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Research Article

Existence of Positive Solutions for Multipoint Boundary Value Problem on the Half-Line with Impulses

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We consider a multi-point boundary value problem on the half-line with impulses. By using a fixed-point theorem due to Avery and Peterson, the existence of at least three positive solutions is obtained.

1. Introduction

Impulsive differential equations are a basic tool to study evolution processes that are subjected to abrupt changes in their state. For instance, many biological, physical, and engineering applications exhibit impulsive effects (see [1–3]). It should be noted that recent progress in the development of the qualitative theory of impulsive differential equations has been stimulated primarily by a number of interesting applied problems [4–24].

In this paper, we consider the existence of multiple positive solutions of the following impulsive boundary value problem (for short BVP) on a half-line:

$$\begin{gathered} u''(t) + q(t)f(t,u) = 0, \quad 0 < t < \infty,\ t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,\ldots,p, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0, \end{gathered} \tag{1.1}$$

where $u'(\infty) = \lim_{t\to+\infty} u'(t), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $0 < t_1 < t_2 < \cdots < t_p < +\infty$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and α_i, f, q, and I_k satisfy

(H_1) $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

(H_2) $f(t,u) \in C([0,\infty)\times[0,+\infty),[0,+\infty))$, $I_k(u) \in C([0,+\infty),[0,+\infty))$, and when $u/(1+t)$ is bounded, $f(t,u)$ and $I_k(u)$ are bounded on $[0,+\infty)$;

(H_3) $q(t) \in C([0,\infty),[0,+\infty))$ and $q(t)$ is not identically zero on any compact subinterval of $(0,+\infty)$. Furthermore $q(t)$ satisfies

$$\sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)q(s)ds < +\infty, \tag{1.2}$$

where

$$G(t,s) = \begin{cases} s, & 0 \le s \le t < +\infty, \\ t, & 0 \le t \le s < +\infty. \end{cases} \tag{1.3}$$

Boundary value problems on the half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and there are many results in this area, see [8, 13, 14, 20, 25–27], for example.

Lian et al. [25] studied the following boundary value problem of second-order differential equation with a p-Laplacian operator on a half-line:

$$\begin{gathered} (\varphi_p(u'(t)))' + \phi(t)f(t,u,u') = 0, \quad 0 < t < +\infty, \\ \alpha u(0) - \beta u'(0) = 0, \qquad u'(\infty) = 0. \end{gathered} \tag{1.4}$$

They showed the existence at least three positive solutions for (1.4) by using a fixed point theorem in a cone due to Avery-Peterson [28].

Yan [20], by using Leray-Schauder theorem and fixed point index theory presents some results on the existence for the boundary value problems on the half-line with impulses and infinite delay.

However to the best knowledge of the authors, there is no paper concerned with the existence of three positive solutions to multipoint boundary value problems of impulsive differential equation on infinite interval so far. Motivated by [20, 25], in this paper, we aim to investigate the existence of triple positive solutions for BVP (1.1). The method chosen in this paper is a fixed point technique due to Avery and Peterson [28].

2. Preliminaries

In this section, we give some definitions and results that we will use in the rest of the paper.

Definition 2.1. Suppose P is a cone in a Banach. The map α is a nonnegative continuous concave functional on P provided $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y) \tag{2.1}$$

for all $x, y \in P$, and $t \in [0,1]$. Similarly, the map β is a nonnegative continuous convex functional on P provided $\beta : P \to [0,\infty)$ is continuous and

$$\beta\big(tx + (1-t)y\big) \le t\beta(x) + (1-t)\beta(y) \tag{2.2}$$

for all $x, y \in P$, and $t \in [0,1]$.

Let γ, θ be nonnegative, continuous, convex functionals on P and α be a nonnegative, continuous, concave functionals on P, and ψ be a nonnegative continuous functionals on P. Then, for positive real numbers a, b, c, and d, we define the convex sets

$$\begin{aligned} P(\gamma, d) &= \{x \in P : \gamma(x) < d\}, \\ P(\gamma, \alpha, b, d) &= \{x \in P : b \le \alpha(x),\ \gamma(x) \le d\}, \\ P(\gamma, \theta, \alpha, b, c, d) &= \{x \in P : b \le \alpha(x),\ \theta(x) \le c,\ \gamma(x) \le d\}, \end{aligned} \tag{2.3}$$

and the closed set

$$R(\gamma, \psi, a, d) = \{x \in P : a \le \psi(x),\ \gamma(x) \le d\}. \tag{2.4}$$

To prove our main results, we need the following fixed point theorem due to Avery and Peterson in [28].

