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Research Article

The Existence of Countably Many Positive Solutions for Nonlinear nth-Order Three-Point Boundary Value Problems

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We consider the existence of countably many positive solutions for nonlinear nth-order three-point boundary value problem $u^{(n)}(t)+a(t)f(u(t))=0$, $t\in(0,1)$, $u(0)=\alpha u(\eta)$, $u'(0)=\cdots=u^{(n-2)}(0)=0$, $u(1)=\beta u(\eta)$, where $n\geq 2$, $\alpha\geq 0$, $\beta\geq 0$, $0<\eta<1$, $\alpha+(\beta-\alpha)\eta^{n-1}<1$, $a(t)\in L^p[0,1]$ for some $p\geq 1$ and has countably many singularities in $[0,1/2)$. The associated Green's function for the nth-order three-point boundary value problem is first given, and growth conditions are imposed on nonlinearity f which yield the existence of countably many positive solutions by using the Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem for operators on a cone.

1. Introduction

The existence of positive solutions for nonlinear second-order and higher-order multipoint boundary value problems has been studied by several authors, for example, see [1–12] and the references therein. However, there are a few papers dealing with the existence of positive solutions for the nth-order multipoint boundary value problems with infinitely many singularities. Hao et al. [13] discussed the existence and multiplicity of positive solutions for the following nth-order nonlinear singular boundary value problems:

$$\begin{gathered} u^{(n)}(t)+a(t)f(t,u)=0, \quad t\in(0,1), \\ u(0)=0, \qquad u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad u(1)=\alpha u(\eta), \end{gathered} \tag{1.1}$$

where $0<\eta<1$, $0<\alpha\eta^{n-1}<1$, $a(t)$ may be singular at $t=0$ and/or $t=1$. Hao et al. established the existence of at least two positive solution for the boundary value problems if f is either superlinear or sublinear by applying the Krasnosel'skii-Guo theorem on cone expansion and compression.

In [14], Kaufmann and Kosmatov showed that there exist countably many positive solutions for the two-point boundary value problems with infinitely many singularities of following form:

$$\begin{gathered} -u''(t) = a(t)f(u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = 0, \end{gathered} \tag{1.2}$$

where $a(t) \in L^p[0,1]$ for some $p \geq 1$ and has countably many singularities in $[0,1/2)$.

In [15], Ji and Guo proved the existence of countably many positive solutions for the nth-order ordinary differential equation

$$u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \tag{1.3}$$

with one of the following m-point boundary conditions:

$$\begin{gathered} u(0) = \sum_{i=1}^{m-2} k_i u(\xi_i), \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = 0, \\ u(0) = 0, \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i), \end{gathered} \tag{1.4}$$

where $n \geq 2$, $k_i > 0$ $(i = 1,2,\ldots,m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $f \in C([0,+\infty),[0,+\infty))$, $a(t) \in L^p[0,1]$ for some $p \geq 1$ and has countably many singularities in $[0,1/2)$.

Motivated by the result of [13–15], in this paper we are interested in the existence of countably many positive solutions for nonlinear nth-order three-point boundary value problem

$$\begin{gathered} u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \alpha u(\eta), \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta), \end{gathered} \tag{1.5}$$

where $n \geq 2$, $\alpha \geq 0$, $\beta \geq 0$, $0 < \eta < 1$, $\alpha + (\beta - \alpha)\eta^{n-1} < 1$, $f \in C([0,+\infty),[0,+\infty))$, $a(t) \in L^p[0,1]$ for some $p \geq 1$ and has countably many singularities in $[0,1/2)$. We show that the problem (1.5) has countably many solutions if a and f satisfy some suitable conditions. Our approach is based on the Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem in cones.

Suppose that the following conditions are satisfied.

(H_1) There exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_{k+1} < t_k (k \in N)$, $t_1 < 1/2$, $\lim_{k\to\infty} t_k = t^* \geq 0$, and $\lim_{t\to t_k} a(t) = +\infty$ for all $k = 1,2,\ldots$.

(H_2) There exists $m > 0$ such that $a(t) \geq m$ for all $t \in [t^*, 1-t^*]$.

Assuming that $a(t)$ satisfies the conditions (H_1)-(H_2) (we cite [15, Example 6.1] to verify existence of $a(t)$) and imposing growth conditions on the nonlinearity f, it will be shown that problem (1.5) has infinitely many solutions.

The paper is organized as follows. In Section 2, we provide some necessary background material such as the Krasnosel'skii fixed-point theorem and Leggett-Williams fixed point theorem in cones. In Section 3, the associated Green's function for the nth-order three-point boundary value problem is first given and we also look at some properties of the Green's function associated with problem (1.5). In Section 4, we prove the existence of countably many positive solutions for problem (1.5) under suitable conditions on a and f. In Section 5, we give two simple examples to illustrate the applications of obtained results.

2. Preliminary Results

Definition 2.1. Let E be a Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$ for all $u \in P$ and for all $a \geq 0$;

(ii) $u, -u \in P$ implies $u = 0$.

Definition 2.2. The map $\alpha : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P provided that α is continuous and

$$\alpha\big(tx + (1-t)y\big) \geq t\alpha(x) + (1-t)\alpha\big(y\big), \tag{2.1}$$

for all $x, y \in P$ and $0 \leq \mathrm{t} \leq 1$. Similarly, we say that the map $\gamma : P \to [0, \infty)$ is a nonnegative continuous convex functional on P provided that γ is continuous and

$$\gamma\big(tx + (1-t)y\big) \leq t\gamma(x) + (1-t)\gamma\big(y\big), \tag{2.2}$$

for all $x, y \in P$ and $0 \leq \mathrm{t} \leq 1$.

