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Research Article

The Problem of Scattering by a Mixture of Cracks and Obstacles

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Consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open crack Γ and a bounded domain D in R^2 as cross section. We assume that the crack Γ is divided into two parts, and one of the two parts is (possibly) coated on one side by a material with surface impedance λ. Different boundary conditions are given on Γ and ∂D. Applying potential theory, the problem can be reformulated as a boundary integral system. We obtain the existence and uniqueness of a solution to the system by using Fredholm theory.

1. Introduction

Crack detection is a problem in nondestructive testing of materials which has been often addressed in literature and more recently in the context of inverse problems. Early works on the direct and inverse scattering problem for cracks date back to 1995 in [1] by Kress. In that paper, Kress considered the direct and inverse scattering problem for a perfectly conducting crack and used Newton's method to reconstruct the shape of the crack from a knowledge of the far-field pattern. In 1997, Mönch considered the same scattering problem for sound-hard crack [2], and in the same year, Alves and Ha Duong discussed the scattering problem but for flat cracks in [3]. Later in 2000, Kress's work was continued by Kirsch and Ritter in [4] who used the factorization method to reconstruct the shape of the crack from the knowledge of the far-field pattern. In 2003, Cakoni and Colton in [5] considered the direct and inverse scattering problem for cracks which (possibly) coated on one side by a material with surface impedance λ. Later in 2008, Lee considered an inverse scattering problem from an impedance crack and tried to recover impedance function from the far field pattern in [6]. However, studying an inverse problem always requires a solid knowledge of the corresponding direct problem. Therefore, in the following we just consider the direct scattering problem for a mixture of a crack Γ and a bounded domain D, and the corresponding inverse scattering problem can be considered by similar methods in [1, 2, 4–12] and the reference therein.

Briefly speaking, in this paper we consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open crack Γ and a bounded domain D in R^2 as cross section. We assume that the cylinder is (possibly) partially coated on one side by a material with surface impedance λ. This corresponds to the situation when the boundary or more generally a portion of the boundary is coated with an unknown material in order to avoid detection. Assuming that the electric field is polarized in the TM mode, this leads to a mixed boundary value problem for the Helmholtz equation defined in the exterior of a mixture in R^2.

Our aim is to establish the existence and uniqueness of a solution to this direct scattering problem. As is known, the method of boundary integral equations has widely applications to various direct and inverse scattering problems (see [13–17] and the reference therein). A few authors have applied such method to study the scattering problem with mixture of cracks and obstacles. In the following, we will use the method of boundary integral equations and Fredholm theory to obtain the existence and uniqueness of a solution. The difficult thing is to prove the corresponding boundary integral operator A which is a Fredholm operator with index zero since the boundary is a mixture and we have complicated boundary conditions.

The outline of the paper is as follows. In Section 2, the direct scattering problem is considered, and we will establish uniqueness to the problem and reformulate the problem as a boundary integral system by using single- and double-layer potentials. The existence and uniqueness of a solution to the corresponding boundary integral system will be given in Section 3. The potential theory and Fredholm theory will be used to prove our main results.

2. Boundary Integral Equations of the Direct Scattering Problem

Consider the scattering of time-harmonic electromagnetic plane waves from an infinite cylinder with a mixture of an open crack Γ and a bounded domain D in R^2 as cross section. For further considerations, we suppose that D has smooth boundary ∂D (e.g., $\partial D \in C^2$), and the crack Γ (smooth) can be extended to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ enclosing a bounded domain Ω such that the normal vector ν on Γ coincides with the outward normal vector on $\partial\Omega$ which we again denote by ν. The bounded domain D is located inside the domain Ω, and $\partial D \bigcap \partial\Omega = \emptyset$.

In the whole paper, we assume that $\partial D \in C^2$ and $\partial\Omega \in C^2$.

Suppose that

$$\Gamma = \{z(s) : s \in [s_0, s_1]\}, \tag{2.1}$$

where $z : [s_0, s_1] \to R^2$ is an injective piecewise C^1 function. We denote the outside of Γ with respect to the chosen orientation by Γ^+ and the inside by Γ^-. Here we suppose that the Γ is divided into two parts Γ_1 and Γ_2 and consider the electromagnetic field E-polarized. Different boundary conditions on $\Gamma_1^{\pm}$, $\Gamma_2^{\pm}$, and ∂D lead to the following problem:

$$\begin{aligned}
\Delta U + k^2 U &= 0 \quad \text{in } R^2 \setminus \left(\overline{D} \cup \Gamma\right), \\
U_{\pm} &= 0 \quad \text{on } \Gamma_1^{\pm}, \\
U_{-} &= 0 \quad \text{on } \Gamma_2^{-}, \\
\frac{\partial U_+}{\partial \nu} + ik\lambda \mathrm{U}_+ &= 0 \quad \text{on } \Gamma_2^{+}, \\
U &= 0 \quad \text{on } \partial D,
\end{aligned} \tag{2.2}$$

where $U_{\pm}(x) = \lim_{h\to 0^+} U(x \pm h\nu)$ for $x \in \Gamma$ and $\partial U_{\pm}/\partial\nu = \lim_{h\to 0^+} \nu \cdot \nabla U(x \pm h\nu)$ for $x \in \Gamma$. The total field U is decomposed into the given incident field $u^i(x) = e^{ikx\cdot d}$, $|d| = 1$, and the unknown scattered field u which is required to satisfy the Sommerfeld radiation condition

$$\lim_{\nu\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{2.3}$$

uniformly in $\widehat{x} = x/|x|$ with $r = |x|$.

We recall some usual Sobolev spaces and some trace spaces on Γ in the following.

Let $\widetilde{\Gamma} \subseteq \Gamma$ be a piece of the boundary. Use $H^1(D)$ and $H^1_{loc}(R^2 \setminus \overline{D})$ to denote the usual Sobolev spaces, $H^{1/2}(\Gamma)$ is the trace space, and we define

$$\begin{aligned}
H^{1/2}\left(\widetilde{\Gamma}\right) &= \left\{ u|_{\widetilde{\Gamma}} : u \in H^{1/2}(\Gamma) \right\}, \\
\widetilde{H}^{1/2}\left(\widetilde{\Gamma}\right) &= \left\{ u \in H^{1/2}(\Gamma) : \operatorname{supp} u \subseteq \widetilde{\Gamma} \right\}, \\
H^{-1/2}\left(\widetilde{\Gamma}\right) &= \left(\widetilde{H}^{1/2}\left(\widetilde{\Gamma}\right)\right)' \quad \text{the dual space of } \widetilde{H}^{1/2}\left(\widetilde{\Gamma}\right), \\
\widetilde{H}^{-1/2}\left(\widetilde{\Gamma}\right) &= \left(H^{1/2}(\widetilde{\Gamma})\right)' \quad \text{the dual space of } H^{1/2}\left(\widetilde{\Gamma}\right).
\end{aligned} \tag{2.4}$$

Just consider the scattered field u, then (2.2) and (2.3) are a special case of the following problem.

