii}_{\alpha,\beta}$), say $\gamma : [a,b] \to \mathbb{R}$, satisfies for all $t \in [a,b] \setminus N$ either $\gamma'(t) = f(t,\gamma(t))$ or (2.4)-(2.5).

For each $(t,x) \in I \times \mathbb{R}$ we define

$$F(t,x) := \begin{cases} f(t,\alpha(t)), & \text{if } x < \alpha(t), \\ f(t,x), & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t,\beta(t)), & \text{if } x > \beta(t). \end{cases} \tag{3.5}$$

Claim 1. The modified problem

$$x' = F(t,x), \quad t \in I, \qquad x(t_0) = x_0, \tag{3.6}$$

satisfies conditions (1) and (2) in Theorem 2.4 with f replaced by F. First we note that (2) is an immediate consequence of ($\text{iii}_{\alpha,\beta}$) and the definition of F.

To show that condition (1) in Theorem 2.4 is satisfied with f replaced by F, let $(t,x) \in (I \setminus N) \times \mathbb{R}$ be fixed. The verification of (2.1) for F at (t,x) is trivial in the following cases: $\alpha(t) < x < \beta(t)$ and f satisfies (2.1) at (t,x), $x < \alpha(t)$, $x > \beta(t)$ and $\alpha(t) = x = \beta(t)$. Let us consider the remaining situations: we start with the case $x = \alpha(t) < \beta(t)$ and f satisfies (2.1) at (t,x), for which we have $F(t,x) = f(t,x)$ and

$$\limsup_{y \to x^-} F(t,y) = f(t,\alpha(t)) = f(t,x) \le \liminf_{y \to x^+} f(t,y) = \liminf_{y \to x^+} F(t,y), \tag{3.7}$$

and an analogous argument is valid when $\alpha(t) < \beta(t) = x$ and f satisfies (2.1).

The previous argument shows that F satisfies (2.1) at every $(t,x) \in (I \setminus N) \times \mathbb{R}$ except, at most, over the graphs of the countable family of admissible nonquasisemicontinuity curves in condition ($\text{ii}_{\alpha,\beta}$) for $x' = f(t,x)$. Therefore it remains to show that if $\gamma : [a,b] \subset I \to \mathbb{R}$ is one of those admissible nqsc curves for $x' = f(t,x)$ then it is also an admissible nqsc curve for $x' = F(t,x)$. As long as the graph of γ remains in the interior of E we have nothing to prove because f and F are the same, so let us assume that $\gamma = \alpha$ on a positive measure set $P \subset [a,b]$, $P \cap N = \emptyset$. Since α and γ are absolutely continuous there is a null measure set $\widehat{N}$ such that $\alpha'(t) = \gamma'(t)$ for all $t \in P \setminus \widehat{N}$, thus for $t \in P \setminus \widehat{N}$ we have

$$\gamma'(t) \le f(t,\gamma(t)) = \limsup_{y \to (\gamma(t))^-} F(t,y), \quad \gamma'(t) \le F(t,\gamma(t)), \tag{3.8}$$

so condition (2.5) with f replaced by F is satisfied on $P \setminus \widehat{N}$. On the other hand, we have to check whether $\gamma'(t) \ge F(t,\gamma(t))$ for those $t \in P \setminus \widehat{N}$ at which we have

$$\gamma'(t) \ge \liminf_{y \to (\gamma(t))^+} F(t,y). \tag{3.9}$$

We distinguish two cases: $\alpha(t) < \beta(t)$ and $\alpha(t) = \beta(t)$. In the first case (3.9) is equivalent to

$$\gamma'(t) \geq \liminf_{y \to (\gamma(t))^+} f(t, y), \tag{3.10}$$

and therefore either $\gamma'(t) = f(t, \gamma(t))$ or condition (2.4) holds, yielding $\gamma'(t) \geq f(t, \gamma(t)) = F(t, \gamma(t))$. If $\alpha(t) = \beta(t)$ then we have $\gamma'(t) = \alpha'(t) = \beta'(t) \geq f(t, \beta(t)) = F(t, \gamma(t))$.

Analogous arguments show that either $\gamma' = F(t, \gamma)$ or (2.4)-(2.5) hold for F at almost every point where γ coincides with β, so we conclude that γ is an admissible nqsc curve for $x' = F(t, x)$.

By virtue of Claim 1 and Theorem 2.4 we can ensure that the functions x_* and x^* defined as

$$\begin{aligned} x_*(t) &= \inf\left\{\tilde{u}(t) : u \text{ upper solution of (3.6)}, \left|\tilde{u}'\right| \leq g + 1 \text{ a.e.}\right\}, \quad t \in I, \\ x^*(t) &= \sup\left\{\tilde{l}(t) : \tilde{l} \text{ lower solution of (3.6)}, \left|\tilde{l}'\right| \leq g + 1 \text{ a.e.}\right\}, \quad t \in I, \end{aligned} \tag{3.11}$$

are absolutely continuous on I and satisfy $x_*(t_0) = x^*(t_0) = x_0$ and $x_*'(t) = F(t, x_*(t))$ for a.a. $t \in I \setminus J$, where $J = \cup_{n,m \in \mathbb{N}} J_{n,m}$ and for all $n, m \in \mathbb{N}$ the set

$$J_{n,m} := \left\{ t \in I : \ x_*'(t) - \frac{1}{n} > \sup\left\{ F(t, y) : x_*(t) - \frac{1}{m} < y < x_*(t) \right\} \right\} \tag{3.12}$$

contains no positive measure set, and $x^{*\prime}(t) = F(t, x^*(t))$ for a.a. $t \in I \setminus K$, where $K = \cup_{n,m \in \mathbb{N}} K_{n,m}$ and for all $n, m \in \mathbb{N}$ the set

$$K_{n,m} := \left\{ t \in I : x^{*\prime}(t) + \frac{1}{n} < \inf\left\{ F(t, y) : x^*(t) < y < x^*(t) + \frac{1}{m} \right\} \right\} \tag{3.13}$$

contains no positive measure set.

Claim 2. For all $t \in I$ we have

$$x_*(t) = \inf\left\{u(t) : u \text{ upper solution of (1.1)}, u \in [\alpha, \beta], \left|u'\right| \leq g + 1 \text{ a.e.}\right\}, \tag{3.14}$$

$$x^*(t) = \sup\left\{l(t) : l \text{ lower solution of (1.1)}, l \in [\alpha, \beta], \left|l'\right| \leq g + 1 \text{ a.e.}\right\}. \tag{3.15}$$

Let $\tilde{u}$ be an upper solution of (3.6) and let us show that $\tilde{u}(t) \geq \alpha(t)$ for all $t \in I$. Reasoning by contradiction, assume that there exist $t_1, t_2 \in I$ such that $t_1 < t_2$, $\tilde{u}(t_1) = \alpha(t_1)$ and

$$\tilde{u}(t) < \alpha(t) \quad \forall t \in (t_1, t_2]. \tag{3.16}$$

For a.a. $t \in (t_1, t_2]$ we have

$$\tilde{u}'(t) \geq F(t, \tilde{u}(t)) = f(t, \alpha(t)) \geq \alpha'(t), \tag{3.17}$$

which together with $\tilde{u}(t_1) = \alpha(t_1)$ imply $\tilde{u} \geq \alpha$ on $[t_1, t_2]$, a contradiction with (3.16). Therefore every upper solution of (3.6) is greater than or equal to α, and, on the other hand, β is an upper solution of (3.6) with $|\beta'| \leq g$ a.e., thus x_* satisfies (3.14).