Definition 2.3. Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on P. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$\begin{gathered} P_r = \{x \in P \mid \|x\| < r\}, \\ P(\alpha, a, b) = \{x \in P \mid a \leq \alpha(x), \|x\| \leq b\}. \end{gathered} \tag{2.3}$$

The following Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem play an important role in this paper.

Theorem 2.4 ([16], Krasnosel'skii fixed point theorem). *Let E be a Banach space and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are bounded open subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : P \bigcap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow P \tag{2.4}$$

is a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|, u \in P \bigcap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|, u \in P \bigcap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|, u \in P \bigcap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|, u \in P \bigcap \partial\Omega_2$.

Then T *has a fixed point in* $P \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 2.5 ([17], Leggett-Williams fixed point theorem). *Let* $A : \overline{P_c} \to \overline{P_c}$ *be a completely continuous operator and let* α *be a nonnegative continuous concave functional on* P *such that* $\alpha(x) \leq \|x\|$ *for all* $x \in \overline{P_c}$. *Suppose there exist* $0 < a < b < d \leq c$ *such that*

(C_1) $\{x \in P(\alpha, b, d) \mid \alpha(x) > b\} \neq \emptyset$, *and* $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, d)$,

(C_2) $\|Ax\| < a$ *for* $\|x\| \leq a$,

(C_3) $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, c)$, *with* $\|Ax\| > d$.

Then A *has at least three fixed points* x_1, x_2, *and* x_3 *such that*

$$\|x_1\| < a, \qquad b < \alpha(x_2), \qquad \|x_3\| > a \quad \text{with } \alpha(x_3) < b. \tag{2.5}$$

In order to establish some of the norm inequalities in Theorems 2.4 and 2.5 we will need Holder's inequality. We use standard notation of $L^p[a,b]$ for the space of measurable functions such that

$$\int_0^1 |f(s)|^p \mathrm{d}s < \infty, \tag{2.6}$$

where the integral is understood in the Lebesgue sense. The norm on $L^p[a,b]$, $\|\cdot\|$, is defined by

$$\|f\|_p = \left(\int_0^1 |f(s)|^p \mathrm{d}s\right)^{1/p}. \tag{2.7}$$

Theorem 2.6 ([18], Holder's inequality). *Let* $f \in L^p[a,b]$ *and* $g \in L^q[a,b]$, *where* $p > 1$ *and* $1/p + 1/q = 1$. *Then* $fg \in L^1[a,b]$ *and, moreover*

$$\int_0^1 |f(s)g(s)| \mathrm{d}s \leq \|f\|_p \|g\|_q. \tag{2.8}$$

Let $f \in L^1[a,b]$ *and* $g \in L^\infty[a,b]$. *Then* $fg \in L^1[a,b]$ *and*

$$\int_0^1 |f(s)g(s)| \mathrm{d}s \leq \|f\|_1 \|g\|_\infty. \tag{2.9}$$

3. Preliminary Lemmas

To prove the main results, we need the following lemmas.

Lemma 3.1 (see [15]). *For $y(t) \in C[0,1]$, the boundary value problem*

$$\begin{aligned} &u^{(n)}(t) + y(t) = 0, \quad t \in (0,1), \\ &u(0) = 0, \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = 0 \end{aligned} \tag{3.1}$$

has a unique solution

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds + t^{n-1}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)ds. \tag{3.2}$$

Lemma 3.2 (see [15]). *The Green's function for the boundary value problem*

$$\begin{aligned} &-u^{(n)}(t) = 0, \quad t \in (0,1), \\ &u(0) = 0, \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = 0 \end{aligned} \tag{3.3}$$

is given by

$$g(t,s) = \frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}, & 0 \le s \le t \le 1, \\ t^{n-1}(1-s)^{n-1}, & 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

Lemma 3.3 (see [15]). *The Green's function $g(t,s)$ defined by* (3.4) *satisfies that*

(i) *$g(t,s) \ge 0$ is continuous on $[0,1] \times [0,1]$;*

(ii) *$g(t,s) \le g(\theta_1(s), s)$ for all $t, s \in [0,1]$ and there exists a constant $\tilde{\gamma}_\tau > 0$ for any $\tau \in (0,1/2)$ such that*

$$\min_{t\in[\tau,1-\tau]} g(t,s) \ge \tilde{\gamma}_\tau g(\theta_1(s), s) \ge \tilde{\gamma}_\tau g(t', s), \quad \forall t', s \in [0,1], \tag{3.5}$$

where

$$\begin{aligned} &\tilde{\gamma}_\tau = \min\left\{ \left(\frac{\tau}{\theta_1(s)}\right)^{n-1}, \frac{\tau}{1-\theta_1(s)} \right\}, \\ &\theta_1(s) = \frac{s}{1-(1-s)^{(n-1)/(n-2)}} \quad (s < \theta_1(s) < 1). \end{aligned} \tag{3.6}$$

