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Research Article

Entire Solutions for a Quasilinear Problem in the Presence of Sublinear and Super-Linear Terms

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We establish new results concerning existence and asymptotic behavior of entire, positive, and bounded solutions which converge to zero at infinite for the quasilinear equation $-\Delta_p u = a(x)f(u)+\lambda b(x)g(u),\ x\in\mathbb{R}^N,\ 1<p<N$, where $f,g:[0,\infty)\to[0,\infty)$ are suitable functions and $a(x),b(x)\geq 0$ are not identically zero continuous functions. We show that there exists at least one solution for the above-mentioned problem for each $0\leq\lambda<\lambda_\star$, for some $\lambda_\star>0$. Penalty arguments, variational principles, lower-upper solutions, and an approximation procedure will be explored.

1. Introduction

In this paper we establish new results concerning existence and behavior at infinity of solutions for the nonlinear quasilinear problem

$$\begin{gathered} -\Delta_p u = a(x)f(u)+\lambda b(x)g(u) \quad \text{in } \mathbb{R}^N, \\ u>0 \quad \text{in } \mathbb{R}^N, \quad u(x)\longrightarrow 0, \quad |x|\longrightarrow\infty, \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $1<p<N$, denotes the p-Laplacian operator; $a,b:\mathbb{R}^N\to[0,\infty)$ and $f,g:[0,\infty)\to[0,\infty)$ are continuous functions not identically zero and $\lambda\geq 0$ is a real parameter.

A solution of (1.1) is meant as a positive function $u\in C^1(\mathbb{R}^N)$ with $u(x)\to 0$ as $|x|\to\infty$ and

$$\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi dx = \int_{\mathbb{R}^N}\left(a(x)f(u)+\lambda b(x)g(u)\right)\varphi\, dx, \quad \forall\varphi\in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.2}$$

The class of problems (1.1) appears in many nonlinear phenomena, for instance, in the theory of quasiregular and quasiconformal mappings [1–3], in the generalized reaction-diffusion theory [4], in the turbulent flow of a gas in porous medium and in the non-Newtonian fluid theory [5]. In the non-Newtonian fluid theory, the quantity p is the characteristic of the medium. If $p < 2$, the fluids are called pseudoplastics; if $p = 2$ Newtonian and if $p > 2$ the fluids are called dilatants.

It follows by the nonnegativity of functions a, b, f, g of parameter λ and a strong maximum principle that all non-negative and nontrivial solutions of (1.1) must be strictly positive (see Serrin and Zou [6]). So, again of [6], it follows that (1.1) admits one solution if and only if $p < N$.

The main objective of this paper is to improve the principal result of Yang and Xu [7] and to complement other works (see, e.g., [8–20] and references therein) for more general nonlinearities in the terms f and g which include the cases considered by them.

The principal theorem in [7] considered, in problem (1.1), $f(u) = u^m$, $u > 0$, and $g(u) = u^n$, $u > 0$ with $0 < m < p - 1 < n$. Another important fact is that, in our result, we consider different coefficients, while in [7] problem (1.1) was studied with $a(x) = b(x)$, $\forall x \in \mathbb{R}^N$.

In order to establish our results some notations will be introduced. We set

$$\begin{aligned} \widetilde{a}(r) &:= \min_{|x|=r} a(x), \quad \widetilde{b}(r) := \min_{|x|=r} b(x), \quad r \geq 0, \\ \widehat{a}(r) &:= \max_{|x|=r} a(x), \quad \widehat{b}(r) := \max_{|x|=r} b(x), \quad r \geq 0. \end{aligned} \tag{1.3}$$

Additionally, we consider

(H_1) (i) $\lim_{s \to 0}(f(s)/s^{p-1}) = \infty$,

(ii) $\lim_{s \to 0}(f(s)/s^{p-1}) = 0$,

(H_2) (i) $\lim_{s \to 0}(g(s)/s^{p-1}) = 0$,

(ii) $\lim_{s \to 0}(g(s)/s^{p-1}) = \infty$.