One can prove by means of analogous arguments that x^* satisfies (3.15).

Claim 3. x_* is the least solution of (1.1) in $[\alpha, \beta]$ and x^* is the greatest one. From (3.14) and (3.15) it suffices to show that x_* and x^* are actually solutions of (3.6). Therefore we only have to prove that J and K are null measure sets.

Let us show that the set J is a null measure set. First, note that

$$J = \left\{ t \in I : x_*'(t) > \limsup_{y \to (x_*(t))^-} F(t, y) \right\}, \tag{3.18}$$

and we can split $J = A \cup B$, where $A = \{t \in J : x_*(t) > \alpha(t)\}$ and $B = J \setminus A = \{t \in J : x_*(t) = \alpha(t)\}$.

Let us show that B is a null measure set. Since α and x_* are absolutely continuous the set

$$\begin{aligned} C = &\{t \in I : \alpha'(t) \text{ does not exist}\} \\ &\cup \{t \in I : x_*'(t) \text{ does not exist}\} \\ &\cup \{t \in I : \alpha(t) = x_*(t),\ \alpha'(t) \neq x_*'(t)\} \end{aligned} \tag{3.19}$$

is null. If $B \not\subset C$ then there is some $t_0 \in B$ such that $\alpha(t_0) = x_*(t_0)$ and $\alpha'(t_0) = x_*'(t_0)$, but then the definitions of B and F yield

$$\alpha'(t_0) > \limsup_{y \to (\alpha(t_0))^-} F(t_0, y) = f(t_0, \alpha(t_0)). \tag{3.20}$$

Therefore $B \setminus C \subset N$ and thus B is a null measure set.

The set A can be expressed as $A = \cup_{k=1}^{\infty} A_k$, where for each $k \in \mathbb{N}$

$$\begin{aligned} A_k &= \left\{ t \in I : x_*(t) > \alpha(t) + \frac{1}{k},\ x_*'(t) > \limsup_{y \to (x_*(t))^-} F(t, y) \right\} \\ &= \bigcup_{n,m=1}^{\infty} A_k \cap J_{n,m}. \end{aligned} \tag{3.21}$$

For $k, m \in \mathbb{N}$, $k < m$, we have $x_*(t) - 1/m > x_*(t) - 1/k$, so the definition of F implies that

$$A_k \cap J_{n,m} = \left\{ t \in I : x_*(t) > \alpha(t) + \frac{1}{k},\ x_*'(t) - \frac{1}{n} > \sup\left\{ f(t, y) : x_*(t) - \frac{1}{m} < y < x_*(t) \right\} \right\} \tag{3.22}$$

which is a measurable set by virtue of Lemma 2.5 and Remark 2.6.

Since $J_{n,m}$ contains no positive measure subset we can ensure that $A_k \cap J_{n,m}$ is a null measure set for all $m \in \mathbb{N}$, $m > k$, and since $J_{n,m}$ increases with n and m, we conclude that $A_k = \cup_{n,m=1}^{\infty}(A_k \cap J_{n,m})$ is a null measure set. Finally A is null because it is the union of countably many null measure sets.

Analogous arguments show that K is a null measure set, thus the proof of Claim 3 is complete.

Claim 4. x_* satisfies (3.3) and x^* satisfies (3.4). Let $U \in [\alpha, \beta]$ be an upper solution of (1.1), let $\tilde{g} = \max\{|U'|, g\}$, and for all $t \in I$ let

$$y_*(t) = \inf\{\tilde{u}(t) : \tilde{u} \text{ upper solution of (3.6)}, \left|\tilde{u}'\right| \leq \tilde{g} + 1 \text{ a.e.}\}. \tag{3.23}$$

Repeating the previous arguments we can prove that also y_* is the least Carathéodory solution of (1.1) in $[\alpha, \beta]$, thus $x_* = y_* \leq U$ on I. Hence x_* satisfies (3.3).

Analogous arguments show that x^* satisfies (3.4). □

Remark 3.2. Problem (3.6) may not satisfy condition (i) in Theorem 2.7 as the compositions $f(\cdot, \alpha(\cdot))$ and $f(\cdot, \beta(\cdot))$ need not be measurable. That is why we used Theorem 2.4, instead of Theorem 2.7, to establish Theorem 3.1.

Next we show that even singular problems may fall inside the scope of Theorem 3.1 if we have adequate pairs of lower and upper solutions.

Example 3.3. Let us denote by $[z]$ the integer part of a real number z. We are going to show that the problem

$$x' = \left[\frac{1}{t + |x|}\right] x + \frac{\operatorname{sgn}(x)}{2}, \quad \text{for a.a. } t \in [0,1], \qquad x(0) = 0 \tag{3.24}$$

has positive solutions. Note that the limit of the right hand side as (t, x) tends to the origin does not exist, so the equation is singular at the initial condition.

In order to apply Theorem 3.1 we consider (1.1) with $t_0 = 0 = x_0$, $L = 1$, and

$$f(t,x) = \begin{cases} \left[\dfrac{1}{t + x}\right] x + \dfrac{1}{2}, & \text{if } x > 0, \\ \dfrac{1}{2}, & \text{if } x \leq 0. \end{cases} \tag{3.25}$$

It is elementary matter to check that $\alpha(t) = 0$ and $\beta(t) = t$, $t \in I$, are lower and upper solutions for the problem. Condition (2.1) only fails over the graphs of the functions

$$\gamma_n(t) = \frac{1}{n} - t, \quad t \in \left[0, \frac{1}{n}\right], \; n \in \mathbb{N}, \tag{3.26}$$

which are a countable family of admissible nqsc curves at which condition (2.13) holds.

Finally note that

$$\left|f(t,x)\right| \le \frac{3}{2} \quad \forall(t,x) \in I \times \mathbb{R},\ 0 \le x \le t, \tag{3.27}$$

so condition ($iii_{\alpha,\beta}$) is satisfied.

Theorem 3.1 ensures that our problem has extremal solutions between α and β which, obviously, are different from zero almost everywhere. Therefore (3.24) has positive solutions.

The result of Theorem 3.1 may fail if we assume that condition ($ii_{\alpha,\beta}$) is satisfied only in the interior of the set E. This is shown in the following example.

Example 3.4. Let us consider problem (1.1) with $t_0 = x_0 = 0$, $L = 1$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined as

$$f(t,x) = \begin{cases} 1, & \text{if } x < 0, \\ \dfrac{1}{2}, & \text{if } x = 0, \\ -1, & \text{if } x > 0. \end{cases} \tag{3.28}$$

It is easy to check that $\alpha(t) = 0$ and $\beta(t) = t$ for all $t \in [0,1]$ are lower and upper solutions for this problem and that all the assumptions of Theorem 3.1 are satisfied in the interior of E. However this problem has no solution at all.

In order to complete the previous information we can say that condition ($ii_{\alpha,\beta}$) in the interior of E is enough if we modify the definitions of lower and upper solutions in the following sense.