Given $f \in H^{1/2}(\Gamma_1)$, $g \in H^{1/2}(\Gamma_2)$, $h \in H^{-1/2}(\Gamma_2)$, and $r \in H^{1/2}(\partial D)$ find $u \in H^1_{loc}(R^2 \setminus (\overline{D} \cup \Gamma))$ such that

$$\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in } R^2 \setminus \left(\overline{D} \cup \Gamma\right), \\
u_{\pm} &= f \quad \text{on } \Gamma_1^{\pm}, \\
u_{-} &= g \quad \text{on } \Gamma_2^{-}, \\
\frac{\partial u_+}{\partial \nu} + ik\lambda u_+ &= h \quad \text{on } \Gamma_2^{+}, \\
u &= r \quad \text{on } \partial D,
\end{aligned} \tag{2.5}$$

and u is required to satisfy the Sommerfeld radiation condition (2.3). For simplicity, we assume that $k > 0$ and $\lambda > 0$.

Theorem 2.1. *The problems* (2.5) *and* (2.3) *have at most one solution.*

Proof. Let u be a solution to the problem (2.5) with $f = g = h = r = 0$, we want to show that $u = 0$ in $R^2 \setminus (\overline{D} \cup \Gamma)$. □

Suppose that B_R (with boundary ∂B_R) is a sufficiently large ball which contains the domain $\overline{\Omega}$. Obviously, to the Helmholtz equation in (2.5), the solution $u \in H^1(B_R \setminus \overline{\Omega}) \bigcup H^1(\Omega \setminus \overline{D})$ satisfies the following transmission boundary conditions on the complementary part $\partial\Omega \setminus \overline{\Gamma}$ of $\partial\Omega$:

$$u_+ = u_-,$$
$$\frac{\partial u_+}{\partial \nu} = \frac{\partial u_-}{\partial \nu}, \tag{2.6}$$

where "±" denote the limit approaching $\partial\Omega$ from outside and inside Ω, respectively. Applying Green's formula for u and $\overline{u}$ in $\Omega \setminus \overline{D}$ and $B_R \setminus \overline{\Omega}$, we have

$$\begin{aligned}
&\int_{\Omega\setminus\overline{D}} (u\Delta\overline{u} + \nabla u \cdot \nabla\overline{u})dx = \int_{\partial\Omega\setminus\overline{\Gamma}} u_- \frac{\partial \overline{u}_-}{\partial \nu} ds + \int_{\Gamma_1} u_- \frac{\partial \overline{u}_-}{\partial \nu} ds + \int_{\Gamma_2} u_- \frac{\partial \overline{u}_-}{\partial \nu} ds, \\
&\int_{B_R\setminus\overline{\Omega}} (u\Delta\overline{u} + \nabla u \cdot \nabla\overline{u})dx \\
&\quad = \int_{\partial B_R} u \frac{\partial \overline{u}}{\partial \nu} ds + \int_{\partial\Omega\setminus\overline{\Gamma}} u_+ \frac{\partial \overline{u}_+}{\partial \nu} ds + \int_{\Gamma_1} u_+ \frac{\partial \overline{u}_+}{\partial \nu} ds + \int_{\Gamma_2} u_+ \frac{\partial \overline{u}_+}{\partial \nu} ds,
\end{aligned} \tag{2.7}$$

where ν is directed into the exterior of the corresponding domain.

Using boundary conditions on Γ_1, Γ_2 and the above transmission boundary condition (2.6), we have

$$\int_{\partial B_R} u \frac{\partial \overline{u}}{\partial \nu} ds = \left(\int_{B_R\setminus\overline{\Omega}} + \int_{\Omega\setminus\overline{D}} \right) \left(|\nabla u|^2 - k^2 |u|^2 \right) dx + \int_{\Gamma_2} ik\lambda |u_+|^2 \mathrm{d}s. \tag{2.8}$$

Hence

$$\operatorname{Im}\left(\int_{\partial B_R} u \frac{\partial \overline{u}}{\partial \nu} ds \right) \geq 0. \tag{2.9}$$

So, from [13, Theorem 2.12] and a unique continuation argument we obtain that $u = 0$ in $R^2 \setminus (\overline{D} \cup \Gamma)$.

We use $[u] = u_- - u_+$ and $[\partial u/\partial \nu] = (\partial u_-/\partial \nu) - (\partial u_+/\partial \nu)$ to denote the jump of u and $\partial u/\partial \nu$ across the crack Γ, respectively. Then we have the following.

Lemma 2.2. *If u is a solution of* (2.5) *and* (2.3)*, then* $[u] \in \widetilde{H}^{1/2}(\Gamma)$ *and* $[\partial u/\partial \nu] \in \widetilde{H}^{-1/2}(\Gamma)$.

The proof of this lemma can be found in [11].

We are now ready to prove the existence of a solution to the above scattering problem by using an integral equation approaching. For $x \in \Omega \setminus \overline{D}$, by Green representation formula

$$u(x) = \int_{\partial\Omega} \left[\frac{\partial u}{\partial \nu}\Phi(x,y) - u\frac{\partial \Phi(x,y)}{\partial \nu} \right] ds_y + \int_{\partial D} \left[\frac{\partial u}{\partial \nu}\Phi(x,y) - u\frac{\partial \Phi(x,y)}{\partial \nu} \right] ds_y \tag{2.10}$$

and for $x \in \mathrm{R}^2 \setminus \overline{\Omega}$

$$u(x) = \int_{\partial\Omega} \left[u\frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu}\Phi(x,y) \right] ds_y, \tag{2.11}$$

where

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|) \tag{2.12}$$

is the fundamental solution to the Helmholtz equation in R^2, and $H_0^{(1)}$ is a Hankel function of the first kind of order zero.