Lemma 3.4. *Suppose $\alpha + (\beta - \alpha)\eta^{n-1} \neq 1$, then for $y(t) \in C[0,1]$, the boundary value problem*

$$\begin{aligned} &u^{(n)}(t) + y(t) = 0, \quad t \in (0,1), \\ &u(0) = \alpha u(\eta), \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta) \end{aligned} \tag{3.7}$$

has a unique solution

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds \\ &- \frac{\alpha}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)ds + \frac{\alpha\eta^{n-1}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)ds \\ &+ \frac{(1-\alpha)t^{n-1}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)ds - \frac{(\beta-\alpha)t^{n-1}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)ds. \end{aligned} \tag{3.8}$$

Proof. The general solution of $u^{(n)}(t) + y(t) = 0$ can be written as

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + At^{n-1} + \sum_{i=1}^{n-2} A_i t^i + B. \tag{3.9}$$

Since $u^{(i)}(0) = 0$ for $i = 1, 2, \ldots, n-2$, we get $A_i = 0$ for $i = 1, 2, \ldots, n-2$. Now we solve for A, B by $u(0) = \alpha u(\eta)$ and $u(1) = \beta u(\eta)$, it follows that

$$\begin{aligned} &B = -\alpha \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + \alpha A\eta^{n-1} + \alpha B \\ &-\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + A + B = -\beta \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + \beta A\eta^{n-1} + \beta B. \end{aligned} \tag{3.10}$$

By solving the above equations, we get

$$\begin{aligned} A &= \frac{1}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \left((1-\alpha) \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s - (\beta-\alpha) \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s \right), \\ B &= \frac{1}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \left(-\alpha \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + \alpha\eta^{n-1} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s \right). \end{aligned} \tag{3.11}$$

Therefore, (3.7) has a unique solution

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s - \frac{\alpha}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + \frac{\alpha\eta^{n-1}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + \frac{(1-\alpha)t^{n-1}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s - \frac{(\beta-\alpha)t^{n-1}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s. \tag{3.12}$$

□

Lemma 3.5. *Suppose $0 < \alpha + (\beta-\alpha)\eta^{n-1} < 1$, the Green's function for the boundary value problem*

$$\begin{gathered} u^{(n)}(t) + y(t) = 0, \quad t \in (0,1), \\ u(0) = \alpha u(\eta), \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta) \end{gathered} \tag{3.13}$$

is given by

$$G(t,s) = g(t,s) + \frac{(\beta-\alpha)t^{n-1} + \alpha}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s), \tag{3.14}$$

where $g(t,s)$ is defined by (3.4).

We omit the proof as it is immediate from Lemma 3.4 and (3.4).

Lemma 3.6. *Suppose $0 < \alpha + (\beta-\alpha)\eta^{n-1} < 1$, the Green's function $G(t,s)$ defined by* (3.14) *satisfies that*

(i) *$G(t,s) \geq 0$ is continuous on $[0,1] \times [0,1]$;*

(ii) *$G(t,s) \leq J(s)$ for all $t,s \in [0,1]$ and there exists a constant $\gamma_\tau > 0$ for any $\tau \in (0,1/2)$ such that*

$$\min_{t\in[\tau,1-\tau]} G(t,s) \geq \gamma_\tau J(s) \geq \gamma_\tau G(t',s), \quad \forall t', s \in [0,1], \tag{3.15}$$

where

$$J(s) = g(\theta_1(s),s) + \frac{\max\{\alpha,\beta\}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s),$$

$$\gamma_\tau = \min\left\{ \tau^{n-1}, \frac{\min\left\{(\beta-\alpha)\tau^{n-1}, (\beta-\alpha)(1-\tau)^{n-1}\right\} + \alpha}{\max\{\alpha,\beta\}} \right\}$$

$$\begin{aligned}&\le \min\left\{\left(\frac{\tau}{\theta_1(s)}\right)^{n-1}, \frac{\tau}{1-\theta_1(s)}, \frac{\min\left\{(\beta-\alpha)\tau^{n-1}, (\beta-\alpha)(1-\tau)^{n-1}\right\}+\alpha}{\max\{\alpha,\beta\}}\right\}\\&= \min\left\{\widetilde{\gamma}_\tau, \frac{\min\left\{(\beta-\alpha)\tau^{n-1}, (\beta-\alpha)(1-\tau)^{n-1}\right\}+\alpha}{\max\{\alpha,\beta\}}\right\}.\end{aligned} \tag{3.16}$$

Proof. (i) From Lemma 3.3 and (3.14), we get

$$G(t,s) \ge 0 \text{ is continuous on } [0,1]\times[0,1]. \tag{3.17}$$

(ii) From Lemma 3.3 and (3.14), we have

$$\begin{aligned}G(t,s) &= g(t,s) + \frac{(\beta-\alpha)t^{n-1}+\alpha}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s)\\&\le g(\theta_1(s),s) + \frac{\max\{\alpha,\beta\}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s) = J(s).\end{aligned} \tag{3.18}$$

Next, we prove that (3.15) holds.