Theorem 3.5. *Suppose that α and β are absolutely continuous functions on I such that $\alpha(t) < \beta(t)$ for all $t \in (t_0, t_0 + L]$, $\alpha(t_0) \le x_0 \le \beta(t_0)$,*

$$\begin{aligned} \alpha'(t) &\le \liminf_{y \to (\alpha(t))^+} f(t,y) \quad \text{for a.a. } t \in I, \\ \beta'(t) &\ge \limsup_{y \to (\beta(t))^-} f(t,y) \quad \text{for a.a. } t \in I, \end{aligned} \tag{3.29}$$

and let $E = \{(t,x) \in I \times \mathbb{R} : \alpha(t) \le x \le \beta(t)\}$.

Suppose that there exists a null-measure set $N \subset I$ such that conditions ($i_{\alpha,\beta}$) and ($iii_{\alpha,\beta}$) hold and, moreover,

($ii^{\circ}_{\alpha,\beta}$) *for every $(t,x) \in \overset{\circ}{E}$, $t \notin N$, one has either* (2.1) *or* (2.13), *and* (2.1) *can fail, at most, over a countable family of admissible non-quasisemicontinuity curves contained in $\overset{\circ}{E}$.*

Then the conclusions of Theorem 3.1 hold true.

Proof (Outline)

It follows the same steps as the proof of Theorem 3.1 but replacing F by

$$\tilde{F}(t,x) = \begin{cases} 0, & \text{if } t = t_0, \\ \liminf\limits_{y \to (\alpha(t))^+} f(t,y), & \text{if } t > t_0,\ x \le \alpha(t), \\ f(t,x), & \text{if } t > t_0,\ \alpha(t) < x < \beta(t), \\ \limsup\limits_{y \to (\beta(t))^-} f(t,y), & \text{if } t > t_0,\ x \ge \beta(t). \end{cases} \tag{3.30}$$

Note that condition (2.1) with f replaced by $\tilde{F}$ is immediately satisfied over the graphs of α and β thanks to the definition of $\tilde{F}$.

Remarks

(i) The function α in Example 3.4 does not satisfy the conditions in Theorem 3.5.

(ii) When $f(t,\cdot)$ satisfies (2.1) everywhere or almost all $t \in I$ then every couple of lower and upper solutions satisfies the conditions in Theorem 3.5, so this result is not really new in that case (which includes the Carathéodory and continuous cases).

4. Discontinuous Lower and Upper Solutions

Another modification of the concepts of lower and upper solutions concerns the possibility of allowing jumps in their graphs. Since the task of finding a pair of lower and upper solutions is by no means easy in general, and bearing in mind that constant lower and upper solutions are the first reasonable attempt, looking for lower and upper solutions "piece by piece" might make it easier to find them in practical situations. Let us consider the following definition.

Definition 4.1. One says that $\alpha : I \to \mathbb{R}$ is a piecewise continuous lower solution of (1.1) if there exist $t_0 < t_1 < \cdots < t_n = t_0 + L$ such that

(a) for all $i \in \{1, 2, \ldots, n\}$, one has $\alpha \in AC(t_{i-1}, t_i)$ and for a.a. $t \in I$

$$\alpha'(t) \le f(t, \alpha(t)), \tag{4.1}$$

(b) $\lim_{t \to t_0^+} \alpha(t) = \alpha(t_0) \le x_0$, for all $i \in \{1, 2, \ldots, n\}$

$$\lim_{t \to t_i^-} \alpha(t) = \alpha(t_i) > \lim_{t \to t_i^+} \alpha(t), \tag{4.2}$$

and $\lim_{t \to t_n^-} \alpha(t) = \alpha(t_n)$.

A piecewise continuous upper solution of (1.1) is defined reversing the relevant inequalities.

The existence of a pair of well-ordered piecewise continuous lower and upper solutions implies the existence of a better pair of continuous lower and upper solutions. We establish this more precisely in our next proposition. Note that the proof is constructive.

Proposition 4.2. *Assume that all the conditions in Theorem 3.1 hold with piecewise continuous lower and upper solutions α and β.*

Then the following statements hold:

(i) *there exist a lower solution $\widetilde{\alpha}$ and an upper solution $\widetilde{\beta}$ such that*

$$\alpha \le \widetilde{\alpha} \le \widetilde{\beta} \le \beta \quad \text{on } I; \tag{4.3}$$

(ii) *if u is an upper solution of (1.1) with $\alpha \le u \le \beta$ then $\widetilde{\alpha} \le u$, and if l is a lower solution with $\alpha \le l \le \beta$ then $l \le \widetilde{\beta}$.*

In particular, the conclusions of Theorem 3.1 remain valid and, moreover, every solution of (1.1) between α and β lies between $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Proof. We will only prove the assertions concerning $\widetilde{\alpha}$ because the proofs for $\widetilde{\beta}$ are analogous.

To construct $\widetilde{\alpha}$ we simply have to join the points $(t_k, \alpha(t_k))$, $k \in \{1, \dots, n-1\}$ with the graph of $\alpha_{|(t_k, t_{k+1}]}$ by means of an absolutely continuous curve with derivative less than or equal to $-g$ a.e., g being the function given in $(\text{iii}_{\alpha,\beta})$. It can be easily proven that this $\widetilde{\alpha}$ is a lower solution of (1.1) that lies between α and β.

Moreover, if u is an upper solution of (1.1) between α and β then we have

$$u'(t) \ge f(t, u(t)) \ge -g(t) \quad \text{a.e. on } [t_0, t_0 + L], \tag{4.4}$$

so it cannot go below $\widetilde{\alpha}$. □

Piecewise continuous lower and upper solutions in the sense of Definition 4.1 were already used in [15, 22]. It is possible to generalize further the concept of lower and upper solutions, as a piecewise continuous lower solution is a particular case of a bounded variation function that has a nonincreasing singular part. Bounded variation lower and upper solutions with monotone singular parts were used in [23, 24], but it is not clear whether Theorem 3.1 is valid with this general type of lower and upper solutions. Anyway, piecewise continuous lower and upper solutions are enough in practical situations, and since these can be transformed into continuous ones which provide better information we will only consider from now on continuous lower and upper solutions as defined in Definition 2.1.

5. Singular Differential Equations

It is the goal of the present section to establish a theorem on existence of solutions for (1.1) between a pair of well-ordered lower and upper solutions and in lack of a local L^1 bound. Solutions will be weak, in the sense of the following definition. By $\text{AC}_{\text{loc}}((t_0, t_0+L])$ we denote the set of functions ξ such that $\xi_{|[t_0+\varepsilon, t_0+L]} \in \text{AC}([t_0+\varepsilon, t_0+L])$ for all $\varepsilon \in (0, L)$, and in a similar way we define $L^1_{\text{loc}}((t_0, t_0+L])$.

Definition 5.1. We say that $\alpha \in C(I) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L])$ is a weak lower solution of (1.1) if $\alpha(t_0) \le x_0$ and $\alpha'(t) \le f(t, \alpha(t))$ for a.a. $t \in I$. A weak upper solution is defined analogously reversing inequalities. A weak solution of (1.1) is a function which is both a weak lower solution and a weak upper solution.

We will also refer to *extremal weak solutions* with obvious meaning.

