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Research Article

On Some Generalizations Bellman-Bihari Result for Integro-Functional Inequalities for Discontinuous Functions and Their Applications

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We present some new nonlinear integral inequalities Bellman-Bihari type with delay for discontinuous functions (integro-sum inequalities; impulse integral inequalities). Some applications of the results are included: conditions of boundedness (uniformly), stability by Lyapunov (uniformly), practical stability by Chetaev (uniformly) for the solutions of impulsive differential and integro-differential systems of ordinary differential equations.

1. Introduction

The first generalizations of the Bihari result for discontinuous functions which satisfy nonlinear impulse inequality (integro-sum inequality) are connected with such types of inequalities:

(a)

$$v(t) \le c + \int_{t_0}^{t} p(\tau)\, v^m(\tau)\, d\tau \ + \sum_{t_0<t_i<t} \beta_i\, v(t_i - 0), \quad m > 0,\ m \ne 1, \tag{1.1}$$

(b)

$$v(t) \le c + \int_{t_0}^{t} p(\tau)\varphi(v(\tau))d\tau \ + \sum_{t_0<t_i<t} \beta_i\, v(t_i - 0), \tag{1.2}$$

Which are studied in the publications by Bainov, Borysenko, Iovane, Laksmikantham, Leela, Martynyuk, Mitropolskiy, Samoilenko ([1–13]), and in many others. In these investigations the method of integral inequalities for continuous functions is generalized to the case of piecewise continuous (one-dimensional inequalities) and discontinuous (multidimensional inequalities) functions.

For the generalization of the integral inequalities method for discontinuous functions and for their applications to qualitative analysis of impulsive systems: existence, uniqueness, boundedness, comparison, stability, and so forth. We refer to the results [2–5, 12, 14] and for periodic boundary value problems we cite [15–17]. More recently, a novel variational approach appeared in [18]. This approach to impulsive differential equations also used the critical point theory for the existence of solutions of a nonlinear Dirichlet impulsive problem and in [19] some new comparison principles and the monotone iterative technique to establish a more general existence theorem for a periodic boundary value problem. Reference [20] is very interesting in that it gives a complete overview of the state-of-the-art of the impulsive differential, inclusions.

In this paper, in Section 2, we investigate new analogies Bihari results for piece-wise continuous functions and, in Section 3, the conditions of boundedness, stability, pract-ical stability of the solutions of nonlinear impulsive differential and integro-differential systems.

2. General Bihari Theorems for Integro-Functional Inequalities for Discontinuous Functions

Let us consider the class $\wp$ of continuous functions $p : R \to R$, $p(t) \le t$, $\lim_{|t|\to\infty} p(t) = \infty$ ($p = p(t)$ is the delaying argument). The following holds.

Theorem 2.1. *(a) Let one suppose that for $x \ge x_0$ the following integro-sum functional inequality holds:*

$$u(x) \le \varphi(x) + q(x) \int_{x_i}^{x} f(\tau)\, W\big(u(p(\tau))\big)d\tau + \sum_{x_0<x_i<x} \beta_i\, u^m(x_i - 0), \tag{2.1}$$

where $q(x) \ge 1$, $\varphi(x)$ is a positive nondecreasing function, $\boldsymbol{\beta}_{\mathbf{i}} = const \ge 0$, $f : R_+ \to R_+$, $m = const > 0$; function $u(x)$ is a nonnegative piecewise-continuous,with I-st kind of discontinuities in the points $x_i : x_0 < x_1 < \cdots \lim_{n\to\infty} x_n = \infty$, $p(t)$ belongs to the class $\wp$.

(b) Function $W(x)$ satisfies such conditions:

(i) $W(\gamma\beta) \le W(\gamma)W(\beta)$;

(ii) $W : R_+ \to R_+$, $W(0) = 0$;

(iii) *W is nondecreasing.*

Then for arbitrary $x \in]x_0, \infty[$ the next estimate holds:

$$u(x) \le \varphi(x)q(x)G_i^{-1}\left[\int_{x_i}^{x} \frac{f(\tau)}{\varphi(\tau)}\, W\big[\varphi(p(\tau))q(p(\tau))\big]d\tau\right] \quad for \quad x \in]x_i, x_{i+1}[$$
$$\int_{x_i}^{x} \frac{f(\tau)}{\varphi(\tau)} W\big[\varphi(p(\tau))q(p(\tau))\big]d\tau \in \mathrm{Dom}\Big(G_i^{-1}\Big), \tag{2.2}$$