Concerning the coefficients a and b,

(H_3) (i) $\int_1^\infty r^{1/(p-1)} \widehat{a}^{1/(p-1)}(r)dr$, $\int_1^\infty r^{1/(p-1)} \widehat{b}^{1/(p-1)}(r)dr < \infty$, if $1 < p \leq 2$,

(ii) $\int_1^\infty r^{((p-2)N+1)/(p-1)} \widehat{a}(r)dr$, $\int_1^\infty r^{((p-2)N+1)/(p-1)} \widehat{b}(r)dr < \infty$, if $p \geq 2$.

Our results will be established below under the hypothesis $N \geq 3$.

Theorem 1.1. *Consider* (H_1)–(H_3), *then there exists one* $\lambda_\star > 0$ *such that for each* $0 \leq \lambda < \lambda_\star$ *there exists at least one* $u = u_\lambda \in C^1(\mathbb{R}^N)$ *solution of problem* (1.1). *Moreover,*

$$C|x|^{-(N-p)/(p-1)} \leq u(x), \quad x \in \mathbb{R}^N, \ |x| \geq 1 \tag{1.4}$$

for some constant $C = C(\lambda) > 0$. *If additionally*

$$\frac{f(t)}{t^{p-1}} \text{ is nonincreasing and } \frac{g(t)}{t^{p-1}} \text{ is nondecreasing for } t > 0, \tag{1.5}$$

then there is a positive constant $D = D(\lambda_\star)$ such that

$$u^2(x) f\left(\frac{u(x)}{4}\right)^{-1/(p-1)} \le D \int_{|x|}^{\infty} \left[t^{1-N} \int_0^t (\widetilde{a}(s) + \widetilde{b}(s)) ds \right]^{1/(p-1)} dt, \quad x \in \mathbb{R}^N. \tag{1.6}$$

Remark 1.2. If we assume (1.5) with $f(t) = t^m,\ t > 0$, where $0 \le m < p-1$, then (1.6) becomes

$$0 < u(x) \le C \left(\int_{|x|}^{\infty} \left[t^{1-N} \int_0^t (\widetilde{a}(s) + \widetilde{b}(s)) ds \right]^{1/(p-1)} dt \right)^{1/(2-m/(p-1))}, \quad x \in \mathbb{R}^N. \tag{1.7}$$

In the sequel, we will establish some results concerning to quasilinear problems which are relevant in itself and will play a key role in the proof of Theorem 1.1.

We begin with the problem of finding classical solutions for the differential inequality

$$\begin{gathered} -\Delta_p v \ge a(x) f(v) + \lambda b(x) g(v) \quad \text{in } \mathbb{R}^N, \\ v > 0 \quad \text{in } \mathbb{R}^N, \quad v(x) \longrightarrow 0, \quad |x| \longrightarrow \infty. \end{gathered} \tag{1.8}$$

Our result is.

Theorem 1.3. *Consider (H_1)–(H_3), then there exists one $\lambda_\star > 0$ such that problem (1.8) admits, for each $0 \le \lambda < \lambda_\star$, at least one radially symmetric solution $v = v_\lambda \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^{1,\nu}_{\text{loc}}(\mathbb{R}^N)$, for some $\nu \in (0,1)$. Moreover, if in additionally one assumes (1.5), then there is a positive constant $D = D(\lambda_\star)$ such that*

$$v^2(x) f\left(\frac{v(x)}{4}\right)^{-1/(p-1)} \le D \int_{|x|}^{\infty} \left[t^{1-N} \int_0^t (\widetilde{a}(s) + \widetilde{b}(s)) ds \right]^{1/(p-1)} dt, \quad x \in \mathbb{R}^N. \tag{1.9}$$

Remark 1.4. Theorems 1.1 and 1.3 are still true with $N = 2$ if (H_3) hypothesis is replaced by

$(H_3)'$ $\int_1^\infty [t^{1-N} \int_0^t (\widetilde{a}(s) + \widetilde{b}(s)) ds]^{1/(p-1)} dt < \infty$.

In fact, (H_3) implies $(H_3)'$, if $N \ge 3$. (see sketch of the proof in the appendix).

Remark 1.5. In Theorem 1.3, it is not necessary to assume that f and g are continuous up to 0. It is sufficient to know that $f, g : (0,\infty) \to (0,\infty)$ are continuous. This includes terms f, g singular in 0.