Theorem 2.2. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on a cone P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$, such that for some positive numbers M and d*

$$\alpha(x) \le \psi(x), \qquad \|x\| \le M\gamma(x) \tag{2.5}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose

$$\Phi : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)} \tag{2.6}$$

is completely continuous and there exist positive numbers a, d, and c with $a < b$ such that

(i) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \ne \emptyset$ and $\alpha(\Phi x) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;*

(ii) *$\alpha(\Phi x) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(\Phi x) > c$;*

(iii) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$, with $\psi(\Phi x) = a$.*

Then Φ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\begin{gathered} \gamma(x_i) \le d, \quad \text{for } i = 1,2,3, \quad \psi(x_1) < a, \\ a < \psi(x_2) \quad \text{with } \alpha(x_2) < b, \quad \alpha(x_3) > b. \end{gathered} \tag{2.7}$$

3. Some Lemmas

Define $PC[0,+\infty) = \{u : [0,+\infty) \to R \mid u(t)$ is continuous at each $t \neq t_k$, left continuous at $t = t_k$, $u(t_k^+)$ exists, $k = 1,\ldots,p\}$.

By a solution of (1.1) we mean a function u in $PC[0,\infty)$ satisfying the relations in (1.1).

Lemma 3.1. *$u(t)$ is a solution of (1.1) if and only if $u(t)$ is a solution of the following equation:*

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)q(s)f(s,u(s))ds + \sum_{0<t_k<t} I_k(u) \\ &\quad + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty} G(\xi_i,s)q(s)f(s,u(s))ds + \sum_{0<t_k<\xi_i} I_k(u)\right] \\ &:= Tu(t), \end{aligned} \tag{3.1}$$

where $G(t,s)$ is defined as (1.3).

The proof is similar to Lemma 3 in [9], and here we omit it.

For $t_p < a^* < b^* < +\infty$, let $c^* = \min\{a^*/(1+a^*), 1/(1+b^*)\}$. Then

$$\frac{G(t,s)}{1+t} \geq c^* \frac{G(r,s)}{1+r}, \quad \frac{1}{1+t} \geq \frac{c^*}{1+r}, \quad \text{for } t \in [a^*,b^*],\ r \in [0,+\infty),\ s \in [0,+\infty). \tag{3.2}$$

It is clear that $0 < c^* < 1$. Consider the space E defined by

$$E = \left\{u \in PC[0,+\infty) : \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t} < +\infty\right\}. \tag{3.3}$$

E is a Banach space, equipped with the norm $\|u\| = \sup_{0\leq t<+\infty}(|u(t)|/(1+t)) < +\infty$. Define the cone $P \subset E$ by

$$P = \left\{u \in E : u(t) \geq 0,\ t \in [0,+\infty),\ \min_{t\in[a^*,b^*]} \frac{u(t)}{1+t} \geq c^*\|u\|\right\}. \tag{3.4}$$

Lemma 3.2 (see [20, Theorem 2.2]). *Let $M \subset PC[0,+\infty)$. Then M is compact in $PC[0,+\infty)$, if the following conditions hold:*

(a) *M is bounded in $PC[0,+\infty)$;*

(b) *the functions belonging to M are piecewise equicontinuous on any interval of $[0,+\infty)$;*

(c) *the functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $\tau(\varepsilon) > 0$ such that $|f(t) - f(+\infty)| < \varepsilon$ for any $t \geq \tau(\varepsilon)$ and $f \in M$.*