Note that (lower/upper) solutions, as defined in Definition 2.1, are weak (lower/upper) solutions but the converse is false in general. For instance the singular linear problem

$$x' = \frac{x}{t} - \frac{\cos(1/t)}{t}, \quad t \in (0,1], \quad x(0) = 0, \tag{5.1}$$

has exactly the following weak solutions:

$$x_a(t) = t \sin\left(\frac{1}{t}\right) + at, \quad t \in (0,1], \qquad x_a(0) = 0 \quad (a \in \mathbb{R}), \tag{5.2}$$

and none of them is absolutely continuous on $[0,1]$. Another example, which uses lower and upper solutions, can be found in [15, Remark 2.4].

However weak (lower/upper) solutions are of Carathéodory type provided they have bounded variation. We establish this fact in the next proposition.

Proposition 5.2. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $h : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and locally absolutely continuous on $(a,b]$.*

A necessary and sufficient condition for h to be absolutely continuous on $[a,b]$ is that h be of bounded variation on $[a,b]$.

Proof. The necessary part is trivial. To estalish the sufficiency of our condition we use Banach-Zarecki's theorem, see [18, Theorem 18.25]. Let $N \subset [a,b]$ be a null measure set, we have to prove that $h(N)$ is also a null measure set. To do this let $n_0 \in \mathbb{N}$ be such that $a + 1/n_0 < b$. Since h is absolutely continuous on $[a + 1/n_0, b]$ the set $h(N \cap [a + 1/n, b])$ is a null measure set for each $n \ge n_0$. Therefore $h(N)$ is also a null measure set because

$$h(N) \subset \{h(a)\} \cup \left(\bigcup_{n=n_0}^{\infty} h\left(N \cap \left[a + \frac{1}{n}, b\right]\right)\right). \tag{5.3}$$

Next we present our main result on existence of weak solutions for (1.1) in absence of integrable bounds. □

Theorem 5.3. *Suppose that* (1.1) *has a weak lower solution α and a weak upper solution β such that $\alpha(t) \le \beta(t)$ for all $t \in I$ and $\alpha(t_0) = x_0 = \beta(t_0)$.*

Suppose that there is a null-measure set $N \subset I$ such that conditions $(i_{\alpha,\beta})$ and $(ii_{\alpha,\beta})$ in Theorem 3.1 hold for $E = \{(t,x) \in I \times \mathbb{R} : \alpha(t) \le x \le \beta(t)\}$, and assume moreover that the following condition holds:

(iii$^*_{\alpha,\beta}$) *there exists* $g \in L^1_{\mathrm{loc}}((t_0, t_0+L])$ *such that for all* $(t,x) \in E$, $t \notin N$, *one has* $|f(t,x)| \le g(t)$.
Then (1.1) *has extremal weak solutions in the set*

$$[\alpha,\beta]_w := \{z \in C(I) \cap AC_{\mathrm{loc}}((t_0, t_0+L]) : \alpha(t) \le z(t) \le \beta(t)\ \forall t \in I\}. \tag{5.4}$$

Moreover the least weak solution of (1.1) *in* $[\alpha,\beta]$ *is given by*

$$x_*(t) = \inf\{u(t) : u \text{ weak upper solution of (1.1)}, u \in [\alpha,\beta]_w\}, \quad t \in I, \tag{5.5}$$

and the greatest weak solution of (1.1) *in* $[\alpha,\beta]_w$ *is given by*

$$x^*(t) = \sup\{l(t) : l \text{ weak lower solution of (1.1)}, l \in [\alpha,\beta]_w\}, \quad t \in I. \tag{5.6}$$

Proof. We will only prove that (5.6) defines the greatest weak solution of (1.1) in $[\alpha,\beta]_w$, as the arguments to show that (5.5) is the least one are analogous.

First note that α is a weak lower solution between α and β, so x^* is well defined.

Let $\{t_n\}_n$ be a decreasing sequence in (t_0, t_0+L) such that $\lim t_n = t_0$. Theorem 2.7 ensures that for every $n \in \mathbb{N}$ the problem

$$y' = f(t,y), \quad t \in [t_n, t_0+L] =: I_n, \qquad y(t_n) = x^*(t_n), \tag{5.7}$$

has extremal Carathéodory solutions between $\alpha_{|I_n}$ and $\beta_{|I_n}$. Let y_n denote the greatest solution of (5.7) between $\alpha_{|I_n}$ and $\beta_{|I_n}$. By virtue of Theorem 2.7 we also know that y_n is the greatest lower solution of (5.7) between $\alpha_{|I_n}$ and $\beta_{|I_n}$.

Next we prove in several steps that $x^* = y_n$ on I_n for each $n \in \mathbb{N}$.

Step 1 ($y_n \ge x^*$ on I_n for each $n \in \mathbb{N}$). The restriction to I_n of each weak lower solution between α and β is a lower solution of (5.7) between $\alpha_{|I_n}$ and $\beta_{|I_n}$, thus y_n is, on the interval I_n, greater than or equal to any weak lower solution of (1.1) between α and β. The definition of x^* implies then that $y_n \ge x^*$ on I_n.

Step 2 ($y_{n+1} \ge y_n$ on I_n for all $n \in \mathbb{N}$). First, since $y_{n+1} \ge x^*$ on I_{n+1} we have $y_{n+1}(t_n) \ge x^*(t_n) = y_n(t_n)$. Reasoning by contradiction, assume that there exists $s \in (t_n, t_0+L)$ such that $y_{n+1}(s) < y_n(s)$. Then there is some $r \in [t_n, s)$ such that $y_{n+1}(r) = y_n(r)$ and $y_{n+1} < y_n$ on (r,s), but then the mapping

$$\tilde{y}(t) = \begin{cases} y_{n+1}(t), & \text{if } t \in [t_{n+1}, r], \\ y_n(t), & \text{if } t \in [r, t_0+L], \end{cases} \tag{5.8}$$

would be a solution of (5.7) (with n replaced by $n+1$) between $\alpha_{|I_{n+1}}$ and $\beta_{|I_{n+1}}$ which is greater than y_{n+1} on (r,s), a contradiction.

The above properties of $\{y_n\}_n$ imply that the following function is well defined:

$$y_\infty(t) = \begin{cases} x_0, & \text{if } t = t_0, \\ \lim y_n(t), & \text{if } t \in (t_0, t_0 + L]. \end{cases} \tag{5.9}$$

Step 3 ($y_\infty \in C(I) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L])$). Let $\varepsilon \in (0, L)$ be fixed. Condition $(\mathrm{iii}^*_{\alpha,\beta})$ implies that for all $n \in \mathbb{N}$ such that $t_n < t_0 + \varepsilon$ we have

$$\left|y_n'(t)\right| = \left|f\left(t, y_n(t)\right)\right| \leq g(t) \quad \text{for a.a. } t \in [t_0 + \varepsilon, t_0 + L], \tag{5.10}$$

with $g \in L^1(t_0 + \varepsilon, t_0 + L)$. Hence for $s, t \in [t_0 + \varepsilon, t_0 + L]$, $s \leq t$, we have

$$\left|y_\infty(t) - y_\infty(s)\right| = \lim_{n \to \infty} \left|\int_s^t y_n'\right| \leq \int_s^t g, \tag{5.11}$$

and therefore $y_\infty \in \mathrm{AC}([t_0 + \varepsilon, t_0 + L])$. Since $\varepsilon \in (0, L)$ was fixed arbitrarily in the previous arguments, we conclude that $y_\infty \in \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L])$.