From Lemma 3.3 and (3.14), for $t \in [\tau, 1-\tau]$, we have

$$\begin{aligned}G(t,s) &= g(t,s) + \frac{(\beta-\alpha)t^{n-1}+\alpha}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s)\\&\ge \widetilde{\gamma}_\tau g(\theta_1(s),s) + \frac{\min\left\{(\beta-\alpha)\tau^{n-1}, (\beta-\alpha)(1-\tau)^{n-1}\right\}+\alpha}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s)\\&= \widetilde{\gamma}_\tau g(\theta_1(s),s) + \frac{\min\left\{(\beta-\alpha)\tau^{n-1}, (\beta-\alpha)(1-\tau)^{n-1}\right\}+\alpha}{\max\{\alpha,\beta\}} \times \frac{\max\{\alpha,\beta\}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s)\\&\ge \gamma_\tau\left(g(\theta_1(s),s) + \frac{\max\{\alpha,\beta\}}{1-\alpha-(\beta-\alpha)\eta^{n-1}} g(\eta,s)\right)\\&= \gamma_\tau J(s)\\&\ge \gamma_\tau G(t',s),\end{aligned} \tag{3.19}$$

for all $t' \in [0,1]$, where $\gamma_\tau = \min\{\tau^{n-1}, (\min\{(\beta-\alpha)\tau^{n-1}, (\beta-\alpha)(1-\tau)^{n-1}\}+\alpha)/\max\{\alpha,\beta\}\}$, $\tau \in (0,1/2)$.

We use inequality (3.15) to define our cones. Let $E = C[0,1]$, then E is a Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$. For a fixed $\tau \in (0,1/2)$, define the cone $P \subset E$ by

$$P = \left\{ u \in E \mid u(t) \geq 0 \text{ on } [0,1], \text{ and } \min_{t\in[\tau,1-\tau]} u(t) \geq \gamma_\tau \|u\| \right\}. \tag{3.20}$$

Define the operator T by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(u(s))\mathrm{d}s, \quad 0 \leq t \leq 1. \tag{3.21}$$

Obviously, $u(t)$ is a solution of (1.5) if and only if $u(t)$ is a fixed point of operator T. □

Theorems 2.4 and 2.5 require the operator T to be completely continuous and cone preserving. If T is continuous and compact, then it is completely continuous. The next lemma shows that $T : P \to P$ for $\tau \in (0,1/2)$ and that T is continuous and compact.

Lemma 3.7. *The operator T is completely continuous and $T : P \to P$ for each $\tau \in (0,1/2)$.*

Proof. Fix $\tau \in (0,1/2)$. Since $a(s)f(u(s)) \geq 0$ for all $s \in [0,1]$, $u \in P$ and since $G(t,s) \geq 0$ for all $t,s \in [0,1]$, then $Tu(t) \geq 0$ for all $t \in [0,1], u \in P$.

Let $u \in P$, by (3.15) and (3.21) we have

$$\begin{aligned}
\min_{t\in[\tau,1-\tau]} u(t) &= \min_{t\in[\tau,1-\tau]} \int_0^1 G(t,s)a(s)f(u(s))\mathrm{d}s \\
&\geq \int_0^1 \min_{t\in[\tau,1-\tau]} G(t,s)a(s)f(u(s))\mathrm{d}s \\
&\geq \gamma_\tau \int_0^1 G(t',s)a(s)f(u(s))\mathrm{d}s \\
&\geq \gamma_\tau Tu(t'),
\end{aligned} \tag{3.22}$$

for all $t' \in [0,1]$. Thus

$$\min_{t\in[\tau,1-\tau]} u(t) \geq \gamma_\tau \|Tu\|. \tag{3.23}$$

Clearly operator (3.21) is continuous. By the Arzela-Ascoli theorem T is compact. Hence, the operator T is completely continuous and the proof is complete. □

4. Main Results

In this section we present that problem (1.5) has countably many solutions if a and f satisfy some suitable conditions.

For convenience, we denote

$$\Lambda_1 = \frac{1}{\max_{t\in[0,1]}\int_{\tau_1}^{1-\tau_1} G(t,s)\mathrm{d}s \cdot m}, \qquad \Lambda_2 = \frac{1}{\|J\|_q \cdot \|a\|_p}. \tag{4.1}$$

Theorem 4.1. *Suppose conditions* (H_1) *and* (H_2) *hold, let* $\{\tau_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1} < \tau_k < t_k, k = 1, 2, \ldots$. *Let* $\{R_k\}_{k=1}^{\infty}$ *and* $\{r_k\}_{k=1}^{\infty}$ *be such that*

$$R_{k+1} < \gamma_{\tau_k} r_k < r_k < R_k, \qquad M r_k < L R_k, \quad k = 1, 2, \ldots, \tag{4.2}$$

where $M \in (\Lambda_1, +\infty)$, $L \in (0, \Lambda_2)$. *Furthermore, for each natural number* k, *assume that* f *satisfies the following two growth conditions:*

(H_3) $f(u) \le L R_k$ *for all* $u \in [0, R_k]$,

(H_4) $f(u) \ge M r_k$ *for all* $u \in [\gamma_{\tau_k} r_k, r_k]$.

Then problem (1.5) *has countably many positive solutions* $\{u_k\}_{k=1}^{\infty}$ *such that* $r_k \le \|u_k\| \le R_k$ *for each* $k = 1, 2, \ldots$.