By making use of the known jump relationships of the single- and double-layer potentials across the boundary $\partial\Omega$ (see [5, 11]) and approaching the boundary $\partial\Omega$ from inside $\Omega \setminus \overline{D}$, we obtain (for $x \in \partial\Omega$)

$$u_-(x) = S_{\Omega\Omega}\frac{\partial u_-}{\partial \nu} - K_{\Omega\Omega}u_- + 2\int_{\partial D} \left[\frac{\partial u(y)}{\partial \nu}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu} \right] ds_y, \tag{2.13}$$

$$\frac{\partial u_-(x)}{\partial \nu} = K'_{\Omega\Omega}\frac{\partial u_-}{\partial \nu} - T_{\Omega\Omega}u_- + 2\frac{\partial}{\partial \nu(x)}\int_{\partial D} \left[\frac{\partial u(y)}{\partial \nu}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu} \right] ds_y, \tag{2.14}$$

where $S_{\Omega\Omega}$, $K_{\Omega\Omega}$, $K'_{\Omega\Omega}$, and $T_{\Omega\Omega}$ are boundary integral operators:

$$\begin{aligned} S_{\Omega\Omega} &: H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega), \qquad & K_{\Omega\Omega} &: H^{1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega) \\ K'_{\Omega\Omega} &: H^{-1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \qquad & T_{\Omega\Omega} &: H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \end{aligned} \tag{2.15}$$

defined by (for $x \in \partial\Omega$)

$$\begin{aligned} S_{\Omega\Omega}\varphi(x) &= 2\int_{\partial\Omega} \varphi(y)\Phi(x,y)ds_y, \qquad & K_{\Omega\Omega}\varphi(x) &= 2\int_{\partial\Omega} \varphi(y)\frac{\Phi(x,y)}{\partial \nu_y}ds_y, \\ K'_{\Omega\Omega}\varphi(x) &= 2\int_{\partial\Omega} \varphi(y)\frac{\partial\Phi(x,y)}{\partial \nu_x}ds_y, \qquad & T_{\Omega\Omega}\varphi(x) &= 2\frac{\partial}{\partial \nu_x}\int_{\partial\Omega} \varphi(y)\frac{\partial\Phi(x,y)}{\partial \nu_y}ds_y. \end{aligned} \tag{2.16}$$

Similarly, approaching the boundary $\partial\Omega$ from inside $R^2 \setminus \overline{\Omega}$ we obtain (for $x \in \partial\Omega$)

$$u_+(x) = -S_{\Omega\Omega}\frac{\partial u_+}{\partial \nu} + K_{\Omega\Omega}u_+, \tag{2.17}$$

$$\frac{\partial u_+(x)}{\partial \nu} = -K'_{\Omega\Omega}\frac{\partial u_+}{\partial \nu} + T_{\Omega\Omega}u_+. \tag{2.18}$$

From (2.13)–(2.18), we have

$$\begin{aligned} u_- + u_+ = {} & S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right) - K_{\Omega\Omega}(u_- - u_+) \\ & + 2\int_{\partial D}\left[\frac{\partial u(y)}{\partial \nu}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right]ds_y, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \frac{\partial u_-}{\partial \nu} + \frac{\partial u_+}{\partial \nu} = {} & K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right) - T_{\Omega\Omega}(u_- - u_+) \\ & + 2\frac{\partial}{\partial \nu(x)}\int_{\partial D}\left[\frac{\partial u(y)}{\partial \nu}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right]ds_y. \end{aligned} \tag{2.20}$$

Restrict u on $\Gamma_1^{\pm}$, from (2.19) we have

$$\begin{aligned} 2f(x) = {} & S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right)\bigg|_{\Gamma_1} - K_{\Omega\Omega}(u_- - u_+)|_{\Gamma_1} \\ & +2\int_{\partial D}\frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y\bigg|_{\Gamma_1} - 2\int_{\partial D}u(y)\frac{\partial \Phi(x,y)}{\partial \nu}ds_y\bigg|_{\Gamma_1}, \end{aligned} \tag{2.21}$$

where $(\cdot)|_{\Gamma_1}$ means a restriction to Γ_1.

Define

$$\begin{gathered} S_{\Omega\Gamma_1}\varphi(x) = 2\int_{\partial\Omega}\varphi(y)\Phi(x,y)ds_y\bigg|_{\Gamma_1}, \\ K_{\Omega\Gamma_1}\varphi(x) = 2\int_{\partial\Omega}\frac{\partial \Phi(x,y)}{\partial \nu}\varphi(y)ds_y\bigg|_{\Gamma_1}, \\ S_{D\Gamma_1}\varphi(x) = 2\int_{\partial D}\varphi(y)\Phi(x,y)ds_y\bigg|_{\Gamma_1}, \\ \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = a, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_{\Gamma_1} = \left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right)\bigg|_{\Gamma_1} = b, \\ \left[\frac{\partial u}{\partial \nu}\right]\bigg|_{\Gamma_2} = \left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right)\bigg|_{\Gamma_2} = c, \qquad [u]|_{\Gamma_2} = (u_- - u_+)|_{\Gamma_2} = d. \end{gathered} \tag{2.22}$$

Then zero extend b, c, and d to the whole $\partial\Omega$ in the following:

$$\widetilde{b} = \begin{cases} 0, & \text{on } \partial\Omega \setminus \Gamma_1, \\ b, & \text{on } \Gamma_1, \end{cases}$$
$$\widetilde{c} = \begin{cases} 0, & \text{on } \partial\Omega \setminus \Gamma_2, \\ c, & \text{on } \Gamma_2, \end{cases} \qquad \widetilde{d} = \begin{cases} 0, & \text{on } \partial\Omega \setminus \Gamma_2, \\ d, & \text{on } \Gamma_2. \end{cases} \tag{2.23}$$

By using the boundary conditions in (2.5), we rewrite (2.21) as

$$S_{D\Gamma_1} a + S_{\Omega\Gamma_1}\left(\widetilde{b} + \widetilde{c}\right) - K_{\Omega\Gamma_1}\widetilde{d} = p_1(x), \tag{2.24}$$

where

$$p_1(x) = 2f(x) + 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(y)} r(y)\,ds_y\bigg|_{\Gamma_1}. \tag{2.25}$$

Furthermore, we modify (2.24) as

$$S_{D\Gamma_1} a + S_{\Gamma_1\Gamma_1} b + S_{\Gamma_2\Gamma_1} c - K_{\Gamma_2\Gamma_1} d = p_1(x), \tag{2.26}$$

where the operator $S_{\Gamma_2\Gamma_1}$ is the operator applied to a function with supp $\subseteq \overline{\Gamma}_2$ and evaluated on Γ_1, with analogous definition for $S_{D\Gamma_1}$, $S_{\Gamma_1\Gamma_1}$, and $K_{\Gamma_2\Gamma_1}$. We have mapping properties (see [5, 11])

$$\begin{aligned} S_{D\Gamma_1} &: \widetilde{H}^{-1/2}(\partial D) \longrightarrow H^{1/2}(\Gamma_1), \\ S_{\Gamma_1\Gamma_1} &: \widetilde{H}^{-1/2}(\Gamma_1) \longrightarrow H^{1/2}(\Gamma_1), \\ S_{\Gamma_2\Gamma_1} &: \widetilde{H}^{-1/2}(\Gamma_2) \longrightarrow H^{1/2}(\Gamma_1), \\ K_{\Gamma_2\Gamma_1} &: \widetilde{H}^{1/2}(\Gamma_2) \longrightarrow H^{1/2}(\Gamma_1). \end{aligned} \tag{2.27}$$