$$G_0(u) = \int_1^u \frac{d\sigma}{W(\sigma)}, \tag{2.3}$$

$$G_i(u) = \int_{c_i}^u \frac{d\sigma}{W(\sigma)}, \quad i = 1, 2, \ldots, \tag{2.4}$$

$$\begin{aligned} c_i &= \left(1 + \beta_i\ \varphi^{m-1}(x_i) q^m(x_i - 0)\right) G_{i-1}^{-1}\left(\int_{x_{i-1}}^{x_i} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau)) q(p(\tau))]\, d\tau\right), \\ &\qquad i = 1, 2, \ldots \ \textit{if } m \in]0, 1],\ \forall x \geq x_0, \\ c_i &= \left(1 + \beta_i\ \varphi^{m-1}(x_i) q^m(x_i - 0)\right) \left[G_{i-1}^{-1}\left(\int_{x_{i-1}}^{x_i} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau)) q(p(\tau))]\ d\tau\right)\right]^m, \\ &\qquad i = 1, 2, \ldots \ \textit{if } m \geq 1,\ \forall x \geq x_0. \end{aligned} \tag{2.5}$$

Proof. It follows from inequality (2.1)

$$\begin{aligned} \frac{u(x)}{\varphi(x)} &\leq\ 1 + q(x) \int_{x_0}^x \frac{f(\tau)\ W(u(p(\tau)))}{\varphi(\tau)} d\tau + \sum_{x_0 < x_i < x} \beta_i\ \frac{u^m(x_i - 0)}{\varphi(x)} \\ &\leq q(x) \left\{1 + \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)}\ W(u(p(\tau))) d\tau + \sum_{x_0 < x_i < x} \beta_i\ \varphi^{m-1}(x_i - 0) \left[\frac{u(x_i - 0)}{\varphi(x_i - 0)}\right]^m\right\}. \end{aligned} \tag{2.6}$$

Denoting by

$$\begin{aligned} u^*(x) &=\ 1 + \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W(u(p(\tau))) d\tau + \sum_{x_0 < x_i < x} \beta_i\ \varphi^{m-1}(x_i - 0) \left[\frac{u(x_i - 0)}{\varphi(x_i - 0)}\right]^m, \\ u^*(x) &=\ 1 \quad \text{for } x = x_0, \end{aligned} \tag{2.7}$$

then

$$\begin{aligned} & u(x_i - 0) \leq \varphi(x_i - 0)\ q(x_i - 0) u^*(x_i - 0), \\ & u(x) \leq \varphi(x)\ q(x) u^*(x), \\ & u(p(\tau)) \leq \varphi(p(\tau))\ q(p(\tau)) u^*(p(\tau)) \leq \varphi(p(\tau)) q(p(\tau)) u^*(\tau), \\ & \Longrightarrow u^*(x) \leq 1 + \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau)) q(p(\tau))] W(u^*(\tau)) d\tau \\ & \qquad + \sum_{x_0 < x_i < x} \beta_i \varphi^{m-1}(x_i - 0) q^m(x_i - 0) u^{*m}(x_i - 0). \end{aligned} \tag{2.8}$$

Let us consider the interval $I_1 = [x_0, x_1[$. Then

$$u^*(x) \le G_0^{-1}\left(\int_{x_0}^{x} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))]\,d\tau\right),$$
$$\text{if only } \int_{x_0}^{x} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))]\,d\tau \in \operatorname{Dom}\left(G_0^{-1}\right), \tag{2.9}$$

where $G_0(\xi) = \int_1^{\xi}(d\tau/W(\tau))$. So it results in

$$u(x) \le \varphi(x)q(x)G_0^{-1}\left[\int_{x_0}^{x} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))]\,d\tau\right], \tag{2.10}$$

and estimate (2.2) is valid in I_1.

Let us suppose that for $x \in I_k = \lfloor x_{k-1}, x_k \lfloor,\ k = 2,3,\ldots$ estimate (2.2) is fulfilled. Then for every $x \in I_{k+1}$ we have

$$u^*(x) \le G_k^{-1}\left(\int_{x_k}^{x} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))]\,d\tau\right)$$
$$\text{with} \int_{x_k}^{x} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))\,q(p(\tau))]\,d\tau \in \operatorname{Dom}\left(G_k^{-1}\right), \tag{2.11}$$

where $G_k(\xi)$ is determined from (2.3)–(2.5).

Taking into account such inequality

$$u(x) \le \varphi(x)q(x)u^*(x), \tag{2.12}$$

we obtain estimate (2.2) for every $x \in [x_0, \infty[$.