The next result improves one result of Goncalves and Santos [21] because it guarantees the existence of radially symmetric solutions in $C^2(B(0,R) \setminus \{0\}) \cap C^1(B(0,R)) \cap C(\overline{B(0,R)})$ for the problem

$$\begin{gathered} -\Delta_p u = \rho(x) h(u) \quad \text{in } B(0,R), \\ u > 0 \quad \text{in } B(0,R), \quad u(x) = 0, \quad x \in \partial B(0,R), \end{gathered} \tag{1.10}$$

where $\rho : B(0,R) \to [0,\infty)$, $h : (0,\infty) \to (0,\infty)$ are continuous and suitable functions and $B(0,R) \subset \mathbb{R}^N$ is the ball in $\mathbb{R}^N$ centered in the origin with radius $R > 0$.

Theorem 1.6. *Assume $\rho(x) = \tilde{\rho}(|x|)$, $x \in \mathbb{R}^N$ where $\tilde{\rho} : [0,\infty) \to [0,\infty)$, with $\tilde{\rho} \neq 0$, is continuous. Suppose that h satisfies (H_1) and additionally*

$$\frac{h(s)}{s^{p-1}}, \quad s > 0 \text{ is nonincreasing}. \tag{1.11}$$

then (1.10) admits at least one radially symmetric solution $u \in C^2(B(0,R) \setminus \{0\}) \cap C^1(B(0,R)) \cap C(\overline{B(0,R)})$. Besides this, $u(x) = \tilde{u}(|x|)$, $x \in B(0,R)$, and $\tilde{u}$ satisfies

$$\tilde{u}(r) = \tilde{u}(0) - \int_0^r \left[t^{1-N} \int_0^t s^{N-1} \tilde{\rho}(s) h(\tilde{u}(s)) ds \right]^{1/(p-1)} dt, \quad r \geq 0. \tag{1.12}$$

The proof of principal theorem (Theorem 1.1) relies mainly on the technics of lower and upper solutions. First, we will prove Theorem 1.3 by defining several auxiliary functions until we get appropriate conditions to define one positive number $\lambda_\star$ and a particular upper solution of (1.1) for each $0 \leq \lambda < \lambda_\star$.

After this, we will prove Theorem 1.6, motivated by arguments in [21], which will permit us to get a lower solution for (1.1). Finally, we will obtain a solution of (1.1) applying the lemma below due to Yin and Yang [22].

Lemma 1.7. *Suppose that $f(x,r)$ is defined on $\mathbb{R}^{N+1}$ and is locally Hölder continuous (with $\gamma \in (0,1)$) in x. Assume also that there exist functions $w, v \in C^{1,\gamma}_{\text{loc}}(\mathbb{R}^N)$ such that*

$$\begin{gathered} -\Delta_p v \geq f(x,v), \quad x \in \mathbb{R}^{\mathbb{N}}, \\ -\Delta_p w \leq f(x,w), \quad x \in \mathbb{R}^{\mathbb{N}}, \\ w(x) \leq v(x), \quad x \in \mathbb{R}^{\mathbb{N}}, \end{gathered} \tag{1.13}$$

and $f(x,r)$ is locally Lipschitz continuous in r on the set

$$\left\{ (x,r) / x \in \mathbb{R}^{\mathbb{N}},\ w(x) \leq r \leq v(x) \right\}. \tag{1.14}$$

Then there exists $u \in C^1(\mathbb{R}^{\mathbb{N}})$ with $w(x) \leq u(x) \leq v(x)$, $x \in \mathbb{R}^{\mathbb{N}}$ satisfying

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f(x,u) \varphi dx, \quad \forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.15}$$

In the two next sections we will prove Theorems 1.3 and 1.6.

2. Proof of Theorem (1.4)

First, inspired by Zhang [20] and Santos [16], we will define functions $F : (0,\infty) \to (0,\infty)$ and $G : (0,\infty)\times(0,\infty) \to (0,\infty)$ by

$$F(s) = \sup_{t\geq s}\frac{f(t)}{t^{p-1}}, \quad s>0, \qquad G(\tau,s) = \begin{cases} \sup\limits_{s\leq t\leq\tau}\dfrac{g(t)}{t^{p-1}}, & s\leq\tau, \\ \dfrac{g(\tau)}{\tau^{p-1}}, & s\geq\tau. \end{cases} \tag{2.1}$$