The continuity of y_∞ at t_0 follows from the continuity of α and β at t_0, the assumption $\alpha(t_0) = x_0 = \beta(t_0)$, and the relation

$$\alpha(t) \leq y_\infty(t) \leq \beta(t) \quad \forall t \in (t_0, t_0 + L]. \tag{5.12}$$

Step 4 (y_∞ is a weak lower solution of (1.1)). For $\varepsilon \in (0, L)$ and $n \in \mathbb{N}$ such that $t_n < t_0 + \varepsilon$ we have (5.10) with $g \in L^1(t_0 + \varepsilon, t_0 + L)$, hence $\limsup y_n' \in L^1(t_0 + \varepsilon, t_0 + L)$, and for $s, t \in (t_0 + \varepsilon, t_0 + L)$, $s < t$, Fatou's lemma yields

$$y_\infty(t) - y_\infty(s) = \lim_{n \to \infty} \int_s^t y_n' \leq \int_s^t \limsup\, y_n'. \tag{5.13}$$

Hence for a.a. $t \in [t_0 + \varepsilon, t_0 + L]$ we have

$$y_\infty'(t) \leq \limsup\, y_n'(t) = \limsup f\left(t, y_n(t)\right). \tag{5.14}$$

Let $J_1 = \cup_{n \in \mathbb{N}} A_n$ where $A_n = \{t \in [t_0 + \varepsilon, t_0 + L] : y_\infty(t) = y_n(t)\}$ and $J_2 = [t_0 + \varepsilon, t_0 + L] \setminus J_1$.

For $n \in \mathbb{N}$ and a.a. $t \in A_n$ we have $y_\infty'(t) = y_n'(t) = f(t, y_n(t)) = f(t, y_\infty(t))$, thus $y_\infty'(t) = f(t, y_\infty(t))$ for a.a. $t \in J_1$.

On the other hand, for a.a. $t \in J_2$ the relation (5.14) and the increasingness of $\{y_n(t)\}$ yield

$$y'_\infty(t) \le \limsup f(t, y_n(t)) \le \limsup_{y \to (y_\infty(t))^-} f(t, y). \tag{5.15}$$

Let $t_0 \in J_2 \setminus N$ be such that (5.15) holds. We have two possibilities: either (2.1) holds for f at $(t_0, y_\infty(t_0))$ and then from (5.15) we deduce $y'_\infty(t_0) \le f(t_0, y_\infty(t_0))$, or $y_\infty(t_0) = \gamma(t_0)$, where γ is an admissible curve of non quasisemicontinuity. In the last case we have that either t_0 belongs to a null-measure set or $y'_\infty(t_0) = \gamma'(t_0)$, which, in turn, yields two possibilities: either $\gamma'(t_0) = f(t, \gamma(t_0))$ and then $y'_\infty(t_0) = f(t, y_\infty(t_0))$, or $\gamma'(t_0) \ne f(t, \gamma(t_0))$ and then (5.15), with $y_\infty(t_0) = \gamma(t_0)$ and $y'_\infty(t_0) = \gamma'(t_0)$, and the definition of admissible curve of non quasisemicontinuity imply that $y'_\infty(t_0) \le f(t_0, y_\infty(t_0))$.

The above arguments prove that $y'_\infty(t) \le f(t, y_\infty(t))$ a.e. on $[t_0 + \varepsilon, t_0 + L]$, and since $\varepsilon \in (0, L)$ was fixed arbitrarily, the proof of Step 4 is complete.

Conclusion

The construction of y_∞ and Step 1 imply that $y_\infty \ge x^*$ and the definition of x^* and Step 4 imply that $x^* \ge y_\infty$. Therefore for all $n \in \mathbb{N}$ we have $x^* = y_n$ on I_n and then x^* is a weak solution of (1.1). Since every weak solution is a weak lower solution, x^* is the greatest weak solution of (1.1) in $[\alpha, \beta]_w$. □

The assumption $\alpha(t_0) = \beta(t_0)$ in Theorem 5.3 can be replaced by other types of conditions. The next theorem generalizes the main results in [7, 12–14] concerning existence of solutions of singular problems of the type of (1.1).

Theorem 5.4. *Suppose that* (1.1) *with* $x_0 = 0$ *has a weak lower solution* α *and a weak upper solution* β *such that* $\alpha(t) \le \beta(t)$ *for all* $t \in I$ *and* $\alpha > 0$ *on* $(t_0, t_0 + L]$.

Suppose that there is a null-measure set $N \subset I$ *such that conditions* $(i_{\alpha,\beta})$ *and* $(ii_{\alpha,\beta})$ *in Theorem 3.1 hold for* $E = \{(t, x) \in I \times \mathbb{R} : \alpha(t) \le x \le \beta(t)\}$, *and assume moreover that the following condition holds:*

$(iii^+_{\alpha,\beta})$ *for every* $r \in (0, 1)$ *there exists* $g_r \in L^1(I)$ *such that for all* $(t, x) \in E$, $t \notin N$, *and* $r \le x \le 1/r$ *one has* $|f(t, x)| \le g_r(t)$.

Then the conclusions of Theorem 5.3 hold true.

Proof. We start observing that there exists a weak upper solution $\tilde{\beta}$ such that $\alpha \le \tilde{\beta} \le \beta$ on I and $\alpha(t_0) = 0 = \tilde{\beta}(t_0)$. If $\beta(t_0) = 0$ then it suffices to take $\tilde{\beta}$ as β. If $\beta(t_0) > 0$ we proceed as follows in order to construct $\tilde{\beta}$: let $\{x_n\}_n$ be a decreasing sequence in $(0, \beta(t_0))$ such that $\lim x_n = 0$ and for every $n \in \mathbb{N}$ let y_n be the greatest solution between α and β of

$$y' = f(t, y), \quad t \in I, \quad y(t_0) = x_n. \tag{5.16}$$

claim [y_n exists]

Let $\varepsilon \in (0, L)$ be so small that $\alpha(t) < x_n - \varepsilon < x_n + \varepsilon < \beta(t)$ for all $t \in [t_0, t_0 + \varepsilon]$. Condition $(\text{iii}^+_{\alpha,\beta})$ implies that there exists $g_\varepsilon \in L^1(t_0, t_0 + \varepsilon)$ such that for a.a. $t \in [t_0, t_0 + \varepsilon]$ and all $x \in [x_n - \varepsilon, x_n + \varepsilon]$ we have $|f(t, x)| \le g_\varepsilon(t)$. Let $v_\pm(t) = x_n \pm \int_0^t g_\varepsilon(s)ds$, $t \in [t_0, t_0 + \varepsilon]$ and let $\delta \in (0, \varepsilon]$ be such that $x_n - \varepsilon \le v_- \le v_+ \le x_n + \varepsilon$ on $[t_0, t_0 + \delta]$. We can apply Theorem 2.7 to the problem

$$\xi' = f(t, \xi), \quad t \in [t_0, t_0 + \delta], \quad \xi(t_0) = x_n, \tag{5.17}$$

and with respect to the lower solution v_- and the upper solution v_+, so there exists ξ_n the greatest solution between v_- and v_+ of (5.17). Notice that if x is a solution of (5.17) then $v_- \le x \le v_+$, so ξ_n is also the greatest solution between α and β of (5.17).