Let us consider the class $\Im$ of functions f such that

(i) $f(x)$*-positive, continuous, nondecreasing for* $x > 0$;

(ii) $\forall u \ge 1,\ v > 0 \Rightarrow u^{-1} f(v) < f(u^{-1} v)$;

(iii) $f(0) = 0$.

The following result is proved. □

Theorem 2.2. *Suppose that the part (a) of Theorem 2.1 is valid and function* $W : [0, \infty[\to [0, \infty[$ *belongs to the class* $\Im$. *Then for arbitrary* $x_0 \le x \le x^*$ *such estimate holds:*

$$u(x) \le \varphi(x)q(x)G_i^{*-1}\left[\int_{x_i}^{x} f(\tau)\,q(p(\tau))\,d\tau\right] \quad \textit{for } I_i = [x_i, x_{i+1}[,\ i = 0,1,\ldots, \tag{2.13}$$

where

$$G_0^*(\eta) = \int_1^\eta \frac{d\sigma}{W(\sigma)}, \qquad G_i^*(\eta) = \int_{c_i^*}^\eta \frac{d\sigma}{W(\sigma)} \quad i = 1,2,\ldots,$$
$$c_i^* = \left(1 + \beta_i\ \varphi^{m-1}(x_i) q^m(x_i)\right)\ {G_{i-1}^*}^{-1}\left(\int_{x_{i-1}}^{x_i} f(\tau) q(p(\tau)) d\tau\right) \quad \text{if } m \in]0,1], \tag{2.14}$$
$$c_i^* = \left(1 + \beta_i\ \varphi^{m-1}(x_i) q^m(x_i)\right)\left[{G_{i-1}^*}^{-1}\left(\int_{x_{i-1}}^{x_i} f(\tau) q(p(\tau)) d\tau\right)\right]^m \quad \text{if } m \geq 1,$$

and $x^* = \sup_x\{\int_{x_{i-1}}^x f(\tau)q(p(\tau))d\tau \in \mathrm{Dom}({G_{i-1}^*}^{-1})\}$, $i = 1,2,\ldots$.

Proof. By using the previous theorem we have $u(x) \leq \varphi(x)g(x)u^*(x)$, $u^*(x) = 1$ $x = x_0$. On the interval I_1

$$\frac{du^*(x)}{dx} = \frac{f(x)}{\varphi(x)} W(u(p(x))). \tag{2.15}$$

Then

$$\begin{aligned} u(p(x)) &\leq \varphi(p(x))q(p(x))u^*(p(x)) \leq \varphi(x)q(p(x))u^*(x), \\ \frac{du^*(x)}{dx} &\leq \frac{f(x)}{\varphi(x)}\ W(q(p(x))\varphi(x)u^*(x)) \\ &\leq \frac{f(x)q(p(x))}{\varphi(x)q(p(x))} W(q(p(x))\varphi(x)u^*(x)) \\ &\leq f(x)q(p(x))W(u^*(x)). \end{aligned} \tag{2.16}$$

Taking into account estimate (2.16), we obtain

$$\begin{aligned} &\int_{x_0}^x \frac{u^{*\prime}(\sigma)}{W(u^*(\sigma))} d\sigma \leq \int_{x_0}^x f(\tau)q(p(\tau))d\tau, \\ &\int_{x_0}^x \frac{u^{*\prime}(\sigma)}{W(u^*(\sigma))} d\sigma = \int_{u^*(x_0)}^{u^*(x)} \frac{du}{W(u)} = G_0^*(u^*(x)) - G_0^*(u^*(x_0)), \\ &u^*(x_0) = 1, \quad u^*(x) \geq 1, \quad G_0^*(u^*(x_0)) = G_0^*(1) = 0, \\ &G_0^*(u^*(x)) \leq \int_{x_0}^x f(\tau)\ q(p(\tau))\ d\tau. \end{aligned} \tag{2.17}$$

Then in I_1 we have

$$u(x) \le \varphi(x)q(x)\ G_0^{*-1}\left[\int_{x_0}^{x} f(\tau)q(p(\tau))d\tau\right] \quad \text{if only} \int_{x_0}^{x} f(\tau)q(p(\tau))d\tau \in \text{Dom}\left(G_0^{*-1}\right). \tag{2.18}$$

As in the previously theorem, the proof is completed by using the inductive method. □