Proof. Consider the sequences $\{\Omega_{1,k}\}_{k=1}^{\infty}$ and $\{\Omega_{2,k}\}_{k=1}^{\infty}$ of open subsets of E defined by

$$\begin{aligned} \Omega_{1,k} &= \{u \in E \mid \|u\| < R_k\}, \\ \Omega_{2,k} &= \{u \in E \mid \|u\| < r_k\}. \end{aligned} \tag{4.3}$$

Let $\{\tau_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $t_0 < t_{k+1} < \tau_k < t_k < 1/2$, for all $k \in N$. For each $k \in N$, define the cone P_k by

$$P_k = \left\{ u \in E \mid u(t) \ge 0 \text{ on } [0,1], \text{ and } \min_{t\in[\tau_k, 1-\tau_k]} u(t) \ge \gamma_{\tau_k} \|u\| \right\}. \tag{4.4}$$

Fixed k and let $u \in P_k \bigcap \partial\Omega_{2,k}$. For $s \in [\tau_k, 1-\tau_k]$, we have

$$\gamma_{\tau_k} r_k = \gamma_{\tau_k} \|u\| \le \min_{s\in[\tau_k, 1-\tau_k]} u(s) \le u(s) \le \|u\| = r_k. \tag{4.5}$$

By condition (H_4), we get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)a(s)f(u(s))\mathrm{d}s \\
&\geq \max_{t\in[0,1]}\int_{\tau_k}^{1-\tau_k} G(t,s)a(s)f(u(s))\mathrm{d}s \\
&\geq \max_{t\in[0,1]}\int_{\tau_k}^{1-\tau_k} G(t,s)a(s)\mathrm{d}s\cdot Mr_k \\
&\geq mMr_k\cdot \max_{t\in[0,1]}\int_{\tau_1}^{1-\tau_1} G(t,s)\mathrm{d}s \\
&\geq r_k = \|u\|.
\end{aligned} \tag{4.6}$$

Now let $u\in P_k\bigcap\partial\Omega_{1,k}$, then $u(s)\leq\|u\|=R_k$ for all $s\in[0,1]$. By condition (H_3), we get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)a(s)f(u(s))\mathrm{d}s \\
&\leq \int_0^1 J(s)a(s)\mathrm{d}s\cdot LR_k \\
&\leq \|J\|_q\|a\|_p\cdot LR_k \\
&\leq R_k = \|u\|.
\end{aligned} \tag{4.7}$$

It is obvious that $0\in\Omega_{2,k}\subset\overline{\Omega}_{2,k}\subset\Omega_{1,k}$. Therefore, by Theorem 2.4, the operator T has at least one fixed point $u_k\in P_k\bigcap(\overline{\Omega}_{1,k}\setminus\Omega_{2,k})$ such that $r_k\leq\|u_k\|\leq R_k$. Since $k\in N$ was arbitrary, Theorem 4.1 is completed. □

Let τ_k is defined by Theorem 4.1. We define the nonnegative continuous concave functionals $\alpha_k(u)$ on P by

$$\alpha_k(u) = \min_{t\in[\tau_k,1-\tau_k]} u(t). \tag{4.8}$$

We observe here that, for each $u\in P$, $\alpha(u)\leq\|u\|$.

For convenience, we denote

$$\Lambda = \|J\|_q\cdot\|a\|_p, \qquad \Gamma_k = \min_{t\in[\tau_k,1-\tau_k]}\int_{\tau_k}^{1-\tau_k} G(t,s)\mathrm{d}s\cdot m. \tag{4.9}$$

Theorem 4.2. *Suppose conditions* (H_1) *and* (H_2) *hold, let* $\{\tau_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1} < \tau_k < t_k, k = 1, 2, \ldots$. *Let* $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$, *and* $\{c_k\}_{k=1}^{\infty}$ *be such that*

$$c_{k+1} < a_k < b_k \leq \min\left\{\gamma_{\tau_k}, \frac{M}{L}\right\} c_k < c_k, \quad k = 1, 2, \ldots, \tag{4.10}$$

where $L \in (\Lambda, +\infty)$, $M \in (0, \Gamma_k)$. *Furthermore, for each natural number* k, *assume that* f *satisfies the following growth conditions:*

(H_5) $f(u) \leq c_k/L$ *for all* $u \in [0, c_k]$,

(H_6) $f(u) < a_k/L$ *for all* $u \in [0, a_k]$,

(H_7) $f(u) \geq b_k/M$ *for all* $u \in [b_k, b_k/\gamma_{\tau_k}]$.

Then problem (1.5) *has three infinite families of solutions* $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$, *and* $\{u_{3k}\}_{k=1}^{\infty}$ *such that*

$$\|u_{1k}\| < a_k, \qquad \min_{t\in[\tau_k, 1-\tau_k]} u_{2k}(t) > b_k, \qquad \|u_{3k}\| > a_k, \quad \text{with} \ \min_{t\in[\tau_k, 1-\tau_k]} u_{3k}(t) < b_k, \tag{4.11}$$

for each $k = 1, 2, \ldots$.

Proof. We note first that $T : \overline{P_{c_k}} \to \overline{P_{c_k}}$ is completely continuous operator. If $u \in P$, then from properties of $G(t,s)$, $Tu(t) \geq 0$, and by Lemma 3.7, $\min_{t\in[\tau_k, 1-\tau_k]} Tu(t) \geq \gamma_{\tau_k}\|Tu\|$. Consequently, $T : P \to P$.