So, for each $\lambda\geq 0$, let $F_\lambda : (0,\infty)\times(0,\infty) \to (0,\infty)$ given by

$$F_\lambda(\tau,s) = F_0(s) + \lambda F(\tau,s), \tag{2.2}$$

where

$$F_0(s) = s^{p-1}F(s), \quad s>0, \qquad F(\tau,s) = s^{p-1}G(\tau,s), \quad \tau,s>0. \tag{2.3}$$

It is easy to check that

$$F_0(s)\geq f(s), \quad s>0, \ \text{for each } \tau>0, \qquad F(\tau,s)\geq g(s), \quad 0<s\leq\tau \tag{2.4}$$

and, as a consequence,

$$F_\lambda(\tau,s)\geq f(s)+\lambda g(s), \quad 0<s\leq\tau. \tag{2.5}$$

Moreover, it is also easy to verify.

Lemma 2.1. *Suppose that* (H_1) *and* (H_2) *hold. Then, for each* $\tau>0$*,*

(i) $F(\tau,s)/s^{p-1}$, $s>0$ *is non-increasing,*

(ii) $F_0(s)/s^{p-1}$, $s>0$ *is non-increasing,*

(iii) $\lim_{s\to 0}(F(\tau,s)/s^{p-1}) = \sup_{0<t\leq\tau}(g(t)/t^{p-1})$,

(iv) $\lim_{s\to 0}(F_0(s)/s^{p-1}) = \infty$,

(v) $\lim_{s\to\infty}(F(\tau,s)/s^{p-1}) = g(\tau)/\tau^{p-1}$,

(vi) $\lim_{s\to\infty}(F_0(s)/s^{p-1}) = 0$.

By Lemma 2.1(iii), (iv), and (2.2), the function $\widetilde{F}_\lambda : (0,\infty)\times(0,\infty) \to (0,\infty)$, given by

$$\widetilde{F}_\lambda(\tau,s) = \frac{s^2}{\int_0^s \left(t/F_\lambda(\tau,t)^{1/(p-1)}\right)dt}, \tag{2.6}$$

is well defined and continuous. Again, by using Lemma 2.1(i) and (ii),

$$\widetilde{F}_\lambda(\tau,s)\geq F_\lambda(\tau,s)^{1/(p-1)}, \quad \forall\tau,s>0. \tag{2.7}$$

Besides this, $\widetilde{F}_\lambda(\tau,\cdot) \in C^1(0,\infty)$, for each $\tau > 0$, and using Lemma 2.1, it follows that $\widetilde{F}_\lambda$ satisfies, for each $\lambda \geq 0$, the following.

Lemma 2.2. *Suppose that* (H_1) *and* (H_2) *hold. Then, for each* $\tau > 0$,

(i) $\widetilde{F}_\lambda(\tau,s)/s$ *is non-increasing in* $s > 0$,

(ii) $\lim_{s\to 0}(\widetilde{F}_\lambda(\tau,s)/s) = \infty$,

(iii) $\lim_{s\to 0}(\widetilde{F}_\lambda(\tau,s)/s) = [\lambda(g(\tau)/\tau^{p-1})]^{1/(p-1)}$, *if* $\lambda > 0$,

(iv) $\lim_{s\to 0}(\widetilde{F}_\lambda(\tau,s)/s) = 0$, *if* $\lambda = 0$.

And, in relation to λ, we have the folowing.

Lemma 2.3. *Suppose that* (H_1) *and* (H_2) *hold. Then, for each* $\tau, s > 0$,

(i) $\widetilde{F}_{\lambda_1}(\tau,s) < \widetilde{F}_{\lambda_2}(\tau,s)$, *if* $\lambda_1 < \lambda_2$,

(ii) $\widetilde{F}_\lambda(\tau,s)/s \to F_0(s)/s$, *as* $\lambda \to 0$.

Finally, we will define, for each $\lambda \geq 0$, $H_\lambda : (0,\infty) \to (0,\infty)$, by

$$H_\lambda(\tau) = \frac{1}{\tau}\int_0^\tau \frac{t}{\widetilde{F}_\lambda(\tau,t)}dt. \tag{2.8}$$

So, H_λ is a continuous function and we have (see proof in the appendix).