Now condition $(\text{iii}^+_{\alpha,\beta})$ ensures that Theorem 2.7 can be applied to the problem

$$z' = f(t, z), \quad t \in [t_0 + \delta, t_0 + L], \quad z(t_0 + \delta) = \xi_n(t_0 + \delta), \tag{5.18}$$

with respect to the lower solution α and the upper solution β (both functions restricted to $[t_0 + \delta, t_0 + L]$). Hence there exists z_n the greatest solution of (5.18) between α and β.

Obviously we have

$$y_n(t) = \begin{cases} \xi_n(t), & \text{if } t \in [t_0, t_0 + \delta], \\ z_n(t), & \text{if } t \in [t_0 + \delta, t_0 + L]. \end{cases} \tag{5.19}$$

Analogous arguments to those in the proof of Theorem 5.3 show that $\widetilde{\beta} = \lim y_n$ is a weak upper solution and it is clear that $\widetilde{\beta}(t_0) = 0 = \alpha(t_0)$.

Finally we show that $(\text{iii}^*_{\alpha,\beta})$ holds with β replaced by $\widetilde{\beta}$. We consider a decreasing sequence $(a_n)_n$ such that $a_0 = t_0 + L$ and $\lim a_n = t_0$. As α and β are positive on $(t_0, t_0 + L]$, we can find $r_i > 0$ such that $r_i \le \alpha \le \beta \le 1/r_i$ on $[a_{i+1}, a_i]$. We deduce then from $(\text{iii}^+_{\alpha,\beta})$ the existence of $\psi_i \in L^1(a_{i+1}, a_i)$ so that $|f(t, x)| \le \psi_i(t)$ for a.e. $t \in [a_{i+1}, a_i]$ and all $x \in [\alpha(t), \beta(t)]$. The function g defined by $g(t) = \psi_i(t)$ for $t \in (a_{i+1}, a_i]$ works.

Theorem 5.3 implies that (1.1) has extremal weak solutions in $[\alpha, \widetilde{\beta}]_w$ which, moreover, satisfy (5.6) and (5.5) with β replaced by $\widetilde{\beta}$. Furthermore if x is a weak solution of (1.1) in $[\alpha, \beta]_w$ then $x \le \widetilde{\beta}$ on I. Assume, on the contrary, that $x(s) > \widetilde{\beta}(s)$ for some $s \in (t_0, t_0 + L)$, then there would exist $n \in \mathbb{N}$ such that $y_n(s) < x(s)$ and then $y = \max\{x, y_n\}$ would be a solution of (5.16) between α and β which is strictly greater than y_n on some subinterval, a contradiction. Hence (1.1) has extremal weak solutions in $[\alpha, \beta]_w$ which, moreover, satisfy (5.6) and (5.5).

□

6. Systems

Let us consider the following system of $n \in \mathbb{N}$ ordinary differential equations:

$$\vec{x}'(t) = \vec{f}(t, \vec{x}) \quad \text{for a. a.}\, t \in I = [t_0, t_0 + L], \quad \vec{x}(t_0) = \vec{x}_0, \tag{6.1}$$

where $t_0 \in \mathbb{R}$, $L > 0$, $\vec{x} = (x_1, \ldots, x_n)$, $\vec{x}_0 = (x_{0,1}, \ldots, x_{0,n})$, and $\vec{f} = (f_1, f_2, \ldots, f_n) : I \times \mathbb{R}^n \to \mathbb{R}^n$.

Our goal is to extend Theorem 5.3 to this multidimensional setting, which, as usual, requires the right hand side $\vec{f}$ to be quasimonotone, as we will define later.

We start extending to the vector case the definitions given before for scalar problems. To do so, let $\mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n)$ denote the set of functions $\vec{y} = (y_1, y_2, \ldots, y_n) : (t_0, t_0 + L] \to \mathbb{R}^n$ such that for each $i \in \{1, 2, \ldots, n\}$ the component y_i is absolutely continuous on $[t_0 + \varepsilon, t_0 + L]$ for each $\varepsilon \in (0, L)$. Also, $C(I, \mathbb{R}^n)$ stands for the class of $\mathbb{R}^n$-valued functions which are defined and continuous on I.

A weak lower solution of (6.1) is a function $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C(I, \mathbb{R}^n) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n)$ such that for each $i \in \{1, 2, \ldots, n\}$ we have $\alpha_i(0) \le x_{0,i}$ and for a.a. $t \in I$ we have $\alpha_i'(t) \le f_i(t, \vec{\alpha}(t))$. Weak upper solutions are defined similarly by reversing the relevant inequalities, and weak solutions of (6.1) are functions which are both weak lower and weak upper solutions.

In the set $C(I, \mathbb{R}^n) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n)$ we define a partial ordering as follows: let $\vec{\alpha}, \vec{\beta} \in C(I, \mathbb{R}^n) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n)$, we write $\vec{\alpha} \le \vec{\beta}$ if every component of $\vec{\alpha}$ is less than or equal to the corresponding component of $\vec{\beta}$ on the whole of I. If $\vec{\alpha}, \vec{\beta} \in C(I, \mathbb{R}^n) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n)$ are such that $\vec{\alpha} \le \vec{\beta}$ then we define

$$\left[\vec{\alpha}, \vec{\beta}\right] := \left\{\vec{\eta} \in C(I, \mathbb{R}^n) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n) : \vec{\alpha} \le \vec{\eta} \le \vec{\beta}\right\}. \tag{6.2}$$

Extremal (least and greatest) weak solutions of (6.1) in a certain subset of $C(I, \mathbb{R}^n) \cap \mathrm{AC}_{\mathrm{loc}}((t_0, t_0 + L], \mathbb{R}^n)$ are defined in the obvious way considering the previous ordering.

Now we are ready to extend Theorem 5.3 to the vector case. We will denote by $\vec{e_i}$ the ith canonical vector. The proof follows the line of that of [8, Theorem 5.1].

Theorem 6.1. *Suppose that* (6.1) *has weak lower and upper solutions* $\vec{\alpha}$ *and* $\vec{\beta}$ *such that* $\vec{\alpha} \le \vec{\beta}$, $\vec{\alpha}(t_0) = \vec{x}_0 = \vec{\beta}(t_0)$, *and let* $E^* = \{(t, \vec{x}) : t \in I, \vec{\alpha}(t) \le \vec{x} \le \vec{\beta}(t)\}$.

Suppose that $\vec{f}$ *is quasimonotone nondecreasing in* E^*, *that is, for* $i \in \{1, 2, \ldots, n\}$ *and* $(t, \vec{x}), (t, \vec{y}) \in E^*$ *the relations* $\vec{x} \le \vec{y}$ *and* $x_i = y_i$ *imply* $f_i(t, \vec{x}) \le f_i(t, \vec{y})$.