If $u \in \overline{P_{c_k}}$, then $\|u\| \leq c_k$, and by condition (H_5), we have

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} |Tu(t)| \\ &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)a(s)f(u(s))\mathrm{d}s \right| \\ &\leq \frac{c_k}{L} \left| \int_0^1 J(s)a(s)\mathrm{d}s \right| \\ &\leq \frac{c_k}{L} \cdot \|J\|_q \cdot \|a\|_p \leq c_k. \end{aligned} \tag{4.12}$$

Therefore, $T : \overline{P_{c_k}} \to \overline{P_{c_k}}$. Standard applications of Arzela-Ascoli theorem imply that T is completely continuous operator.

In a completely analogous argument, condition (H_6) implies that condition (C_2) of Theorem 2.5 is satisfied.

We now show that condition (C_1) of Theorem 2.5 is satisfied. Clearly,

$$\left\{ u \in P\left(\alpha_k, b_k, \frac{b_k}{\gamma_{\tau_k}}\right) \mid \alpha_k(u) > b_k \right\} \neq \emptyset. \tag{4.13}$$

If $u \in P(\alpha_k, b_k, b_k/\gamma_{\tau_k})$, then $b_k \le u(s) \le b_k/\gamma_{\tau_k}$, for $s \in [\tau_k, 1-\tau_k]$. By condition (H_7), we get

$$\begin{aligned}\alpha_k(Tu) &= \min_{t\in[\tau_k,1-\tau_k]} \int_0^1 G(t,s)a(s)f(u(s))\mathrm{d}s \\ &\ge \min_{t\in[\tau_k,1-\tau_k]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)f(u(s))\mathrm{d}s \\ &\ge m\cdot\frac{b_k}{M}\min_{t\in[\tau_k,1-\tau_k]}\int_{\tau_k}^{1-\tau_k} G(t,s)\mathrm{d}s \ge b_k.\end{aligned} \tag{4.14}$$

Therefore, condition (C_1) of Theorem 2.5 is satisfied.

Finally, we show that condition (C_3) of Theorem 2.5 is also satisfied.

If $u \in P(\alpha_k, b_k, c_k)$ and $\|Tu\| > b_k/\gamma_{\tau_k}$, then

$$\alpha_k(Tu) = \min_{t\in[\tau_k,1-\tau_k]} Tu(t) \ge \gamma_{\tau_k}\|Tu\| > b_k. \tag{4.15}$$

Therefore, condition (C_3) is also satisfied. By Theorem 2.5, There exist three infinite families of solutions $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$, and $\{u_{3k}\}_{k=1}^{\infty}$ for problem (1.5) such that

$$\|u_{1k}\| < a_k, \qquad \min_{t\in[\tau_k,1-\tau_k]} u_{2k}(t) > b_k, \qquad \|u_{3k}\| > a_k, \quad \text{with} \min_{t\in[\tau_k,1-\tau_k]} u_{3k}(t) < b_k, \tag{4.16}$$

for each $k = 1, 2, \ldots$. Thus, Theorem 4.2 is completed. □

5. Example

In this section, we cite an example (see [15]) to verify existence of $a(t)$, and two simple examples are presented to illustrate the applications for obtained conclusion of Theorems 4.1 and 4.2.

Example 5.1. As an example of problem (1.5), we mention the boundary value problem

$$\begin{aligned}&u^{(3)}(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ &u(0) = \frac{1}{2}u\left(\frac{1}{2}\right), \qquad u'(0) = 0, \qquad u(1) = u\left(\frac{1}{2}\right),\end{aligned} \tag{5.1}$$

where $a(t)$ is defined by [15, Example 6.1] and $\varepsilon = 1/4$,

$$f(u) = \begin{cases} \dfrac{32 \times 10^{-(4k+2)} - (1/2) \times 10^{-4(k+1)}}{(1/625) \times 10^{-(8k+4)} - 10^{-8(k+1)}} \left(u^2 - 10^{-8(k+1)}\right) \\ \quad + \dfrac{1}{2} \times 10^{-4(k+1)}, & u \in \left[10^{-4(k+1)}, \dfrac{1}{25} \times 10^{-(4k+2)}\right], \\ 18 \times 10^{-(4k+2)} \times \sin \pi \dfrac{u - (1/25) \times 10^{-(4k+2)}}{10^{-(4k+2)} - (1/25) \times 10^{-(4k+2)}} \\ \quad +32 \times 10^{-(4k+2)}, & u \in \left[\dfrac{1}{25} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right], \\ \dfrac{32 \times 10^{-(4k+2)} - (1/2) \times 10^{-4k}}{10^{-(8k+4)} - 10^{-8k}} \left(u^2 - 10^{-8k}\right) \\ \quad + \dfrac{1}{2} \times 10^{-4k}, & u \in \left[10^{-(4k+2)}, 10^{-4k}\right] \ (k = 1, 2, \ldots), \\ \dfrac{1}{2} \times 10^{-4}, & u \in \left[10^{-4}, +\infty\right). \end{cases} \tag{5.2}$$

We notice that $n = 3$, $\alpha = 1/2$, $\beta = 1$, $\eta = 1/2$.

If we take $t_0 = 5/16$, $t_k = t_0 - \sum_{i=0}^{k-1} 1/(i+2)^4$, $\tau_k = (1/2)(t_k + t_{k+1})$, $k = 1, 2, \ldots$, then $t_{k+1} < \tau_k < t_k$, and $1/5 < t^* < \tau_k < \tau_1 = 1/4 - 1/(2 \times 3^4) < 1/4$, $\gamma_{\tau_k} = \min\{\tau_k^2, (\min\{(\beta-\alpha)\tau_k^2, (\beta-\alpha)(1-\tau_k)^2\} + \alpha)/\max\{\alpha, \beta\}\} > 1/25$, $k = 1, 2, \ldots$.