The following result is easily to obtain

Theorem 2.3. *Suppose that for $x \ge x_0$ the next inequality holds:*

$$\begin{aligned} u(x) \le u_0 + q(x)&\left[\int_{x_0}^{x} f(s)\,u(p(s))ds + \int_{x_0}^{x} f(s)\left(\int_{x_0}^{x} g(\tau)u(p(\tau))d\tau\right)ds\right] \\ &+ \int_{x_0}^{x} h(s)\,W(u(\sigma(s)))ds + \sum_{x_0<x_i<x} \beta_i\ u^m(x_i - 0), \end{aligned} \tag{2.19}$$

where functions $u(x), f(x), q(x), g(x), h(x), p(x), \sigma(x)$ are real nonnegative for $x \ge x_0 > 0, p(x), \sigma(x) \in \Im, q(x) \ge 1, \beta_i \ge 0$, function W satisfies conditions (i),...,(iii) of Theorem 2.1.

Then for $x \ge x_0$ it results in

$$\begin{aligned} u(x) \le &\prod_{x_0<x_i<x}\left(1+\beta_i q^m(x_i)u_0^{m-1}\right)\exp\left(\int_{x_0}^{x} q(p(\tau))\left[f(\tau)+g(\tau)\right]d\tau\right) \\ &\cdot \psi_0^{-1}\left(\int_{x_0}^{x} h(\tau)\ W\left[\prod_{x_0<x_i<\sigma(\tau)}\left(1+\beta_i q^m(x_i)u_0^{m-1}\right)\right]W \right.\\ &\qquad \left.\times\left[q(\sigma(\tau))\exp\left(\int_{x_0}^{\sigma(\tau)} q(p(s))\left[f(s)+g(s)\right]ds\right)\right]d\tau\right), \quad \text{if } \ m \in\,]0,1] \\ &\int_{x_0}^{x} h(\tau)W\left[\prod_{x_0<x_i<\sigma(\tau)}\left(1+\beta_i q^m(x_i)u_0^{m-1}\right)\right]W \\ &\qquad \times\left[q(\sigma(\tau))\exp\left(\int_{x_0}^{\sigma(\tau)} q(p(s))\left[f(s)+g(s)\right]ds\right)\right]d\tau \in \text{Dom}\left(\psi_0^{-1}\right), \end{aligned} \tag{2.20}$$

where $\psi_0(u) = \int_{u_0}^{u}(dv/W(v))$;

$$u(x) \leq \prod_{x_0<x_i<x}\left(1+\beta_i q^m(x_i)u_0^{m-1}\right)\exp\left(m\int_{x_0}^{x} q(p(\tau))[f(\tau)+g(\tau)]\right)$$
$$\cdot\psi_0^{-1}\left(\int_{x_0}^{x} h(\tau)\left[\prod_{x_0<x_i<\sigma(\tau)}\left(1+\beta_i q^m(x_i)u_0^{m-1}\right)\right]\right.$$
$$\left.\cdot W\left[q(\sigma(\tau))\exp\left(m\int_{x_0}^{\sigma(\tau)} q(p(s))[f(s)+g(s)]ds\right)\right]d\tau\right), \quad \text{if } m \geq 1, \tag{2.21}$$
$$\int_{x_0}^{x} h(\tau)W\left[\prod_{x_0<x_i<\sigma(\tau)}\left(1+\beta_i q^m(x_i)u_0^{m-1}\right)\right]W$$
$$\times\left[q(\sigma(\tau))\exp\left(m\int_{x_0}^{\sigma(\tau)} q(p(s))[f(s)+g(s)]ds\right)\right]d\tau \in \mathrm{Dom}\left(\psi_0^{-1}\right).$$

The proof the same procedure as that of (Iovane [21, Theorems 2.1 and 3.1]).

Corollary 2.4. *Suppose that*

(a) $m = 1$, *then the result of Theorem 2.1 coincides with the result [22, Theorem 3.7.1, page 232];*

(b) $m = 1$, $\varphi(x) = c$, $q(x) = 1, p(t) = t$, *then the result of Theorem 2.1 coincides with result [12, Proposition 2.3, page 2143];*

(c) $q(x) = 1$, $W(u) = u$, $p(t) = t$, *then one obtains the analogy of Gronwall- Bellman result for discontinuous functions [23, Lemma 1] and estimate* (2.2) *reduces in the following form:*

$$u(x) \leq \varphi(x)\prod_{x_0<x_i<x}\left(1+\beta_i\,\varphi^{m-1}(x_i)\right)\exp\left(\int_{x_0}^{x} f(\tau)\,d\tau\right) \quad \text{if } m \in]0,1],\ \forall x \geq x_0,$$
$$u(x) \leq \varphi(x)\prod_{x_0<x_i<x}\left(1+\beta_i\,\varphi^{m-1}(x_i)\right)\exp\left(m\int_{x_0}^{x} f(\tau)\,d\tau\right) \quad \text{if } m \geq 1,\ \forall x \geq x_0. \tag{2.22}$$