Again from (2.13)–(2.18), restricting u to boundary Γ_2^- we have

$$\begin{aligned} 2g(x) = {} & S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu}\right)\bigg|_{\Gamma_2} - K_{\Omega\Omega}(u_- - u_+)|_{\Gamma_2} + (u_- - u_+)|_{\Gamma_2} \\ & + 2\int_{\partial D} \frac{\partial u(y)}{\partial\nu(y)}\Phi(x,y)\,ds_y\bigg|_{\Gamma_2} - 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(y)} r(y)\,ds_y\bigg|_{\Gamma_2} \end{aligned} \tag{2.28}$$

or

$$
\begin{aligned}
&2\int_{\partial D}\frac{\partial u(y)}{\partial \nu(y)}\Phi(x,y)ds_y\bigg|_{\Gamma_2}+S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu}-\frac{\partial u_+}{\partial \nu}\right)\bigg|_{\Gamma_2}-K_{\Omega\Omega}(u_- - u_+)|_{\Gamma_2}+(u_- - u_+)|_{\Gamma_2}\\
&\quad=2g(x)+2\int_{\partial D}\frac{\partial \Phi(x,y)}{\partial \nu(y)}r(y)ds_y\bigg|_{\Gamma_2}.
\end{aligned}
\tag{2.29}
$$

Like previous, define

$$
\begin{aligned}
S_{\Omega\Gamma_2}\varphi(x)&=2\int_{\partial\Omega}\varphi(y)\Phi(x,y)ds_y\bigg|_{\Gamma_2},\\
K_{\Omega\Gamma_2}\varphi(x)&=2\int_{\partial\Omega}\frac{\partial \Phi(x,y)}{\partial \nu}\varphi(y)ds_y\bigg|_{\Gamma_2},\\
S_{D\Gamma_2}\varphi(x)&=2\int_{\partial D}\varphi(y)\Phi(x,y)ds_y\bigg|_{\Gamma_2}.
\end{aligned}
\tag{2.30}
$$

Then we can rewrite (2.29) as

$$
S_{D\Gamma_2}a+S_{\Omega\Gamma_2}\left(\tilde{b}+\tilde{c}\right)-K_{\Omega\Gamma_2}\tilde{d}+d=p_2(x),\quad x\in\Gamma_2^-,
\tag{2.31}
$$

where

$$
p_2(x)=2g(x)+2\int_{\partial D}\frac{\partial \Phi(x,y)}{\partial \nu(y)}r(y)ds_y\bigg|_{\Gamma_2^-}.
\tag{2.32}
$$

Similar to (2.26), we modify (2.31) as

$$
S_{D\Gamma_2}a+S_{\Gamma_1\Gamma_2}b+S_{\Gamma_2\Gamma_2}c+(I-K_{\Gamma_2\Gamma_2})d=p_2(x)
\tag{2.33}
$$

and we have mapping properties:

$$
\begin{aligned}
S_{D\Gamma_2}&:\widetilde{H}^{-1/2}(\partial D)\longrightarrow H^{1/2}(\Gamma_2),\\
S_{\Gamma_2\Gamma_2}&:\widetilde{H}^{-1/2}(\Gamma_2)\longrightarrow H^{1/2}(\Gamma_2),\\
S_{\Gamma_1\Gamma_2}&:\widetilde{H}^{-1/2}(\Gamma_1)\longrightarrow H^{1/2}(\Gamma_2),\\
K_{\Gamma_2\Gamma_2}&:\widetilde{H}^{1/2}(\Gamma_2)\longrightarrow H^{1/2}(\Gamma_2).
\end{aligned}
\tag{2.34}
$$

Combining (2.13) and (2.14),

$$
\begin{aligned}
&-ik\lambda\left(S_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}-K_{\Omega\Omega}u_-\right)\\
&\quad=-ik\lambda\left\{u_- -2\int_{\partial D}\left[\frac{\partial u(y)}{\partial\nu}\Phi(x,y)-u\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right]ds_y\right\}\\
&\quad=-ik\lambda u_- +2ik\lambda\int_{\partial D}\frac{\partial u(y)}{\partial\nu}\Phi(x,y)ds_y-2ik\lambda\int_{\partial D}r(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}ds_y,
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
&-K'_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}+T_{\Omega\Omega}u_-\\
&\quad=-\frac{\partial u_-}{\partial\nu}+2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial u(y)}{\partial\nu}\Phi(x,y)ds_y-2\frac{\partial}{\partial\nu(x)}\int_{\partial D}r(y)\frac{\partial\Phi(x,y)}{\partial\nu}ds_y,\\
&-ik\lambda u_- -\frac{\partial u_-}{\partial\nu}\\
&\quad=-ik\lambda(u_- -u_+)-\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)-ik\lambda u_+ -\frac{\partial u_+}{\partial\nu}.
\end{aligned}
\tag{2.36}
$$

Using (2.17) and (2.18),

$$
\begin{aligned}
\frac{\partial u_+}{\partial\nu}+ik\lambda u_+ &= -K'_{\Omega\Omega}\frac{\partial u_+}{\partial\nu}+T_{\Omega\Omega}u_+ +ik\lambda\left(K_{\Omega\Omega}u_+ -S_{\Omega\Omega}\frac{\partial u_+}{\partial\nu}\right)\\
&=K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)-T_{\Omega\Omega}(u_- -u_+)+ik\lambda S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\\
&\quad-ik\lambda K_{\Omega\Omega}(u_- -u_+)-ik\lambda\left(S_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}-K_{\Omega\Omega}u_-\right)-K'_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}+T_{\Omega\Omega}u_-\\
&=K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)-T_{\Omega\Omega}(u_- -u_+)+ik\lambda S_{\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\\
&\quad-ik\lambda K_{\Omega\Omega}(u_- -u_+)-ik\lambda u_- -\frac{\partial u_-}{\partial\nu}\\
&\quad+2ik\lambda\int_{\partial D}\frac{\partial u(y)}{\partial\nu}\Phi(x,y)ds_y+2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial u(y)}{\partial\nu}\Phi(x,y)ds_y\\
&\quad-2ik\lambda\int_{\partial D}r(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}ds_y-2\frac{\partial}{\partial\nu(x)}\int_{\partial D}r(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}ds_y.
\end{aligned}
\tag{2.37}
$$