Lemma 2.4. *Suppose that* (H_1) *and* (H_2) *hold. Then,*

(i) $\lim_{\tau\to 0} H_\lambda(\tau) = 0$, *for any* $\lambda \geq 0$,

(ii) $\lim_{\tau\to\infty} H_\lambda(\tau) = \infty$, *if* $\lambda = 0$,

(iii) $\lim_{\tau\to\infty} H_\lambda(\tau) = 0$, *if* $\lambda > 0$,

(iv) $H_{\lambda_1}(\tau,s) > H_{\lambda_2}(\tau,s)$, *if* $\lambda_1 < \lambda_2$,

(v) $\lim_{\lambda\to 0} H_\lambda(\tau) = H_0(\tau)$, *for each* $\tau > 0$.

By Lemma 2.4(ii), there exists a $\tau_\infty > 0$ such that $H_0(\tau_\infty) > \alpha + 1$, where by either (H_3) or $(H_3)'$, we have

$$0 < \alpha := \int_0^\infty \left[t^{1-N}\int_0^t (\widetilde{a}(s) + \widetilde{b}(s))ds\right]^{1/(p-1)} dt < \infty. \tag{2.9}$$

So, by Lemma 2.4(v), there exists a $\lambda^\star > 0$ such that $H_{\lambda^\star}(\tau_\infty) > \alpha$. That is,

$$\frac{1}{\tau_\infty}\int_0^{\tau_\infty} \frac{t}{\widetilde{F}_{\lambda^\star}(\tau_\infty,t)}dt > \alpha. \tag{2.10}$$

Let $P : (0,\infty) \times [0,\tau_\infty] \to \mathbb{R}^N$ by

$$P(t,s) = \widetilde{\omega}(t) - \frac{1}{\tau_\infty}\int_0^s \frac{\varsigma}{\widetilde{F}_{\lambda^\star}(\tau_\infty,\varsigma)}d\varsigma, \tag{2.11}$$

where $\tilde{\omega} : (0,\infty) \to (0,\infty)$, $\tilde{\omega} \in C^2((0,\infty)) \cap C^1([0,\infty))$ is given by $\omega(x) = \tilde{\omega}(|x|)$, $x \in \mathbb{R}^N$ where $\omega \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^1(\mathbb{R}^N)$ is the unique positive and radially symmetric solution of problem

$$\begin{gathered} -\Delta_p \omega = \tilde{a}(|x|) + \tilde{b}(|x|) \quad \text{in } \mathbb{R}^N, \\ \omega > 0 \quad \text{in } \mathbb{R}^N, \quad \omega(x) \longrightarrow 0, \quad |x| \longrightarrow \infty. \end{gathered} \tag{2.12}$$

More specifically, by DiBenedetto [23], $\omega \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^{1,\nu}_{\text{loc}}(\mathbb{R}^N)$, for some $\nu \in (0,1)$. In fact, $\tilde{\omega}$ satisfies

$$\tilde{\omega}(r) = \alpha - \int_0^r \left[t^{1-N} \int_0^t (\tilde{a}(s) + \tilde{b}(s)) ds \right]^{1/(p-1)} dt, \quad r \geq 0. \tag{2.13}$$

So, by (2.10), (2.11), and (2.13), we have for each $t > 0$,

$$P(t,0) = \tilde{\omega}(t) > 0, \qquad P(t,\tau_\infty) < \alpha - \frac{1}{\tau_\infty} \int_0^{\tau_\infty} \frac{t}{\tilde{F}_{\lambda^*}(\tau_\infty, t)} dt < 0. \tag{2.14}$$

Hence, after some pattern calculations, we show that there is a $\vartheta \in C^2((0,\infty)) \cap C^1([0,\infty))$ such that $\vartheta(r) \leq \tau_\infty, r \geq 0$ and

$$\tilde{\omega}(r) = \frac{1}{\tau_\infty} \int_0^{\vartheta(r)} \frac{t}{\tilde{F}_{\lambda^*}(\tau_\infty, t)} dt, \quad r \geq 0. \tag{2.15}$$