Suppose, moreover, that for each $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in [\vec{\alpha}, \vec{\beta}]$ *the following conditions hold:*

(H_1) *the function* $\vec{f}(\cdot, \vec{\eta}(\cdot))$ *is measurable;*

(H_2) *for all* $i \in \{1, \ldots, n\}$ *and a.a.* $t \in I$ *one has either*

$$\begin{aligned}\limsup_{s \to x^-} f_i\left(t, \vec{\eta}(t) + (s - \eta_i(t))\vec{e_i}\right) &\le f_i\left(t, \vec{\eta}(t) + (x - \eta_i(t))\vec{e_i}\right) \\ &\le \liminf_{s \to x^+} f_i\left(t, \vec{\eta}(t) + (s - \eta_i(t))\vec{e_i}\right),\end{aligned} \tag{6.3}$$

or

$$\liminf_{s\to x^-} f_i\left(t,\overrightarrow{\eta}(t)+(s-\eta_i(t))\overrightarrow{e_i}\right) \ge f_i\left(t,\overrightarrow{\eta}(t)+(x-\eta_i(t))\overrightarrow{e_i}\right) \\ \ge \limsup_{s\to x^+} f_i\left(t,\overrightarrow{\eta}(t)+(s-\eta_i(t))\overrightarrow{e_i}\right), \tag{6.4}$$

and (6.3) *fails, at most, over a countable family of admissible nqsc curves of the scalar differential equation* $x' = f_i(t,\overrightarrow{\eta}(t)+(x-\eta_i(t))\overrightarrow{e_i})$ *contained in the sector* $E_i := \{(t,x) : t\in I,\ \alpha_i(t)\le x\le \beta_i(t)\}$*;*

(H_3) *there exists* $g\in L^1_{\text{loc}}((t_0,t_0+L])$ *such that for* $i\in\{1,2,\ldots,n\}$ *and a.a.* $t\in I$ *one has* $|f_i(t,\overrightarrow{\eta}(t))|\le g(t)$*.*

Then (6.1) *has extremal weak solutions in* $[\alpha,\beta]$*. Moreover the least weak solution* $\overrightarrow{x_*}=(x_{*,1},\ldots,x_{*,n})$ *is given by*

$$x_{*,i}(t)=\inf\left\{u_i(t):(u_1,\ldots,u_n)\text{ weak upper solution of (6.1) in }\left[\overrightarrow{\alpha},\overrightarrow{\beta}\right]\right\} \tag{6.5}$$

and the greatest weak solution $\overrightarrow{x^*}=(x_1^*,\ldots,x_n^*)$ *is given by*

$$x_i^*(t)=\sup\left\{l_i(t):(l_1,\ldots,l_n)\text{ weak lower solution of (6.1) in }\left[\overrightarrow{\alpha},\overrightarrow{\beta}\right]\right\}. \tag{6.6}$$

Proof. Let $\overrightarrow{L}=(L_1,\ldots,L_n)\in[\overrightarrow{\alpha},\overrightarrow{\beta}]$ be a weak lower solution of (6.1), and let $g\in L^1_{\text{loc}}((t_0,t_0+L])$ be as in (H_3) and such that $|L_i'|\le g$ a.e. on I for all $i\in\{1,\ldots,n\}$. Now let $\overrightarrow{\xi^*}=(\xi_1^*,\ldots,\xi_n^*)$ be defined for $i\in\{1,\ldots,n\}$ as

$$\xi_i^*(t)=\sup\left\{l_i(t):(l_1,\ldots,l_n)\text{ weak lower solution in }\left[\overrightarrow{\alpha},\overrightarrow{\beta}\right],\ \left|l_i'\right|\le g\text{ a.e. on }I\right\}. \tag{6.7}$$

In particular, $\overrightarrow{\xi^*}\ge\overrightarrow{L}$. Further, every possible solution of (6.1) in $[\overrightarrow{\alpha},\overrightarrow{\beta}]$ is less than or equal to $\overrightarrow{\xi^*}$ by (6.7) and ($H3$), independently of $\overrightarrow{L}$.

Claim 1 ($\overrightarrow{\xi^*}\in C(I,\mathbb{R}^n)\cap \text{AC}_{\text{loc}}((t_0,t_0+L],\mathbb{R}^n)$). If $(l_1,\ldots,l_n)$ is a weak lower solution in $[\overrightarrow{\alpha},\overrightarrow{\beta}]$ with $|l_i'|\le g$ a.e. on I then for $s,t\in(t_0,t_0+L]$, $s<t$, we have

$$|l_i(t)-l_i(s)|\le\int_s^t g(r)dr, \tag{6.8}$$

which implies

$$\left|\xi_i^*(t)-\xi_i^*(s)\right|\le\int_s^t g(r)dr, \tag{6.9}$$

and, therefore, $\xi_i^*\in\text{AC}_{\text{loc}}((t_0,t_0+L])$. Further ξ_i^* is continuous at t_0 because $\alpha_i(t)\le\xi_i^*(t)\le\beta_i(t)$ for all $t\in I$, α_i and β_i are continuous at t_0 and $\alpha_i(t_0)=\beta_i(t_0)$.

Claim 2. $\overrightarrow{\xi^*}$ is the greatest weak solution of (6.1) in $[\vec{\alpha}, \vec{\beta}]$. For each weak lower solution $\vec{l} \in [\vec{\alpha}, \vec{\beta}]$ such that $|l_i'| \le g$ a.e., the quasimonotonicity of $\vec{f}$ yields

$$l_i'(t) \le f_i\left(t, \overrightarrow{\xi^*}(t) + \left(l_i(t) - \xi_i^*(t)\right)\vec{e_i}\right) \quad \text{for a. a. } t \in I. \tag{6.10}$$

Hence l_i is a weak lower solution between α_i and β_i of the scalar problem

$$x' = f_i\left(t, \overrightarrow{\xi^*}(t) + \left(x - \xi_i^*(t)\right)\vec{e_i}\right) \quad \text{for a. a. } t \in I, \quad x(t_0) = x_{0,i}, \tag{6.11}$$

and then Theorem 5.3 implies that $l_i \le y_i^*$, where y_i^* is the greatest weak solution of (6.11) in $[\alpha_i, \beta_i]$. Then $\overrightarrow{\xi^*} \le \overrightarrow{y^*} = (y_1^*, \ldots, y_n^*)$.

On the other hand, we have

$$y_i^{*\prime}(t) = f_i\left(t, \overrightarrow{\xi^*}(t) + \left(y_i^*(t) - \xi_i^*(t)\right)\vec{e}_i\right) \le f_i\left(t, \overrightarrow{y^*}\right) \quad \text{for a. a. } t \in I, \tag{6.12}$$

hence $\overrightarrow{y^*}$ is a weak lower solution of (6.1) in $[\vec{\alpha}, \vec{\beta}]$ with $|y_i'| \le g$ a.e. on I, thus $\overrightarrow{\xi^*} \ge \overrightarrow{y^*}$. Therefore $\overrightarrow{\xi^*}$ is a weak solution of (6.1), and, by (6.7) and $(H3)$, it is the greatest one in $[\vec{\alpha}, \vec{\beta}]$. In particular, the greatest weak solution of (6.1) in $[\vec{\alpha}, \vec{\beta}]$ exists and it is greater than or equal to $\vec{L}$.

Claim 3. The greatest weak solution of (6.1) in $[\vec{\alpha}, \vec{\beta}]$, $\overrightarrow{x^*}$, satisfies (6.6). The weak lower solution $\vec{L} \in [\vec{\alpha}, \vec{\beta}]$ was fixed arbitrarily, so $\overrightarrow{x^*}$ is greater than or equal to any weak lower solution in $[\vec{\alpha}, \vec{\beta}]$. On the other hand, $\overrightarrow{x^*}$ is a weak lower solution.