It follows from a direct calculation that

$$\begin{aligned} \int_{\tau_1}^{1-\tau_1} G(t,s)\mathrm{d}s > \int_{1/4}^{1-1/4} G(t,s)\mathrm{d}s &= \int_{1/4}^{3/4} g(t,s)\mathrm{d}s + \frac{4}{3}\left(1+t^2\right)\int_{1/4}^{3/4} g\left(\frac{1}{2}, s\right)\mathrm{d}s \\ &= \frac{1}{2}\left\{\int_{1/4}^{t}\left[t^2(1-s)^2 - (t-s)^2\right]\mathrm{d}s + \int_{t}^{3/4} t^2(1-s)^2\mathrm{d}s \right. \\ &\quad \left. + \frac{4}{3}\left(1+t^2\right)\left[\int_{1/4}^{1/2}\left(\frac{1}{4}(1-s)^2 - \left(\frac{1}{2}-s\right)^2\right)\mathrm{d}s + \int_{1/2}^{3/4}\frac{1}{4}(1-s)^2\mathrm{d}s\right]\right\} \\ &= \frac{1}{576}\left(-96t^3 + 122t^2 - 18t + \frac{25}{2}\right), \end{aligned} \tag{5.3}$$

so

$$\max_{t\in[0,1]}\int_{\tau_1}^{1-\tau_1} G(t,s)\mathrm{d}s \ge \max_{t\in[1/4,1-1/4]}\int_{1/4}^{1-1/4} G(t,s)\mathrm{d}s = \frac{31\times 7}{24\times 6\times 32} > \frac{1}{32},$$
$$\|J_1\|_2 = \left(\int_0^1 J_1^2(s)\mathrm{d}s\right)^{1/2} \le \frac{5}{6}, \qquad \|a\|_2 = \sqrt{\sqrt{2}\left(\frac{\pi^2}{3}-\frac{9}{4}\right)}. \tag{5.4}$$

In addition, if we take $r_k = 10^{-(4k+2)}$, $R_k = 10^{-4k}$, $M = 32$, $L = 1/2$, $m = (4/3)^{1/4}$, then

$$a(t) \ge \left(\frac{4}{3}\right)^{1/4} = m, \quad t\in[t^*, 1-t^*],$$
$$R_{k+1} = 10^{-4(k+1)} < \frac{1}{25}\times 10^{-(4k+2)} < \gamma_{\tau_k}\cdot r_k < r_k = 10^{-(4k+2)} < R_k = 10^{-4k},$$
$$Mr_k = 32\times 10^{-(4k+2)} < LR_k = \frac{1}{2}\times 10^{-4k}, \quad k = 1,2,\ldots, \tag{5.5}$$
$$\Lambda_1 = \frac{1}{\max_{t\in[0,1]}\int_{\tau_1}^{1-\tau_1} G_1(t,s)\mathrm{d}s\cdot m} \le \frac{1}{(1/32)\times(4/3)^{1/4}} < 32 = M,$$
$$\Lambda_2 = \frac{1}{\|J_1\|_2\cdot\|a\|_2} \ge \frac{1}{(5/6)\times\sqrt{\sqrt{2}(\pi^2/3-9/4)}} > L = \frac{1}{2},$$

and $f(u)$ satisfies the following growth conditions:

$$f(u) \le LR_k = \frac{1}{2}\times 10^{-4k}, \quad u\in\left[0, 10^{-4k}\right],$$
$$f(u) \ge Mr_k = 32\times 10^{-(4k+2)}, \quad u\in\left[\frac{1}{25}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right]. \tag{5.6}$$

Then all the conditions of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1 we know that problem (5.1) has countably many positive solutions $\{u_k\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \le \|u_k\| \le 10^{-4k}$ for each $k = 1,2,\ldots$.

Example 5.2. As another example of problem (1.5), we mention the boundary value problem

$$u^{(3)}(t) + a(t)f(u(t)) = 0, \quad t\in(0,1),$$
$$u(0) = \frac{1}{2}u\left(\frac{1}{2}\right), \qquad u'(0) = 0, \qquad u(1) = u\left(\frac{1}{2}\right), \tag{5.7}$$

where $a(t)$ is defined by [15, Example 6.1] and $\varepsilon = 1/4$,

$$f(u) = \begin{cases} \dfrac{u}{2} & u \in \left[10^{-4(k+1)}, 10^{-(4k+3)}\right], \\ \dfrac{(1/2)\times 10^{-(4k+3)} - 45\times 10^{-(4k+2)}}{10^{-(8k+6)} - 10^{-(8k+4)}}\left(u^2 - 10^{-(8k+4)}\right) & \\ \quad +45\times 10^{-(4k+2)}, & u \in \left[10^{-(4k+3)}, 10^{-(4k+2)}\right], \\ 5\times 10^{-(4k+2)} \times \sin\pi \dfrac{u - 10^{-(4k+2)}}{25\times 10^{-(4k+2)} - 10^{-(4k+2)}} & \\ \quad +45\times 10^{-(4k+2)}, & u \in \left[10^{-(4k+2)}, 25\times 10^{-(4k+2)}\right], \\ \dfrac{45\times 10^{-(4k+2)} - (1/2)\times 10^{-4k}}{625\times 10^{-(8k+4)} - 10^{-8k}}\left(u^2 - 10^{-8k}\right) & \\ \quad +\dfrac{1}{2}\times 10^{-4k}, & u \in \left[25\times 10^{-(4k+2)}, 10^{-4k}\right],\ (k=1,2,\ldots), \\ \dfrac{1}{2}\times 10^{-4}, & u \in \left[10^{-4}, +\infty\right). \end{cases} \tag{5.8}$$

We notice that $n = 3$, $\alpha = 1/2$, $\beta = 1$, $\eta = 1/2$.