As consequences of (2.9), (2.13) and (2.15), we have $\vartheta(r) \to 0$, $r \to \infty$ and

$$\begin{aligned} \left(r^{N-1} |\tilde{\omega}'(r)|^{p-1} \tilde{\omega}'(r) \right)' = {} & \frac{1}{\tau_\infty^{p-1}} \left(\frac{\vartheta(r)}{\tilde{F}_{\lambda^*}(\tau_\infty, \vartheta(r))} \right)^{p-1} \left(r^{N-1} |\vartheta'(r)|^{p-1} \vartheta'(r) \right)' \\ & + \frac{p-1}{\tau_\infty^{p-1}} \left(\frac{\vartheta(r)}{\tilde{F}_{\lambda^*}(\tau_\infty, \vartheta(r))} \right)^{p-2} \frac{d}{ds} \left(\frac{s}{\tilde{F}_{\lambda^*}(\tau_\infty, s)} \right) r^{N-1} |\vartheta'(r)|^p \end{aligned} \tag{2.16}$$

and hence, by Lemma 2.2 (i), (2.7) and $\vartheta(r) \leq \tau_\infty$, $r \geq 0$, we obtain

$$\begin{aligned} -\left(r^{N-1} |\vartheta'(r)|^{p-1} \vartheta'(r) \right)' & \geq \left(\frac{\tau_\infty}{\vartheta(r)} \right)^{p-1} \tilde{F}_{\lambda^*}(\tau_\infty, \vartheta(r))^{p-1} \left[-\left(r^{N-1} |\tilde{\omega}'(r)|^{p-1} \tilde{\omega}'(r) \right)' \right] \\ & = r^{N-1} F_{\lambda^*}(\tau_\infty, \vartheta(r)) \left(\tilde{a}(r) + \tilde{b}(r) \right), \end{aligned} \tag{2.17}$$

that is, by using (2.2), we have

$$-\left(r^{N-1}\left|\vartheta'(r)\right|^{p-1}\vartheta'(r)\right)' \geq r^{N-1}\widetilde{a}(r)F_0(\vartheta(r)) + \lambda^{\star}r^{N-1}\widetilde{b}(r)F(\tau_\infty,\vartheta(r)), \quad r \geq 0. \tag{2.18}$$

In particular, making $v(x) = \vartheta(|x|)$, $x \in \mathbb{R}^N$, we get from (2.15), Lemma 2.2(i) and $\omega \in C^2(\mathbb{R}^N\setminus\{0\})\cap C^{1,\nu}_{\text{loc}}(\mathbb{R}^N)$ that $v \in C^2(\mathbb{R}^N\setminus\{0\})\cap C^{1,\nu}_{\text{loc}}(\mathbb{R}^N)$ and satisfies (1.8), for each $0 \leq \lambda \leq \lambda^{\star}$. That is, v is an upper solution to (1.1).

To prove (1.9), first we observe, using Lemma 2.2(i) and (2.15), that

$$\begin{aligned}\widetilde{\omega}(r) &\geq \frac{1}{\tau_\infty}\int_0^{\vartheta(r)/2}\frac{t}{F_{\lambda_\star}(\tau_\infty,t)^{1/(p-1)}}dt \geq \frac{1}{\tau_\infty}\int_{\vartheta(r)/4}^{\vartheta(r)/2}\frac{t}{F_{\lambda_\star}(\tau_\infty,t)^{1/(p-1)}}dt \\ &\geq \frac{1}{\tau_\infty}\left[\frac{1}{F(\vartheta(r)/4)+\lambda_\star G(\tau_\infty,\vartheta(r)/4)}\right]^{1/(p-1)}(\vartheta(r)/4), \quad r \geq 0.\end{aligned} \tag{2.19}$$

So, by definition of F, $G(\tau_\infty,\cdot)$ and hypothesis (1.5), we have

$$F\left(\frac{\vartheta(r)}{4}\right)+\lambda_\star G\left(\tau_\infty,\frac{\vartheta(r)}{4}\right) = \frac{f(\vartheta(r)/4)}{(\vartheta(r)/4)^{p-1}}+\lambda_\star\frac{g(\tau_\infty)}{\tau_\infty^{p-1}}, \quad r \geq 0. \tag{2.20}$$

Thus,

$$\frac{(\vartheta(r)/4)^{2(p-1)}}{f(\vartheta(r)/4)} \leq \tau_\infty^{p-1}\left[1+\lambda_\star\frac{g(\tau_\infty)}{\tau_\infty^{p-1}}\frac{(\vartheta(r)/4)^{(p-1)}}{f(\vartheta(r)/4)}\right]\widetilde{\omega}(r)^{p-1}, \quad r \geq 0. \tag{2.21}$$