(d) $q(x) = 1$, $W(u) = u$, *then one obtains the result [21, Theorem 2.1] and estimate* (2.2) *are as follows:*

$$u(x) \leq \varphi(x)\prod_{x_0<x_i<x}\left(1+\beta_i\,\varphi^{m-1}(x_i)\right)\exp\left(\int_{x_0}^{x} f(\tau)\,\frac{\varphi(p(\tau))}{\varphi(\tau)}d\tau\right), \quad \text{if } m \in]0,1],\ \forall x \geq x_0;$$
$$u(x) \leq \varphi(x)\prod_{x_0<x_i<x}\left(1+\beta_i\,\varphi^{m-1}(x_i)\right)\exp\left(m\int_{x_0}^{x} f(\tau)\,\frac{\varphi(p(\tau))}{\varphi(\tau)}d\tau\right) \quad \text{if } m \geq 1,\ \forall x \geq x_0. \tag{2.23}$$

(e) $q(x) = 1, W(u) = u^m, m > 0,\ p(t) = t$, *then one obtains the analogy of Bihari result for discontinuous functions [23, Lemma 2] and estimate* (2.2) *reduces as follows are reduced:*

$$u(x) \le \varphi(x) \prod_{x_0<x_i<x} \left(1 + \beta_i\ \varphi^{m-1}(x_i)\right) \left[1 + (1-m)\int_{x_0}^{x} \varphi^{m-1}(\tau) f(\tau) d\tau\right]^{1/(1-m)},$$

$$\text{if } 0 < m < 1,\ \forall x \ge x_0,$$

$$u(x) \le \varphi(x) \prod_{x_0<x_i<x} \left(1 + \beta_i m\varphi^{m-1}(x_i)\right) \left[1 - (m-1)\left[\prod_{x_0<x_i<x}\left(1 + \beta_i\ m\varphi^{m-1}(x_i)\right)\right]^{m-1}\right.$$

$$\left. \times \int_{x_0}^{x} \varphi^{m-1}(\tau)\ f(\tau)\ d\tau\right]^{-1/(m-1)} \quad \forall x \ge x_0, \tag{2.24}$$

such that

$$\int_{x_0}^{x} \varphi^{m-1}(\tau) f(\tau)\ d\tau \le \frac{1}{m}, \quad m > 1, \quad \prod_{x_0<x_i<x}\left(1 + \beta_i\ \varphi^{m-1}(x_i)\right) < \left(1 + \frac{1}{m-1}\right)^{1/(m-1)}. \tag{2.25}$$

(f) *W(u) = u^m, m> 0, then estimate* (2.2) *reduces as follows (see [21, Theorem 2.2]):*

$$u(x) \le \varphi(x)\ q(x) \prod_{x_0<x_i<x}\left(1 + \beta_i\ \varphi^{m-1}(x_i)\ \ q^m(x_i)\right)$$

$$\times \left[1 + (1-m)\int_{x_0}^{x} \varphi^{m-1}(\tau) f(\tau) q^m(p(\tau)) \left[\frac{\varphi(p(\tau))}{\varphi(\tau)}\right]^m d\tau\right]^{1/(1-m)}$$

$$\text{if } 0 < m < 1,\ \forall x \ge x_0,$$

$$u(x) \le \varphi(x)\ q(x) \prod_{x_0<x_i<x}\left(1 + \beta_i m\varphi^{m-1}(x_i)\ q^m(x_i)\right)$$

$$\times \left\{1 - (m-1)\left[\prod_{x_0<x_i<x}\left(1 + \beta_i\ m\varphi^{m-1}(x_i) q^m(x_i)\right)\right]^{m-1}\right.$$

$$\left. \times \int_{x_0}^{x} \varphi^{m-1}(\tau)\ f(\tau) q^m(p(\tau)) \left[\frac{\varphi(p(\tau))}{\varphi(\tau)}\right]^m d\tau\right\}^{-1/(m-1)} \forall x \ge x_0$$

such that

$$\int_{x_0}^{x} \varphi^{m-1}(\tau)\ f(\tau) q^m(p(\tau)) \left[\frac{\varphi(p(\tau))}{\varphi(\tau)}\right]^m d\tau \le \frac{1}{m}, \quad m > 1,$$

$$\prod_{x_0<x_i<x} \left(1+\beta_i m \varphi^{m-1}(x_i) q^m(x_i)\right) < \left(1+\frac{1}{m-1}\right)^{-1/(m-1)}. \tag{2.26}$$