Then using (2.36),

$$
\begin{aligned}
2\left(\frac{\partial u_+}{\partial \nu}+ik\lambda u_+\right) &= K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu}-\frac{\partial u_+}{\partial \nu}\right)-T_{\Omega\Omega}(u_- - u_+)+ik\lambda S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu}-\frac{\partial u_+}{\partial \nu}\right)\\
&\quad -ik\lambda K_{\Omega\Omega}(u_- - u_+)-ik\lambda(u_- - u_+)-\left(\frac{\partial u_-}{\partial \nu}-\frac{\partial u_+}{\partial \nu}\right)\\
&\quad +2ik\lambda\int_{\partial D}\frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y+2\frac{\partial}{\partial \nu(x)}\int_{\partial D}\frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y\\
&\quad -2ik\lambda\int_{\partial D}r(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds_y-2\frac{\partial}{\partial \nu(x)}\int_{\partial D}r(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds_y.
\end{aligned}
\tag{2.38}
$$

From (2.29), we have

$$
\begin{aligned}
2ik\lambda g(x) = ik\lambda\Bigg[&S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial \nu}-\frac{\partial u_+}{\partial \nu}\right)\Bigg|_{\Gamma_2}-K_{\Omega\Omega}(u_- - u_+)|_{\Gamma_2}+(u_- - u_+)|_{\Gamma_2}\\
&+2\int_{\partial D}\frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y\Bigg|_{\Gamma_2}-2\int_{\partial D}r(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds_y\Bigg|_{\Gamma_2}\Bigg].
\end{aligned}
\tag{2.39}
$$

Restricting (2.38) to Γ_2^+ and using (2.39), we modify (2.38) as

$$
\begin{aligned}
&2\frac{\partial}{\partial \nu(x)}\int_{\partial D}\frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y\Bigg|_{\Gamma_2^+}+K'_{\Omega\Omega}\left(\frac{\partial u^-}{\partial \nu}-\frac{\partial u^+}{\partial \nu}\right)\Bigg|_{\Gamma_2^+}\\
&\quad -T_{\Omega\Omega}(u^- - u^+)|_{\Gamma_2^+}-\left(\frac{\partial u^-}{\partial \nu}-\frac{\partial u^+}{\partial \nu}\right)\Bigg|_{\Gamma_2^+}-2ik\lambda(u_- - u_+)|_{\Gamma_2^+}=p_3(x),
\end{aligned}
\tag{2.40}
$$

where

$$
p_3(x)=2h(x)-2ik\lambda g(x)+\int_{\partial D}r(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds_y\Bigg|_{\Gamma_2^+}
\tag{2.41}
$$

for $x\in\Gamma^+$.

Define

$$
K'_{D\Gamma_2}\varphi(x)=2\frac{\partial}{\partial \nu(x)}\int_{\partial D}\varphi(y)\Phi(x,y)ds_y\Bigg|_{\Gamma_2^+},
\tag{2.42}
$$

and using the notation in previous, we can rewrite (2.40) as

$$
K'_{D\Gamma_2}a+K'_{\Omega\Gamma_2}\left(\widetilde{b}+\widetilde{c}\right)-T_{\Omega\Gamma_2}\widetilde{d}-c-2ik\lambda d=p_3(x)
\tag{2.43}
$$

or

$$K'_{D\Gamma_2}a + K'_{\Gamma_1\Gamma_2}b + \left(K'_{\Gamma_2\Gamma_2} - I\right)c - (T_{\Gamma_2\Gamma_2} + 2ik\lambda I)d = p_3(x), \tag{2.44}$$

where the operators $K'_{\Gamma_1\Gamma_2}$, $K'_{\Gamma_2\Gamma_2}$, and $T_{\Gamma_2\Gamma_2}$ are restriction operators (see (2.29)). As before, we have mapping properties:

$$\begin{aligned} K'_{D\Gamma_2} &: \widetilde{H}^{-1/2}(\partial D) \longrightarrow H^{-1/2}(\Gamma_2),\\ K'_{\Gamma_1\Gamma_2} &: \widetilde{H}^{-1/2}(\Gamma_1) \longrightarrow H^{-1/2}(\Gamma_2),\\ K'_{\Gamma_2\Gamma_2} &: \widetilde{H}^{-1/2}(\Gamma_2) \longrightarrow H^{-1/2}(\Gamma_2),\\ T_{\Gamma_2\Gamma_2} &: \widetilde{H}^{1/2}(\Gamma_2) \longrightarrow H^{-1/2}(\Gamma_2). \end{aligned} \tag{2.45}$$

By using Green formula and approaching the boundary ∂D from inside $\Omega \setminus \overline{D}$ we obtain (for $x \in \partial D$)

$$\begin{aligned} u(x) = {} & 2\int_{\partial D} \frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y + 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu}u(y)ds_y\\ & + 2\int_{\partial \Omega}\left[\frac{\partial u_-(y)}{\partial \nu}\Phi(x,y) - u_-(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right]ds_y. \end{aligned} \tag{2.46}$$

The last term in (2.46) can be reformulated as

$$\begin{aligned} & 2\int_{\partial \Omega}\left[\frac{\partial u_-(y)}{\partial \nu}\Phi(x,y) - u_-(y)\frac{\partial \Phi(x,y)}{\partial \nu}u(y)\right]ds_y\\ & \quad = 2\int_{\partial \Omega}\left[\left(\frac{\partial u_-(y)}{\partial \nu} - \frac{\partial u_+(y)}{\partial \nu}\right)\Phi(x,y) - (u_-(y) - u_+(y))\frac{\partial \Phi(x,y)}{\partial \nu}\right]ds_y\\ & \qquad + 2\int_{\partial \Omega}\left[\frac{\partial u_+(y)}{\partial \nu}\Phi(x,y) - u_+(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right]ds_y. \end{aligned} \tag{2.47}$$

Since $x \in \partial D$ and $y \in \partial\Omega$ in (2.47), we have the following result (see [13]).

Lemma 2.3. *By using Green formula and the Sommerfeld radiation condition* (2.3), *one obtains*

$$\int_{\partial \Omega}\left[\frac{\partial u_+(y)}{\partial \nu}\Phi(x,y) - u_+(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right]ds_y = 0. \tag{2.48}$$

Proof. Denote by B_R a sufficiently large ball with radius R containing $\overline{\Omega}$ and use Green formula inside $B_R \setminus \overline{\Omega}$. Furthermore noticing $x \in \partial D$, $y \in \partial\Omega$, and the Sommerfeld radiation condition (2.3), we can prove this lemma. □