Lemma 3.3. *$T : P \to P$ is completely continuous.*

Proof. Firstly, for $u \in P$, from (H_1)–(H_3), it is easy to check that Tu is well defined, and $Tu(t) \geq 0$ for all $t \in [0,+\infty)$. For $t \in [a^*, b^*]$

$$\begin{aligned}
\frac{1}{1+t}(Tu)(t) &= \frac{1}{1+t}\int_0^{+\infty} G(t,s)q(s)f(s,u(s))ds + \frac{1}{1+t}\sum_{k=1}^{p} I_k(u) \\
&\quad + \frac{1}{1+t}\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty} G(\xi_i,s)q(s)f(s,u(s))ds + \sum_{0<t_k<\xi_i} I_k(u)\right] \\
&\geq c^*\left[\int_0^{+\infty}\frac{G(r,s)}{1+r}q(s)f(s,u(s))ds + \frac{1}{1+r}\sum_{k=1}^{p} I_k(u)\right] \\
&\quad + c^*\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty}\frac{G(\xi_i,s)}{1+r}q(s)f(s,u(s))ds + \frac{1}{1+r}\sum_{0<t_k<\xi_i} I_k(u)\right] \\
&\geq c^*\frac{Tu(r)}{1+r}, \quad \text{for } r \in [0,+\infty)
\end{aligned} \tag{3.5}$$

so

$$\min_{t\in[a^*,b^*]}\frac{Tu(t)}{1+t} \geq c^*\|Tu\|, \tag{3.6}$$

which shows $TP \subseteq P$.

Now we prove that T is continuous and compact, respectively. Let $u_n \to u$ as $n \to +\infty$ in P. Then there exists r_0 such that $\sup_{n\in N\setminus\{0\}}\|u_n\| < r_0$. By (H_2) we have $f(t,u)$ is bounded on $[0,+\infty)\times[0,r_0]$. Set $B_0 = \sup\{f(t,u) : (t, u/(1+t)) \in [0,+\infty)\times[0,r_0]\}$, and we have

$$\int_0^{+\infty}\frac{G(t,s)}{1+t}q(s)\left|f(s,u_n)-f(s,u)\right|ds \leq 2B_0\int_0^{+\infty}\frac{G(t,s)}{1+t}q(s)ds. \tag{3.7}$$

Therefore by the Lebesgue dominated convergence theorem and continuity of f and I_k, one arrives at

$$\begin{aligned}
&\|Tu_n - Tu\| \\
&\quad \leq \sup_{t\in[0,+\infty)}\frac{1}{1+t}\left\{\int_0^{+\infty} G(t,s)q(s)\left|f(s,u_n)-f(s,u)\right|ds + \sum_{0<t_k<t}|I_k(u_n)-I_k(u)| + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right. \\
&\qquad \left.\times\left[\int_0^{+\infty} G(\xi_i,s)q(s)\left|f(s,u_n)-f(s,u)\right|ds + \sum_{0<t_k<\xi_i}|I_k(u_n)-I_k(u)|\right]\right\} \\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.
\end{aligned} \tag{3.8}$$

Therefore T is continuous.

Let Ω be any bounded subset of P. Then there exists $r > 0$ such that $\|u\| \le r$ for all $u \in \Omega$. Set $B_1 = \sup\{f(t,u) : (t, u/(1+t)) \in [0,+\infty)\times[0,r]$, $B_{2k} = \sup\{I_k(u) : u/(1+t) \in [0,r]\}$, then

$$\begin{aligned}\|Tu\| &= \sup_{t\in[0,+\infty)} \frac{1}{1+t}\left\{\int_0^{+\infty} G(t,s)q(s)\left|f(s,u)\right|ds + \sum_{0<t_k<t}|I_k(u)|\right.\\ &\left.\quad+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty}G(\xi_i,s)q(s)\left|f(s,u)\right|ds+\sum_{0<t_k<\xi_i}|I_k(u)|\right]\right\}\\ &\le B_1\left[\int_0^{+\infty}G(t,s)q(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty}G(\xi_i,s)q(s)ds\right]\\ &\quad+\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\sum_{k=1}^{p}B_{2k}.\end{aligned} \tag{3.9}$$

So $T\Omega$ is bounded.