(g) *Suppose that in Theorem 2.3* $q(x)=1, W(u)=u, \sigma(s)=p(s)=s$, *then estimates* (2.20), (2.21) *reduce as shown:*

$$u(x) \le u_0 \prod_{x_0<x_i<x} \left(1+\beta_i\ u_0^{m-1}(x_i)\right) \exp\left[\int_{x_0}^{x} [f(\xi)+g(\xi)+h(\xi)]\, d\xi\right] \quad \text{if } m \in]0,1],\ \forall x \ge x_0;$$

$$u(x) \le u_0 \prod_{x_0<x_i<x} \left(1+\beta_i\ u_0^{m-1}(x_i)\right) \exp\left[m\int_{x_0}^{x} [f(\xi)+g(\xi)+h(\xi)]\, d\xi\right] \quad \text{if } m \ge 1,\ \forall x \ge x_0, \tag{2.27}$$

which coincide with result of [21, Theorem 3.1] for $h(t) = u_0$.

3. Applications

Let us consider the following system of differential equations

$$\frac{dx}{dt} = F(t,x), \quad t \ne t_i, \qquad \Delta x|_{t=t_i} = I_i(x) \tag{3.1}$$

where $x \in \mathfrak{R}^n$, $F \in \mathfrak{R}^n$, $I_i(x) \in \mathfrak{R}^n$ $(i=1,2,\ldots)$, $t \ge t_0 \ge 0$, $\lim_{i\to\infty} t_i = \infty$, $t_{i-1} < t_i$ for all $i = 1,2,\ldots$.

Let us assume that $F(t,x)$ and $I_i(x)$ are defined in the domain $D = \{(t,x)\ :\ t \in \mathfrak{I} = [t_0, T],\ T \le \infty,\ \|x\| \le h\ \}$ and satisfy such conditions:

(a) $\|F(t,x)\| \le f(t) W(\|x\|)$, $f : \mathfrak{R}_+ \to \mathfrak{R}_+$,

W satisfies conditions (i)–(iii) of Theorem 2.1;

(b) $\|I_i(x)\| \le \beta_i \|x\|^m$, $\beta_i = \text{const} > 0, m > 0$.

Consider $x(t) = x(t, t_0, x_0)$ the solution of Cauchy problem for system (3.1). Then

$$x(t,t_0,x_0) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau,t_0,x_0))\, d\tau + \sum_{t_0<t_i<t} I_i(x(t_i - 0, t_0, x_0)), \tag{3.2}$$

from which it follows

$$\|x(t,t_0,x_0\,)\| \leq \|x_0\| + \int_{t_0}^{t} f(\tau)\; W(\|x(\tau,t_0,x_0)\|)d\tau + \sum_{t_0<t_i<t} \beta_i \|x(t_i-0,t_0,x_0)\|^m. \tag{3.3}$$

By using the result of Theorem 2.1 and estimate (2.2) we obtain

$$\begin{gathered}\|x(t,t_0,x_0)\| \leq \|x_0\| G_i^{-1}\left[\int_{x_i}^{x} f(\tau)\; \frac{W(\|x_0\|)}{\|x_0\|}d\tau\right] \quad \text{for } x \in \,]x_i, x_{i+1}[\;, \\ \int_{x_i}^{x} f(\tau)\; \frac{W(\|x_0\|)}{\|x_0\|}d\tau \in \operatorname{Dom}\left(G_i^{-1}\right),\end{gathered} \tag{3.4}$$

where

$$\begin{gathered} G_0(u) = \int_1^u \frac{d\sigma}{W(\sigma)}, \qquad G_i(u) = \int_{c_i}^u \frac{d\sigma}{W(\sigma)}, \quad i = 1,2,\ldots, \\ c_i = \left(1+\beta_i\|x_0\|^{m-1}\right)G_{i-1}^{-1}\left(\int_{x_{i-1}}^{x_i} f(\tau)\; \frac{W(\|x_0\|)}{\|x_0\|}d\tau\right), \\ i = 1,2,\ldots \quad \text{if } m \in \,]0,1],\; \forall x \geq x_0, \\ c_i = \left(1+\beta_i\|x_0\|^{m-1}\right)\left[G_{i-1}^{-1}\left(\int_{x_{i-1}}^{x_i} f(\tau)\; \frac{W(\|x_0\|)}{\|x_0\|}d\tau\right)\right]^m, \\ i = 1,2,\ldots \quad \text{if } m \geq 1,\; \forall x \geq x_0. \end{gathered} \tag{3.5}$$

Let us consider some particular cases of W.