If we take $t_0 = 5/16$, $t_k = t_0 - \sum_{i=0}^{k-1} 1/(i+2)^4$, $\tau_k = (1/2)(t_k + t_{k+1})$, $k = 1, 2, \ldots$, then $t_{k+1} < \tau_k < t_k$, and $1/5 < t^* < \tau_k < \tau_1 = 1/4 - 1/(2\times 3^4) < 1/4$, $\gamma_{\tau_k} = \min\{\tau_k^2, (\min\{(\beta-\alpha)\tau_k^2, (\beta-\alpha)(1-\tau_k)^2\} + \alpha)/\max\{\alpha,\beta\}\} > 1/25$, $k = 1, 2, \ldots$.

It follows from a direct calculation that

$$\begin{gathered} \Lambda = \|J\|_2 \cdot \|a\|_2 \le \frac{5}{6}\sqrt{\sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right)}, \\ \min_{t\in[\tau_k, 1-\tau_k]} \int_{\tau_k}^{1-\tau_k} G(t,s)\,ds \ge \min_{t\in[1/5, 1-1/5]} \int_{1/4}^{1-1/4} G(t,s)\,ds = \frac{3253}{3200\times 45} > \frac{1}{45}. \end{gathered} \tag{5.9}$$

In addition, if we take $a_k = 10^{-(4k+3)}$, $b_k = 10^{-(4k+2)}$, $c_k = 10^{-4k}$, $M = 1/45$, $L = 2$, $m = (4/3)^{1/4}$, then

$$\begin{aligned} a(t) &\ge \left(\frac{4}{3}\right)^{1/4} = m, \quad t \in [t_0, 1 - t_0], \\ c_{k+1} &= 10^{-4(k+1)} < a_k = 10^{-(4k+3)} < b_k = 10^{-(4k+2)} \\ &< \frac{1}{90}\times 10^{-4k} = \min\left\{\gamma_{\tau_k}, \frac{M}{L}\right\} c_k < c_k = 10^{-4k}, \end{aligned}$$

$$M = \frac{1}{45} < \frac{1}{45} \times \left(\frac{4}{3}\right)^{1/4} < \min_{t\in[\tau_k,1-\tau_k]} \int_{\tau_k}^{1-\tau_k} G(t,s)ds \cdot m = \Gamma_k, \quad k = 1,2,\ldots,$$

$$\Lambda = \|J\|_2 \cdot \|a\|_2 \leq \frac{5}{6}\sqrt{\sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right)} < 2 = L, \tag{5.10}$$

and $f(u)$ satisfies the following growth conditions:

$$\begin{aligned} f(u) &\leq \frac{c_k}{L} = \frac{10^{-4k}}{2}, \quad u \in \left[0, 10^{-4k}\right], \\ f(u) &< \frac{a_k}{L} = \frac{10^{-(4k+3)}}{2}, \quad u \in \left[0, 10^{-(4k+3)}\right], \\ f(u) &\geq \frac{b_k}{M} = \frac{10^{-(4k+2)}}{1/45} = 45 \times 10^{-(4k+2)}, \quad u \in \left[10^{-(4k+2)}, 25 \times 10^{-(4k+2)}\right]. \end{aligned} \tag{5.11}$$

Then all the conditions of Theorem 4.2 are satisfied. Therefore, by Theorem 4.2 we know that problem (5.7) has countably many positive solutions $\{u_k\}_{k=1}^{\infty}$ such that

$$\begin{aligned} \|u_{1k}\| &< 10^{-(4k+3)}, \quad \min_{t\in[\tau_k,1-\tau_k]} u_{2k}(t) > 10^{-(4k+2)} \\ \|u_{3k}\| &> 10^{-(4k+3)}, \quad \text{with} \min_{t\in[\tau_k,1-\tau_k]} u_{3k}(t) < 10^{-(4k+2)}, \end{aligned} \tag{5.12}$$

for each $k = 1,2,\ldots$.

Remark 5.3. In [8–12], the existence of solutions for local or nonlocal boundary value problems of higher-order nonlinear ordinary (fractional) differential equations that has been treated did not discuss problems with singularities. In [13], the singularity only allowed to appear at $t = 0$ and/or $t = 1$, the existence and multiplicity of positive solutions were asserted under suitable conditions on f. Although, [14, 15] seem to have considered the existence of countably many positive solutions for the second-order and higher-order boundary value problems with infinitely many singularities in $[0, 1/2)$. However, in [15], only the boundary conditions $u(0) = 0$ or $u(1) = 0$ have been considered. It is clear that the boundary conditions of Examples 5.1 and 5.2 are $u(0) \neq 0$ and $u(1) \neq 0$. Hence, we generalize second-order and higher-order multipoint boundary value problem.

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