Moreover, for any $v \in (0,+\infty)$ and $t', t'' \in (t_k, t_{k+1}] \subset [0,v](t' < t'')$, and $u \in \Omega$, then

$$\begin{aligned}\left|\frac{Tu(t'')}{1+t''}-\frac{Tu(t')}{1+t'}\right| &\le \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[B_1\int_0^{+\infty}G(\xi_i,s)q(s)ds+\sum_{0<t_k<\xi_i}B_{2k}\right]\left|\frac{1}{1+t''}-\frac{1}{1+t'}\right|\\ &\quad+B_1\int_0^{+\infty}\left|\frac{G(t'',s)}{1+t''}-\frac{G(t',s)}{1+t'}\right|q(s)ds+\sum_{0<t_k<t'}B_{2k}\left|\frac{1}{1+t''}-\frac{1}{1+t'}\right|\\ &\longrightarrow 0, \quad \text{uniformly as}\, t' \longrightarrow t''.\end{aligned} \tag{3.10}$$

So $T\Omega$ is quasi-equicontinuous on any compact interval of $[0,+\infty)$.

Finally, we prove for any ε, there exists sufficiently large $N_1 > 0$ such that

$$\left|\frac{Tu(t'')}{1+t''}-\frac{Tu(t')}{1+t'}\right| < \varepsilon, \quad \forall t', t'' \ge N_1,\, u \in \Omega. \tag{3.11}$$

Since $\int_0^{+\infty}G(t,s)q(s)ds < +\infty$, we can choose $N_1 > 0$ such that

$$\begin{aligned}&\frac{\sum_{i=1}^{m-2}\alpha_i}{N_1\left(1-\sum_{i=1}^{m-2}\alpha_i\right)}\left[B_1\int_0^{+\infty}G(\xi_i,s)q(s)ds+\sum_{0<t_k<\xi_i}B_{2k}\right]<\frac{\varepsilon}{6},\\ &\frac{B_1\int_0^{+\infty}G(t,s)q(s)ds}{N_1}<\frac{\varepsilon}{6}, \qquad \sum_{k=1}^{p}\frac{B_{2k}}{N_1}\le\frac{\varepsilon}{6}.\end{aligned} \tag{3.12}$$

For $t', t'' \geq N_1$, it follows that

$$\begin{aligned}
\left|\frac{(Tu)(t')}{1+t'} - \frac{(Tu)(t'')}{1+t''}\right| &\leq \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[B_1\int_0^{+\infty}G(\xi_i,s)q(s)ds + \sum_{0<t_k<\xi_i}B_{2k}\right]\left(\frac{1}{1+t''}+\frac{1}{1+t'}\right) \\
&\quad + B_1\int_0^{+\infty}\frac{G(t',s)}{1+t'}q(s)ds + B_1\int_0^{+\infty}\frac{G(t'',s)}{1+t''}q(s)ds \\
&\quad + \sum_{k=1}^{p}B_{2k}\left(\frac{1}{1+t''}+\frac{1}{1+t'}\right) \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \tag{3.13}$$

That is (3.11) holds. By Lemma 3.2, $T\Omega$ is relatively compact. In sum, $T : P \to P$ is completely continuous. □

4. Existence of Three Positive Solutions

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals γ and θ, and the nonnegative continuous functionals ψ be defined on the cone P by

$$\gamma(u) = \psi(u) = \theta(u) = \sup_{t\in[0,+\infty)}\frac{u(t)}{1+t}, \qquad \alpha(u) = \min_{t\in[a^*,b^*]}\frac{u(t)}{1+t}. \tag{4.1}$$

For notational convenience, we denote by

$$\begin{aligned}
M &= \min_{t\in[a^*,b^*]}\int_0^{+\infty}\frac{G(t,s)}{1+t}q(s)ds, \\
M_1 &= \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty}G(\xi_i,s)q(s)ds.
\end{aligned} \tag{4.2}$$

The main result of this paper is the following.