Recalling that $\vartheta(r) \leq \tau_\infty, r \geq 0$ and using (1.5) again, we obtain

$$\vartheta(r)^2\left[f\left(\frac{\vartheta(r)}{4}\right)\right]^{-1/(p-1)} \leq 16\tau_\infty\left[1+\lambda_\star\frac{g(\tau_\infty)}{\tau_\infty^{p-1}}\frac{(\tau_\infty/4)^{(p-1)}}{f(\tau_\infty/4)}\right]^{1/(p-1)}\widetilde{\omega}(r), \quad r \geq 0. \tag{2.22}$$

Thus by (2.9), (2.13), and $v(x) = \vartheta(r)$, $r = |x|$, for all $x \in \mathbb{R}^N$, there is one positive constant $D = D(\lambda_\star)$ such that (1.9) holds. This ends the proof of Theorem 1.3.

3. Proof of Theorem (1.5)

To prove Theorem (1.5), we will first show the existence of a solution, say $u_k \in C^2(B(0,R)\setminus\{0\})\cap C^1(B(0,R))\cap C(\overline{B(0,R)})$, for each $k = 1,2,\ldots$, for the auxiliary problem

$$\begin{gathered}-\Delta_p u = \rho(x)h_k(u) \quad \text{in } B(0,R), \\ u > 0 \quad \text{in } B(0,R), \quad u(x) = 0, \quad x \in \partial B(0,R),\end{gathered} \tag{3.1}$$

where $h_k(s) = h(s+1/k)$, $s \geq 0$. In next, to get a solution for problem (1.10), we will use a limit process in k.

For this purpose, we observe that

(i) $\liminf_{s\to 0} h_k(s) = h(1/\mathrm{k}) > 0$,

(ii) $\lim_{s\to\infty}(h_k(s)/s^{p-1}) = \lim_{s\to\infty}(h(s+1/k)/(s+1/k)^{p-1})(1+1/ks)^{p-1} = 0$, by (H_1) and by (1.11), it follows that

(iii) $h_k(s)/s^{p-1} = (h(s+1/k)/(s+1/k)^{p-1})(1+1/ks)^{p-1},\ s > 0$ is non-increasing, for each $k+1, 2, \ldots$.

By items (i)–(iii) above, ρ and h_k fulfill the assumptions of Theorem 1.3 in [21]. Thus (3.1) admits one solution $u_k \in C^2(B(0,R)\setminus\{0\}) \cap C^1(B(0,R)) \cap C(\overline{B(0,R)})$, for each $k = 1, 2, \ldots$. Moreover, $u_k(x) = \widetilde{u}_k(|x|),\ x \in \mathbb{R}^N$ with $\widetilde{u}_k \in C^2((0,R)) \cap C^1([0,R)) \cap C([0,R])$ satisfying

$$\widetilde{u}_k(r) = \widetilde{u}_k(0) - \int_0^r \left[t^{1-N}\int_0^t s^{N-1}\widetilde{\rho}(s)h(\widetilde{u}_k(s)+1/k)ds\right]^{1/(p-1)} dt, \quad 0 \le r \le R. \tag{3.2}$$

Adapting the arguments of the proof of Theorem 1.3 in [21], we show

$$c\varphi_1(r) \le \widetilde{u}_{k+1}(r) + \frac{1}{k+1} \le \widetilde{u}_k(r) + \frac{1}{k}, \quad 0 \le r \le R, \tag{3.3}$$

where $\varphi_1 \in C^2(\overline{B(0,R)})$ is the positive first eigenfunction of problem

$$\begin{gathered} -\left(r^{N-1}|\varphi'|^{p-2}\varphi'\right)' = \lambda r^{N-1}\widetilde{\rho}(r)|\varphi|^{p-2}\varphi \quad \text{in } B(0,R), \\ \varphi = 0 \quad \text{on } \partial B(0,R), \end{gathered} \tag{3.4}$$

and $c > 0$, independent of k, is chosen (using (H_1)) such that

$$\frac{h(c\|\varphi_1\|_\infty)}{(c\|\varphi_1\|_\infty)^{p-1}} > \lambda_1, \tag{3.5}$$

with $\lambda_1 > 0$ denoting the first eigenvalue of problem (3.4) associated to the φ_1.