Combining (2.46), (2.47), and Lemma 2.3 and restricting x to ∂D we have

$$\begin{aligned} &2\int_{\partial D} \frac{\partial u(y)}{\partial \nu}\Phi(x,y)ds_y\bigg|_{\partial D} + 2\int_{\partial\Omega}\left[\frac{\partial u_-(y)}{\partial \nu} - \frac{\partial u_+(y)}{\partial \nu}\right]\Phi(x,y)ds_y\bigg|_{\partial D} \\ &\quad - 2\int_{\partial\Omega}\left[u_-(y) - u_+(y)\right]\frac{\partial\Phi(x,y)}{\partial \nu}ds_y\bigg|_{\partial D} = p_0(x), \end{aligned} \tag{2.49}$$

where

$$p_0(x) = r(x) - 2\int_{\partial D} r(y)\frac{\partial\Phi(x,y)}{\partial \nu}ds_y\bigg|_{\partial D}. \tag{2.50}$$

Define

$$S_{DD}\varphi(x) = 2\int_{\partial D}\varphi(y)\Phi(x,y)ds_y\bigg|_{\partial D}, \tag{2.51}$$

and then we can rewrite (2.49) as

$$S_{DD}a + S_{\Gamma_1 D}b + S_{\Gamma_2 D}c - K_{\Gamma_2 D}d = p_0(x). \tag{2.52}$$

Similarly, $S_{\Gamma D}$ and $K_{\Gamma D}$ are restriction operators as before, and we have mapping properties:

$$\begin{aligned} S_{DD} &: \widetilde{H}^{-1/2}(\partial D) \longrightarrow H^{1/2}(\partial D), \\ S_{\Gamma D} &: \widetilde{H}^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\partial D), \\ K_{\Gamma D} &: \widetilde{H}^{1/2}(\Gamma) \longrightarrow H^{1/2}(\partial D). \end{aligned} \tag{2.53}$$

Combining (2.52), (2.26), (2.33), and (2.44), we have

$$\begin{gathered} S_{DD}a + S_{\Gamma_1 D}b + S_{\Gamma_2 D}c - K_{\Gamma_2 D}d = p_0(x), \\ S_{D\Gamma_1}a + S_{\Gamma_1\Gamma_1}b + S_{\Gamma_2\Gamma_1}c - K_{\Gamma_2\Gamma_1}d = p_1(x), \\ S_{D\Gamma_2}a + S_{\Gamma_1\Gamma_2}b + S_{\Gamma_2\Gamma_2}c + (I - K_{\Gamma_2\Gamma_2})d = p_2(x), \\ K'_{D\Gamma_2}a + K'_{\Gamma_1\Gamma_2}b + \left(K'_{\Gamma_2\Gamma_2} - I\right)c - (T_{\Gamma_2\Gamma_2} + 2ik\lambda I)d = p_3(x). \end{gathered} \tag{2.54}$$

If we define

$$A = \begin{pmatrix} S_{DD} & S_{\Gamma_1 D} & S_{\Gamma_2 D} & -K_{\Gamma_2 D} \\ S_{D\Gamma_1} & S_{\Gamma_1\Gamma_1} & S_{\Gamma_2\Gamma_1} & -K_{\Gamma_2\Gamma_1} \\ S_{D\Gamma_2} & S_{\Gamma_1\Gamma_2} & S_{\Gamma_2\Gamma_2} & I - K_{\Gamma_2\Gamma_2} \\ K'_{D\Gamma_2} & K'_{\Gamma_1\Gamma_2} & K'_{\Gamma_2\Gamma_2} - I & -(T_{\Gamma_2\Gamma_2} + 2ik\lambda I) \end{pmatrix},$$
$$\overrightarrow{p} = \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{pmatrix}, \tag{2.55}$$

then (2.54) can be rewritten as a boundary integral system:

$$A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \overrightarrow{p}. \tag{2.56}$$

Remark 2.4. If the above system (2.56) has a unique solution, our problem (2.5) with (2.3) will have a unique solution (see [13, 14]).

3. Existence and Uniqueness

Based on the Fredholm theory, we show the existence and uniqueness of a solution to the integral system (2.56).

Define

$$H = H^{-1/2}(\partial D) \times \widetilde{H}^{-1/2}(\Gamma_1) \times \widetilde{H}^{-1/2}(\Gamma_2) \times \widetilde{H}^{1/2}(\Gamma_2) \tag{3.1}$$

and its dual space

$$H^* = H^{1/2}(\partial D) \times \widetilde{H}^{1/2}(\Gamma_1) \times \widetilde{H}^{1/2}(\Gamma_2) \times \widetilde{H}^{-1/2}(\Gamma_2). \tag{3.2}$$

Theorem 3.1. *The operator A maps H continuously into H^* and is Fredholm with index zero.*

Proof. As is known, the operator S_{DD} is positive and bounded below up to a compact perturbation (see [18]); that is, there exists a compact operator

$$L_D : H^{-1/2}(\partial D) \longrightarrow H^{1/2}(\partial D) \tag{3.3}$$

such that

$$\mathrm{Re}\big(\langle (S_{DD}+L_D)\varphi,\overline{\varphi}\rangle\big) \geq C\|\varphi\|^2_{H^{-1/2}(\partial D)}, \quad \text{for } \varphi \in H^{-1/2}(\partial D) \tag{3.4}$$

where $\langle , \rangle$ denote the duality between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$.

For convenience, in the following discussion we define

$$S_0 = S_{DD} + L_D. \tag{3.5}$$

Similarly, the operators $S_{\Omega\Omega}$ and $-T_{\Omega\Omega}$ are positive and bounded below up to compact perturbations (see [18]), that is, there exist compact operators

$$\begin{aligned} L_\Omega &: H^{-1/2}(\partial\Omega) \longrightarrow \mathrm{H}^{1/2}(\partial\Omega) \\ L_T &: H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega) \end{aligned} \tag{3.6}$$

such that

$$\begin{aligned} \mathrm{Re}\big(\langle (S_{\Omega\Omega}+L_\Omega)\varphi,\overline{\varphi}\rangle\big) &\geq C\|\varphi\|^2_{H^{-1/2}(\partial D)}, \quad \text{for } \varphi \in H^{-1/2}(\partial\Omega), \\ \mathrm{Re}\big(\langle -(T_{\Omega\Omega}+L_T)\varphi,\overline{\varphi}\rangle\big) &\geq C\|\varphi\|^2_{H^{1/2}(\partial D)}, \quad \text{for } \varphi \in H^{1/2}(\partial\Omega). \end{aligned} \tag{3.7}$$

Define $S_1 = S_{\Omega\Omega} + L_\Omega$ and $T_1 = -(T_{\Omega\Omega} + L_T)$, then S_1 and T_1 are bounded below up and positive.

Take $a \in H^{-1/2}(\partial D)$, and let $\widetilde{b} \in H^{-1/2}(\partial\Omega)$, $\widetilde{c} \in H^{-1/2}(\partial\Omega)$, and $\widetilde{d} \in H^{1/2}(\partial\Omega)$ be the extension by zero to $\partial\Omega$ of $b \in \widetilde{H}^{-1/2}(\Gamma_1)$, $c \in \widetilde{H}^{-1/2}(\Gamma_2)$, and $d \in \widetilde{H}^{1/2}(\Gamma_2)$, respectively.