Theorem 4.1. *Assume* (H_1)–(H_3) *hold. Let* $a_k \geq 0$, $0 < a < b/c^* < c = d$, $b/M < c^*d/2(M+M_1)$ *and suppose that* f, I_k *satisfy the following conditions:*

(A_1) $f(t,u) \leq c^*d/2(M+M_1)$, $I_k(u) \leq dc^*/2M_2$ for $(t, u/(1+t)) \in [0,+\infty)\times[0,d]$,

(A_2) $f(t,u) > b/M$ *for* $(t, u/(1+t)) \in [a^*,b^*]\times[b,c]$,

(A_3) $f(t,u) < c^*a/2(M+M_1)$, $I_k(u) \leq ac^*a_k/2M_2$ for $t \in (t, u/(1+t)) \in [0,+\infty)\times[0,a]$, *where* $M_2 = \sum_{k=1}^{p} a_k / (1 - \sum_{i=1}^{m-2} \alpha_i)$. *Then* (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\gamma(u_i) \le d, \quad \text{for } i = 1,2,3, \qquad \psi(u_1) < a, \quad a < \psi(u_2) \quad \text{with } \alpha(u_2) < b,\ \alpha(u_3) > b. \tag{4.3}$$

Proof.
Step 1. From the definition α, ψ, and γ, we easily show that

$$\alpha(u) \le \psi(u), \quad \|u\| \le \gamma(u) \quad \text{for } u \in \overline{P(\gamma, d)}. \tag{4.4}$$

Next we will show that

$$T : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)}. \tag{4.5}$$

In fact, for $u \in \overline{P(\gamma, d)}$, then

$$\sup_{t \in [0,+\infty)} \frac{u(t)}{1+t} \le d. \tag{4.6}$$

From condition (A_1), we obtain

$$f(t,u) \le \frac{dc^*}{2(M + M_1)}, \qquad I_k(u) \le \frac{dc^*}{2M_2}. \tag{4.7}$$

It follows that

$$\begin{aligned}
\gamma(Tu) &= \sup_{t \in [0,+\infty)} \frac{Tu(t)}{1+t} \le \frac{1}{c^*} \min_{t \in [a^*,b^*]} \frac{Tu(t)}{1+t} \\
&\le \frac{1}{c^*} \min_{t \in [a^*,b^*]} \left[\frac{1}{1+t} \int_0^{+\infty} G(t,s) q(s) f(s,u)\, ds + \frac{1}{1+t} \sum_{k=1}^{p} I_k(u) \right. \\
&\qquad \left. + \frac{1}{1+t} \cdot \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \left(\int_0^{+\infty} G(\xi_i, s) q(s) f(s,u)\, ds + \sum_{0 < t_k < \xi_i} I_k(u) \right) \right] \\
&\le \frac{1}{c^*} \cdot \frac{c^* d}{2(M + M_1)} \left[\min_{t \in [a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s)\, ds + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^{+\infty} G(\xi_i, s) q(s)\, ds \right] \\
&\quad + \frac{1}{c^*} \cdot \frac{c^* d \sum_{k=1}^{p} a_k}{2M_2} \left[1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \right] \\
&\le \frac{d}{2} + \frac{d}{2} = d.
\end{aligned} \tag{4.8}$$

Thus (4.5) holds.

Step 2. We show that condition (i) in Theorem 2.2 holds. Taking $u(t) = (1+t)((b+d)/2)$, then $u \in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha(u) > b$, which shows $\{u \in P(\gamma,\theta,\alpha,b,c,d) \mid \alpha(u) > b\} \neq \emptyset$. Thus for $u \in P(\gamma,\theta,\alpha,b,c,d)$, there is

$$b \leq \frac{u(t)}{1+t} \leq \quad \text{for } t \in [a^*,b^*]. \tag{4.9}$$

Hence by (A_2), we have

$$\begin{aligned}\alpha(Tu) &= \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t}\\ &\geq \min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s) f(s,u) ds\\ &> \frac{b}{M} \cdot \min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s) ds = b.\end{aligned} \tag{4.10}$$

Therefore we have

$$\alpha(Tu) > b, \quad \forall u \in P(\gamma,\theta,\alpha,b,c,d). \tag{4.11}$$

This shows the condition (i) in Theorem 2.2 is satisfied.
Step 3. We now prove (ii) in Theorem 2.2 holds. For $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$, we have

$$\alpha(Tu) = \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t} \geq c^* \|Tu\| = c^*\theta(Tu) > c^*c > b. \tag{4.12}$$