If $W(u) = u$, $m = 1$, estimate (3.4) is reduced in such form

$$\|x(t,t_0,x_0)\| \leq \|x_0\| \prod_{t_0<t_i<t} (1+\beta_i) \exp\left[\int_{t_0}^{t} f(\tau)d\tau\right]. \tag{3.6}$$

Then such result holds.

Proposition 3.1. *Let the following conditions be fulfilled for system* (3.1) *:*

(i) $\|F(t,x)\| \leq f(t)\|x\|$;

(ii) $\|I_i(x)\| \leq \beta_i\|x\|$;

(iii) $\exists m_1(t_0) = const. > 0 : \prod_{t_0<t_i<t}(1+\beta_i) \;\leq m_1(t_0) < \infty$;

(iv) $\exists m_2(t_0) = \; const. > 0 : \int_{t_0}^{t} f(\tau)\; d\tau \leq m_2(t_0) < \infty, \forall t \geq t_0$.

Then one has:

(a) *All solutions of system* (3.1) *are bounded (uniformly, if* $m_i(t_0)$ *are independent of* t_0*) and such estimate is valid:*

$$\|x(t,t_0,x_0)\| \le m_1(t_0)\exp[m_2(t_0)]\|x_0\|. \tag{3.7}$$

(b) *The trivial solution of system* (3.1) *is stable by Lyapunov (uniformly stable relative* t_0*, if* $m_i(t_0) = m_i$*,* $i = 1,2$*).*

Remark 3.2. If conditions I–IV of Proposition 3.1 are valid and $\lambda/\Lambda < (m_1(t_0)\exp[m_2(t_0)])^{-1}$, then the trivial solution is $(\lambda,\Lambda,\Im)$-stable by Chetaev (uniformly $(\lambda,\Lambda,\Im)$-stable, if $m_i(t_0)$, $i = 1,2$ is independent of t_0).

If $W(u) = u^l$, $l \neq 1$, $m = 1$ the estimate (3.4) is reduced in such form

$$\|x(t,t_0,x_0)\| \le \prod_{t_0<t_i<t}(1+\beta_i)\left[\|x_0\|^{1-l} + (1-l)\int_{t_0}^{t} f(\tau)d\tau\right]^{1/(1-l)} \quad \forall t \ge t_0, \text{ if } 0<l<1, \tag{3.8}$$

$$\begin{aligned}\|x(t,t_0,x_0)\| &\le \|x_0\|\prod_{t_0<t_i<t}(1+\beta_i)\\ &\times\left[1-\ (l-1)\ \|x_0\|^{l-1}\ \cdot\left[\prod_{t_0<t_i<t}(1+\beta_i)\right]^{l-1}\int_{t_0}^{t} f(\tau)d\tau\right]^{-\,1/(l-1)} \quad \forall t \ge t_0,\end{aligned} \tag{3.9}$$

$$\int_{t_0}^{t} f(\tau)d\tau < \left((l-1)\left[\|x_0\|\prod_{t_0<t_i<t}(1+\beta_i)\right]^{l-1}\right)^{-1}, \quad \text{if } l>1. \tag{3.10}$$

From estimate (3.8) the next propositions follow.

Proposition 3.3. *Suppose that such conditions occur:*

(a) $\|F(t,x) - F(t,y)\| \le f(t)\ \|x-y\|^l$, $0<l<1$ *for all* $x,y \in D$

(b) *estimates ii–iv of Proposition 3.1 be fulfilled.*

Then all the solutions of system (3.1) *are bounded (uniformly if* $m_i(t_0) = m_i$*,* $i = 1,2$*).*

Remark 3.4. Suppose that conditions (a), (b) of Proposition 3.3 are valid and

$$\lambda^{1-l} + (1-l)\ m_2(t_0) < \left[\frac{\Lambda}{m_1(t_0)}\right]^{1-l}. \tag{3.11}$$

Then trivial solution of system (3.1) is $(\lambda, \Lambda, \Im)$-stable by Chetaev (uniformly if $m_i(t_0)$ is independent of t_0).