Denote $\overrightarrow{\xi} = (a,b,c,d)^T$.

It is easy to check that the operators $S_{\Gamma_1 D}$, $S_{\Gamma_2 D}$, $S_{D\Gamma_1}$, $S_{D\Gamma_2}$, $K_{\Gamma_2 D}$, and $K'_{D\Gamma_2}$ are compact operators, and then we can rewrite $A\overrightarrow{\xi}$ as the following:

$$A\overrightarrow{\xi} = A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = A_0\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + A_C\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = A_0\overrightarrow{\xi} + A_C\overrightarrow{\xi} \tag{3.8}$$

with

$$
A_C\overrightarrow{\xi} = \begin{pmatrix} -L_D a|_{\partial D} + S_{\Gamma_1 D} b + S_{\Gamma_2 D} c - K_{\Gamma_2 D} d \\ S_{D\Gamma_1} a - \left(L_\Omega \widetilde{b} + L_\Omega \widetilde{c} \right)\Big|_{\Gamma_1} \\ S_{D\Gamma_2} a - \left(L_\Omega \widetilde{b} + L_\Omega \widetilde{c} \right)\Big|_{\Gamma_2} \\ K'_{D\Gamma_2} + L_T \widetilde{d}\Big|_{\Gamma_2} \end{pmatrix},
$$
$$
A_0\overrightarrow{\xi} = \begin{pmatrix} S_0 a \\ \left(S_1 \widetilde{b} + S_1 \widetilde{c} \right)\Big|_{\Gamma_1} - K_{\Gamma_2\Gamma_1} d \\ \left(S_1 \widetilde{b} + S_1 \widetilde{c} \right)\Big|_{\Gamma_2} + (I - K_{\Gamma_2\Gamma_2}) d \\ K'_{\Gamma_1\Gamma_2} b + \left(K'_{\Gamma_2\Gamma_2} - I \right) c + T_1 \widetilde{d}\Big|_{\Gamma_2} - 2ik\lambda d \end{pmatrix}, \tag{3.9}
$$

where $A_C : H \to H^*$ is compact and $A_0 : H \to H^*$ defines a sesquilinear form, that is,

$$
\begin{aligned}
\left\langle A_0\overrightarrow{\xi}, \overline{\overrightarrow{\xi}} \right\rangle_{H,H^*} &= (S_0 a, a) + \left(S_1 \widetilde{b} + S_1 \widetilde{c}, \widetilde{b} \right) - (K_{\Gamma_2\Gamma_1} d, b)|_{\Gamma_1} \\
&\quad + \left(S_1 \widetilde{b} + S_1 \widetilde{c}, \widetilde{c} \right) + (d, c) - (K_{\Gamma_2\Gamma_2} d, c)|_{\Gamma_2} \\
&\quad + (K'_{\Gamma_1\Gamma_2} b, d)\Big|_{\Gamma_2} + (K'_{\Gamma_2\Gamma_2} c, d)\Big|_{\Gamma_2} - (c, d) + \left(T_1 \widetilde{d}, \widetilde{d} \right) - 2ik\lambda(d, d).
\end{aligned} \tag{3.10}
$$

Here (u, v) denotes the scalar product on $L^2(\partial D)$ or $L^2(\partial\Omega)$ defined by $\int_{\partial D} u\overline{v}\, ds$ or $\int_{\partial\Omega} u\overline{v}\, ds$, and $(u, v)|_{\Gamma_i}$ is the scalar product on $L^2(\Gamma_i)$ $(i = 1, 2)$.

By properties of the operators S_0, S_1, and T_1, we have

$$
\begin{aligned}
\operatorname{Re}(S_0 a, a) &\geq C\|a\|^2_{H^{-1/2}(\partial D)}, \\
\operatorname{Re}\left[\left(S_1 \widetilde{b} + S_1 \widetilde{c}, \widetilde{b} \right) + \left(S_1 \widetilde{b} + S_1 \widetilde{c}, \widetilde{c} \right) + \left(T_1 \widetilde{d}, \widetilde{d} \right) \right] &= \operatorname{Re}\left[\left(S_1 \left(\widetilde{b} + \widetilde{c} \right), \widetilde{b} + \widetilde{c} \right) + \left(T_1 \widetilde{d}, \widetilde{d} \right) \right] \\
&\geq C\left(\left\| \widetilde{b} + \widetilde{c} \right\|_{H^{-1/2}(\partial\Omega)} + \left\| \widetilde{d} \right\|_{H^{1/2}(\partial\Omega)} \right) \\
&= C\left(\|b\|^2_{\widetilde{H}^{-1/2}(\Gamma_1)} + \|c\|^2_{\widetilde{H}^{-1/2}(\Gamma_2)} + \|d\|^2_{\widetilde{H}^{1/2}(\Gamma_2)} \right).
\end{aligned} \tag{3.11}
$$

Similarly,

$$\begin{aligned}
\operatorname{Re}\left[(K'_{\Gamma_1\Gamma_2}b,d)\Big|_{\Gamma_2} - (K_{\Gamma_2\Gamma_1}d,b)|_{\Gamma_1}\right] &= \operatorname{Re}\left[(b,K_{\Gamma_2\Gamma_1}d)|_{\Gamma_1} - (K_{\Gamma_2\Gamma_1}d,b)|_{\Gamma_1}\right] \\
&= \operatorname{Re}\left[\overline{(K_{\Gamma_2\Gamma_1}d,b)}\Big|_{\Gamma_1} - (K_{\Gamma_2\Gamma_1}d,b)|_{\Gamma_1}\right] \\
&= 0, \\
\operatorname{Re}\left[(K'_{\Gamma_2\Gamma_2}c,d)\Big|_{\Gamma_2} - (K_{\Gamma_2\Gamma_2}d,c)|_{\Gamma_2}\right] &= 0, \\
\operatorname{Re}[(d,c)-(c,d)-2id\lambda(d,d)] &= \operatorname{Re}\left[(d,c)-\overline{(d,c)}-2ik\lambda(d,d)\right] \\
&= 0.
\end{aligned} \tag{3.12}$$

So the operator A_0 is coercive, that is,

$$\operatorname{Re}\left(\left\langle (A-A_c)\xi,\overline{\xi}\right\rangle_{H,H^*}\right) \geq C\|\xi\|_H^2 \quad \text{for } \xi \in H, \tag{3.13}$$

whence the operator A is Fredholm with index zero. □

Theorem 3.2. *The operator A has a trivial kernel if $-k^2$ is not Dirichlet eigenvalue of the Laplace operator in D.*

Proof. In this part, we show that $\operatorname{Kern}A = \{0\}$. To this end let $\overrightarrow{\varphi} = (a,b,c,d)^T \in H$ be a solution of the homogeneous system $A\overrightarrow{\varphi} = \overrightarrow{0}$, and we want to prove that $\overrightarrow{\varphi} \equiv \overrightarrow{0}$.