Hence, condition (ii) in Theorem 2.2 is satisfied.
Step 4. Finally, we prove (iii) in Theorem 2.2 is satisfied. Since $\psi(0) = 0 < a$, so $0 \notin R(\gamma,\psi,a,d)$. Suppose that $u \in R(\gamma,\theta,a,d)$ with $\psi(u) = a$, then

$$0 \leq \frac{u(t)}{1+t} \leq a, \tag{4.13}$$

by the condition (A_3) of this theorem,

$$\begin{aligned}\psi(Tu) &= \sup_{t\in[0,+\infty)} \frac{Tu(t)}{1+t} \leq \frac{1}{c^*} \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t}\\ &\leq \frac{1}{c^*} \cdot \frac{c^*a}{2(M+M_1)} \left[\min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s) ds + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i} \int_0^{+\infty} G(\xi_i,s) q(s) ds \right]\\ &\quad + \frac{1}{c^*} \cdot \frac{c^*a\sum_{k=1}^{p} a_k}{2M_2} \left[1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i} \right]\\ &\leq \frac{a}{2} + \frac{a}{2} = a.\end{aligned} \tag{4.14}$$

Thus condition (iii) in Theorem 2.2 holds. Therefore an application of Theorem 2.2 implies the boundary value problem (1.1) has at least three positive solutions such that

$$\sup_{t\in[0,+\infty)} \frac{u_i(t)}{1+t} \le d, \quad i = 1,2,3,$$

$$\sup_{t\in[0,+\infty)} \frac{u_1(t)}{1+t} < a, \quad a < \sup_{t\in[0,+\infty)} \frac{u_2(t)}{1+t} \quad \text{with} \min_{t\in[a^*,b^*]} \frac{u_2(t)}{1+t} < b, \tag{4.15}$$

$$\min_{t\in[a^*,b^*]} \frac{u_3(t)}{1+t} > b.$$

□

5. An Example

Now we consider the following boundary value problem

$$\begin{gathered}
u''(t) + q(t)f(t,u) = 0, \quad 0 < t < +\infty, t \neq t_1, \\
\Delta u(t_1) = I_1(u(t_1)), \quad t_1 = 1, \\
u(0) = \frac{1}{4}u\left(\frac{1}{4}\right) + \frac{1}{4}u(4), \qquad u'(\infty) = 0 \\
f(t,u) = \begin{cases} \dfrac{1}{100}e^{-t} + 4\left(\dfrac{u}{1+t}\right)^7, & \dfrac{u}{1+t} \le 1, \\ \dfrac{1}{100}e^{-t} + 4, & \dfrac{u}{1+t} \ge 1, \end{cases} \\
I_1(u) = \begin{cases} \dfrac{1}{2}\left(\dfrac{u}{1+t}\right)^4, & \dfrac{u}{1+t} \le 1, \\ \dfrac{1}{2}, & \dfrac{u}{1+t} \ge 1, \end{cases}
\end{gathered} \tag{5.1}$$

$q(t) = e^{-t}$. Choose $a_1 = 1/2$, $a = 1/2$, $b = 3/4$, $c = d = 48$. If taking $a^* = 2$, $b^* = 3$, then $c^* = 1/4$, and $M = (1 - e^{-3})/4$, $M_1 = 2 - e^{-1/4} - e^{-4}$, $M_2 = 1$. Consequently, $f(t,u)$, $I_k(u)$ satisfies the following:

(1) $f(t,u) \le 1/100+4 < c^*d/2(M+M_1)$, $I_1(u) \le 1/2 < 3 = c^*a_1d/2M_2$, for $(t, u/(1+t)) \in [0,+\infty) \times [0,48]$;

(2) $f(t,u) > 4 > b/M$, for $(t, u/(1+t)) \in [2,3] \times [3/4,48]$;

(3) $f(t,u) < 1/100 + (1/2)^5 \le c^*a/2(M + M_1)$, $I_1(u) \le 1/32 = c^*a_1a/2M_2$, for $(t, u/(1 + t)) \in [0,+\infty) \times [0,1/2]$.

Then all conditions of Theorem 4.1 hold, so by Theorem 4.1, boundary value problem (5.1) has at least three positive solutions.

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