Proposition 3.5. *Let conditions ii–iv of Proposition 3.1 be fulfilled for system* (3.1)*, inequality* (3.10) *holds and*

$$\|F(t,x)\| \leq f(t)\|x\|^l, \quad l > 1. \tag{3.12}$$

Then trivial solution of system (3.1) *is stable by Lyapunov (uniformly if $m_i(t_0) = m_i,\ i = 1, 2$).*

Remark 3.6. If $W(u) = u^1, 1 > 0$, and $m \neq 1$ the conditions of boundedness, stability, $(\lambda, \Lambda, \Im)$-stability is investigated in [14, see Theorems 3.4–3.6]; the estimates of the solutions of system (3.1) with non-Lipschitz type of discontinuities are investigated in [23, see Proposition 1, Proposition 2].

Let us consider the following impulsive system of integro-differential equations:

$$\begin{aligned} \frac{dx}{dt} &= F(t, x, K[x(t)]), \quad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x), \end{aligned} \tag{3.13}$$

where $x \in \Re^n$, $F \in \Re^n$, $I_i(x) \in \Re^n$ $(i = 1, 2, \ldots)$ and defined in the domain D, $K[x(t)] = \int_{t_0}^{t} k(t, \tau, x(\tau))\, d\tau$.

We suppose that such conditions are valid:

(i) $\|F(t,x,y)\| \leq f(t)[\|x\| + \|y\|]$ for all $x, y \in D, f : \Re_+ \to \Re_+$;

(ii) $\|k(t,s,x)\| \leq g(t)\|x\|$ for all $s \in [t_0, t],\ g : \Re_+ \to \Re_+$;

(iii) $\|I_i(x)\| \leq \beta_i\|x\|^m$ for all $x, y \in D, \beta_i = \text{const} > 0, m > 0\ m \neq 1$.

It is easy to see that

$$\begin{aligned} \|x(t, t_0, x_0)\| &\leq \|x_0\| + \int_{t_0}^{t} f(\tau)\|x(\tau, t_0, x_0)\| d\tau \\ &\quad + \int_{t_0}^{t} f(\tau)\left(\int_{t_0}^{\tau} g(\xi)\|x(\xi, t_0, x_0)\| d\xi\right) d\tau + \sum_{t_0 < t_i < t} \beta_i \|x(t_i - 0, t_0, x_0)\|^m \\ \Longrightarrow \|x(t, t_0, x_0)\| &\leq \|x_0\| \prod_{t_0 < t_i < t}\left(1 + \beta_i\|x_0\|^{m-1}\right) \exp\int_{t_0}^{t}[f(\xi) + g(\xi)]d\xi, \\ &\qquad \text{if } 0 < m \leq 1,\ t \geq t_0 \end{aligned} \tag{3.14}$$

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| \prod_{t_0 < t_i < t}\left(1 + \beta_i\|x_0\|^{m-1}\right) \exp\left(m\int_{t_0}^{t}[f(\xi) + g(\xi)]d\xi\right), \\ &\qquad \text{if } 0\, m \geq 1,\ t \geq t_0. \end{aligned} \tag{3.15}$$

From estimate (3.15) such result follows.

Proposition 3.7. *Let one suppose that for system* (3.13) *conditions (i)–(iii) take place for* $m > 1$ *and the following estimates are fulfilled:*

(a) $\exists\ m_3(t_0) = \ const. > 0 : \prod_{t_0<t_i<t}(1 + \beta_i\|x_0\|^{m-1}) \ \leq m_3(t_0) < \infty;$

(b) $\exists\ m_4(t_0) = \ const. > 0 : \int_{t_0}^{t}[f(\xi) + g(\xi)]\ d\xi \leq m_4(t_0) < \infty$ *for all* $t \geq t_0$.

Then we have:

(i) *All solutions of system* (3.13) *are bounded and satisfy the estimate:*

$$\|x(t)\| \ \leq m_3(t_0)\exp[m_4(t_0)]\|x_0\|. \tag{3.16}$$

(ii) *The trivial solution of system* (3.13) *is stable by Lyapunov (uniformly, if* $m_i(t_0) = m_i$, $i = 3, 4$*).*

(iii) *The trivial solution of system* (3.13) *is* $(\lambda, \Lambda, \Im)$*-stable by Chetaev (uniformly if* $m_i(t_0)$ *is independent of* t_0*) and* $m_3(t_0)\exp[m_4(t_0)] < \Lambda/\lambda$.

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