However, $A\overrightarrow{\varphi} = \overrightarrow{0}$ means that

$$\begin{gathered}
S_{DD}a + S_{\Gamma_1 D}b + S_{\Gamma_2 D}c - K_{\Gamma_2 D}d = 0, \\
S_{D\Gamma_1}a + S_{\Gamma_1\Gamma_1}b + S_{\Gamma_2\Gamma_1}c - K_{\Gamma_2\Gamma_1}d = 0, \\
S_{D\Gamma_2}a + S_{\Gamma_1\Gamma_2}b + S_{\Gamma_2\Gamma_2}c + (I - K_{\Gamma_2\Gamma_2})d = 0, \\
K'_{D\Gamma_2}a + K'_{\Gamma_1\Gamma_2}b + \left(K'_{\Gamma_2\Gamma_2} - I\right)c - (T_{\Gamma_2\Gamma_2} + 2ik\lambda I)d = 0.
\end{gathered} \tag{3.14}$$

Define a potential

$$v(x) = S_D a + S_\Omega \tilde{b} + S_\Omega \tilde{c} - K_\Omega \tilde{d} \tag{3.15}$$

where $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ have the same meaning as before and

$$S_D\varphi(x) = \int_{\partial D} \varphi(y)\Phi(x,y)ds_y,$$
$$S_\Omega\varphi(x) = \int_{\partial\Omega} \varphi(y)\Phi(x,y)ds_y, \tag{3.16}$$
$$K_\Omega\varphi(x) = \int_{\partial\Omega} \varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}ds_y.$$

This potential $v(x)$ satisfies Helmholtz equation in $R^2 \setminus (D\cup\overline{\Gamma})$ and the Sommerfeld radiation condition (see [13, 14]).

Considering the potential $v(x)$ inside $\Omega \setminus \overline{D}$ and approaching the boundary ∂D $(x \to \partial D)$, we have

$$v(x) = \frac{1}{2}\{S_{DD}a + S_{\Gamma_1 D}b + S_{\Gamma_2 D}c - K_{\Gamma_2 D}d\} \tag{3.17}$$

and (3.14) implies that

$$v(x)|_{\partial D} = 0. \tag{3.18}$$

Similarly, considering the potential $v(x)$ inside $\Omega \setminus \overline{D}$ and approaching the boundary $\partial\Omega$ $(x \to \partial\Omega)$, then restricting $v(x)$ to the partial boundary Γ_1^-:

$$v(x)|_{\Gamma_1^-} = \frac{1}{2}\{S_{D\Gamma_1}a + S_{\Gamma_1\Gamma_1}b + S_{\Gamma_2\Gamma_1}c - K_{\Gamma_2\Gamma_1}d\} = 0, \quad x \in \Gamma_1^-, \tag{3.19}$$

and restricting $v(x)$ to the partial boundary Γ_2^-, we have

$$v(x)|_{\Gamma_2^-} = \frac{1}{2}\{S_{D\Gamma_2}a + S_{\Gamma_1\Gamma_2}b + S_{\Gamma_2\Gamma_2}c + d - K_{\Gamma_2\Gamma_2}d\} = 0, \quad x \in \Gamma_2^-. \tag{3.20}$$

Now, we consider the potential $v(x)$ in the region $R^2 \setminus \overline{\Omega}$ and approach the boundary $\partial\Omega$ $(x \to \partial\Omega)$, and then restricting $v(x)$ to the partial boundary Γ_1^+, similar to (3.19), we have

$$v(x)|_{\Gamma_1^+} = 0, \quad x \in \Gamma_1^-. \tag{3.21}$$

Refering to (3.20),

$$\begin{aligned} v(x)|_{\Gamma_2^+} &= -[v]|_{\Gamma_2^+} = 0, \quad x \in \Gamma_2^+, \\ \left.\frac{\partial v(x)}{\partial \nu}\right|_{\Gamma_2^+} &= \frac{1}{2}\left\{K'_{D\Gamma_2}a + K'_{\Gamma_1\Gamma_2}b + \left(K'_{\Gamma_2\Gamma_2} - I\right)c - T_{\Gamma_2\Gamma_2}d\right\}, \quad x \in \Gamma_2^+. \end{aligned} \tag{3.22}$$

Combining (3.22), from (3.14) we have

$$2\left(\frac{\partial v(x)}{\partial \nu} + ik\lambda v(x)\right)\Big|_{\Gamma_2^+} = K'_{D\Gamma_2}a + K'_{\Gamma_1\Gamma_2}b + \left(K'_{\Gamma_2\Gamma_2} - I\right)c - T_{\Gamma_2\Gamma_2}d - 2ik\lambda d$$
$$= 0. \tag{3.23}$$

From (3.18)–(3.23), the potential $v(x)$ satisfies the following boundary value problem:

$$\begin{aligned}
\Delta v + k^2 v = 0 &\quad \text{in } R^2 \setminus \left(\overline{D} \cup \Gamma\right),\\
v_{\pm} = 0 &\quad \text{on } \Gamma_1^{\pm},\\
v_{-} = 0 &\quad \text{on } \Gamma_2^{-},\\
\frac{\partial v_+}{\partial \nu} + ik\lambda v_+ = 0 &\quad \text{on } \Gamma_2^{+},\\
v = 0 &\quad \text{on } \partial D,
\end{aligned} \tag{3.24}$$

and the Sommerfeld radiation condition:

$$\lim_{\nu \to \infty} \sqrt{r}\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{3.25}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$.

The uniqueness result Theorem 2.1 in Section 2 implies that

$$v(x) = 0, \quad x \in R^2 \setminus (D \cup \Gamma). \tag{3.26}$$

Notice that $-k^2$ is not Dirichlet eigenvalue of the Laplace operator in D, and so

$$v(x) = 0, \quad x \in D. \tag{3.27}$$

Therefore, the well-known jump relationships (see [13, 14]) imply that

$$\vec{\varphi} = (a, b, c, d)^T = (0, 0, 0, 0)^T. \tag{3.28}$$

So we complete the proof of the theorem.

Combining Theorems 3.1 and 3.2, we have the following □

Theorem 3.3. *The boundary integral system* (2.56) *has a unique solution.*

Remark 3.4. If we remove the condition that "$-k^2$ is not Dirichlet eigenvalue of the Laplace operator in D," instead of it by the assumption that $\operatorname{Im} k > 0$, then Theorem 2.1 in Section 2 and Theorem 3.3 in Section 3 are also true.

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