Analogously, the least weak solution of (6.1) in $[\vec{\alpha}, \vec{\beta}]$ is given by (6.5). □

7. Examples

Example 7.1. Let us show that the following singular and non-quasisemicontinous problem has a unique positive Carathéodory solution:

$$x' = f(x) = \left[\frac{1}{x}\right], \quad t \in I = \left[0, \frac{1}{2}\right], \qquad x(0) = 0. \tag{7.1}$$

Here square brackets mean integer part, and by positive solution we mean a solution which is positive on $(0, 1/2]$.

First note that (7.1) has at most one positive weak solution because the right hand side in the differential equation is nonincreasing with respect to the unknown on $(0, +\infty)$, thus at no point can solutions bifurcate.

For all $x > 0$ we have $[1/x] \le 1/x$ and therefore $\beta(t) = \sqrt{2t}$, $t \in I$, is an upper solution of (7.1) as it solves the majorant problem

$$x' = \frac{1}{x}, \qquad x(0) = 0. \tag{7.2}$$

On the other hand it is easy to check that for $0 < x \leq 1$ we have $[1/x] \geq 1/(2x)$ and then $\alpha(t) = \sqrt{t}$, $t \in I$, is a lower solution.

The function f is continuous between the graphs of α and β except over the lines $\gamma_n(t) = 1/n$, $t \in [n^{-2}/2, n^{-2}]$, which are admissible nqsc curves for all $n \in \mathbb{N}$, $n \geq 2$ (note that γ_1 is not an admissible nqsc curve but it does not lie between α and β).

Finally, for $r \in (0,1)$ we have

$$|f(x)| \leq n-1 \quad \text{for } \max\{r, \alpha(t)\} \leq x \leq \beta(t), \tag{7.3}$$

where $n \in \mathbb{N}$ is such that $1/n < r$.

Therefore Theorem 5.4 implies the existence of a weak solution of (7.1) between α and β. Moreover, this weak solution between α and β is increasing, so Proposition 5.2 ensures that it is, in fact, a Carathéodory solution on I.

It is possible to extend the solution on the right of $t = 1/2$ to some t^* where the solution will assume the value 1. The solution cannot be extended further on the right of t^*, as (7.1) with $x(0) = 0$ replaced by $x(0) = 1$ has no solution on the right of 0.

We owe to the anonymous referee the following remarks. Problem (7.1) is autonomous, so it falls inside the scope of the results in [21], which ensure that if we find $\alpha > 0$ such that

$$\int_0^{\alpha} \frac{ds}{[1/s]} < +\infty, \tag{7.4}$$

then (7.1) has a positive absolutely continuous solution defined implicitly by

$$\int_0^{x(t)} \frac{ds}{[1/s]} = t \quad \forall t \in \left(0, \int_0^{\alpha} \frac{ds}{[1/s]}\right). \tag{7.5}$$

Since

$$\int_0^1 \frac{ds}{[1/s]} = \frac{\pi^2 - 6}{6} \approx 0.644934, \tag{7.6}$$

we deduce that the solution $x(t)$ is defined at least on $[0, t^*]$, where $t^* = (\pi^2 - 6)/6$ and $x(t^*) = 1$.

Example 7.2. Let $\varepsilon : [0,1] \to \mathbb{R}$ be measurable and $0 < \varepsilon(t) \leq 1$ for a.a. $t \in [0,1]$. We will prove that for each $k \in \mathbb{R}$, $k \geq 1$, the problem

$$x' = f(t,x) = \left[\frac{1}{x^k}\right] - \left[\frac{1}{t^k}\right] + \varepsilon(t), \quad t \in I = [0,1], \qquad x(0) = 0 \tag{7.7}$$

has a unique positive Carathéodory solution.

Note that the equation is not separable and f assumes positive and negative values on every neighborhood of the initial condition. Moreover the equation is singular at the initial condition with respect to both of its variables.

Once again the right hand side in the differential equation is nonincreasing with respect to the unknown x on $(0,+\infty)$, thus we have at most one positive weak solution.

Lower and upper solutions are given by, respectively, $\alpha(t) = t/3$ and $\beta(t) = t$ for $t \in I$.

For each $t \in (0,1]$ the function $f(t,\cdot)$ is continuous between the graphs of α and β except over the lines $\gamma_n(t) = n^{-1/k}$, $t \in [n^{-1/k}, T]$, where $T = \min\{3n^{-1/k}, 1\}$, $n \in \mathbb{N}$. Let us show that f is positive between α and β, thus γ_n will be an admissible nqsc curve for each $n \in \mathbb{N}$. For $t \in (0,1]$ and $(n+1)^{-1/k} < x \le n^{-1/k}$, $n \in \mathbb{N}$, we have

$$f(t,x) = n - \left[\frac{1}{t^k}\right] + \varepsilon(t), \tag{7.8}$$

and if, moreover, we restrict our attention to those $t > 0$ such that $\alpha(t) \le x \le \beta(t)$ then we have $(n+1)^{-1/k} < t \le 3n^{-1/k}$ which implies

$$\left[\frac{n}{3}\right] \le \left[\frac{1}{t^k}\right] \le n, \tag{7.9}$$

and thus for $t \in (0,1]$, $(n+1)^{-1/k} < x \le n^{-1/k}$, and $\alpha(t) \le x \le \beta(t)$, we have

$$\varepsilon(t) \le f(t,x) \le n - \left[\frac{n}{3}\right] + 1 \le \frac{2}{3}n + 2. \tag{7.10}$$

This shows that f is positive between α and β and, moreover, we can say that for $r \in (0,1)$ it suffices to take $n \in \mathbb{N}$ such that $(n+1)^{-1/k} < r$ to have $|f(t,x)| \le (2/3)n + 2$ for all (t,x) between the graphs of α and β and $r \le x \le 1/r$.

Therefore Theorem 5.4 implies the existence of a weak solution of (7.7) between α and β. Moreover, since f is positive between α and β the solution is increasing and, therefore, it is a Carathéodory solution.

The previous two examples fit the conditions of Theorems 5.3 and 5.4. Next we show an example where Theorem 5.3 can be used but it is not clear whether or not we can also apply Theorem 5.4.

Example 7.3. Let $a > 0$ be fixed and consider the problem

$$x' = f(t,x) = \left[\frac{1}{x+at}\right], \quad t \in I = [0,1], \qquad x(0) = 0. \tag{7.11}$$

Lower and upper solutions are given by $\alpha \equiv 0$ and $\beta(t) = \sqrt{2t}$, $t \in I$. Since f is nonnegative between α and β the lines $\gamma_n(t) = -at + 1/n$, $n \in \mathbb{N}$, are admissible nqsc curves for the differential equation. Finally it is easy to check that $0 \le f(t,x) \le n+1$ if $a^{-1}(n+1)^{-1} \le t \le 1$ and $x \ge 0$, thus one can construct $g \in L^1_{\text{loc}}((0,1])$ such that $|f(t,x)| \le g(t)$ for a.a. $t \in I$ and $0 \le x \le \sqrt{2t}$.

Theorem 5.3 ensures that (7.11) has extremal weak solutions between α and β. Moreover (7.11) has a unique solution between α and β as f is nonincreasing with respect to the unknown. Further, the unique solution is monotone and therefore it is a Carathéodory solution.

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