Hence, by (3.3),

$$\widetilde{u}_k(r) \longrightarrow \widetilde{u}(r) \quad \text{with } c\varphi_1(r) \le \widetilde{u}(r) \le |\widetilde{u}_1|_\infty + 1,\ 0 \le r \le R. \tag{3.6}$$

Using (H_1), (3.3), the above convergence and Lebesgue's theorem, we have, making $k \to \infty$ in (3.2), that

$$\widetilde{u}(r) = \widetilde{u}(0) - \int_0^r \left[t^{1-N}\int_0^t s^{N-1}\widetilde{\rho}(s)h(\widetilde{u}(s))ds\right]^{1/(p-1)} dt, \quad 0 \le r \le R. \tag{3.7}$$

So, making $u(x) = \widetilde{u}(|x|),\ x \in \mathbb{R}^N$, after some calculations, we obtain that $u \in C^2(B(0,R)\setminus\{0\}) \cap C^1(B(0,R)) \cap C(\overline{B(0,R)})$. This completes the proof of Theorem 1.6.

4. Proof of Main Result: Theorem 1.1

To complete the proof of Theorem 1.1, we will first obtain a classical and positive lower solution for problem (1.1), say w, such that $w \leq v$, where v is given by Theorem 1.3. After this, the existence of a solution for the problem (1.1) will be obtained applying Lemma 1.7.

To get a lower solution for (1.1), we will proceed with a limit process in u_n, where u_n is a classical solution of problem (1.10) (given by Theorem 1.6) with $\widetilde{\rho} = \widetilde{a}$, h is a suitable function and $R = n$ for $n \geq n_0$ and n_0 is such that $\widetilde{a} \neq 0$ in $[0, n_0)$.

Let

$$f_\infty(s) = s^{p-1}\widetilde{f}_\infty(s), \quad s > 0, \quad \text{where } \widetilde{f}_\infty(s) = \inf_{0<t\leq s} \frac{f(t)}{t^{p-1}}, \quad s > 0. \tag{4.1}$$

Thus, it is easy to check the following lemma.

Lemma 4.1. *Suppose that* (H_1) *and* (H_2) *hold. Then,*

(i) $0 < f_\infty(s) \leq f(s) \leq F_0(s) + \lambda_\star F(\tau_\infty, s), \ s > 0,$

(ii) $f_\infty(s)/s^{p-1}, \ s > 0$ *is non-increasing,*

(iii) $\lim_{s\to 0}(f_\infty(s)/s^{p-1}) = \infty$ *and* $\lim_{s\to\infty}(f_\infty(s)/s^{p-1}) = 0$.

Hence, Lemma 4.1 shows that f_∞ fulfills all assumptions of Theorem 1.6. Thus, for each $n \in \mathbb{N}$ such that $n \geq n_0$ there exists one $\varpi_n \in C^2(B(0,n) \setminus \{0\}) \cap C^1(B(0,n)) \cap C(\overline{B(0,n)})$ with $\varpi_n(x) = \widetilde{\varpi}_n(|x|), x \in B(0,n)$ and $\widetilde{\varpi}_n$ satisfying

$$\begin{gathered} -\left(r^{N-1}|\widetilde{\varpi}_n'|^{p-2}\widetilde{\varpi}_n'\right)' = r^{N-1}\widetilde{a}(r) f_\infty(\widetilde{\varpi}_n(r)) \quad \text{in } 0 < r < n, \\ \widetilde{\varpi}_n > 0 \quad \text{in } [0,n), \quad \widetilde{\varpi}_n(n) = 0, \end{gathered} \tag{4.2}$$

equivalently,

$$\widetilde{\varpi}_n(r) = \widetilde{\varpi}_n(0) - \int_0^r \left[t^{1-N}\int_0^t s^{N-1}\widetilde{a}(s) f_\infty(\widetilde{\varpi}_n(s))ds\right]^{1/(p-1)} dt, \quad 0 \leq r \leq n. \tag{4.3}$$

Consider $\widetilde{\varpi}_n$ extended on $[n, \infty)$ by 0. We claim that

$$0 \leq \cdots \leq \widetilde{\varpi}_n \leq \widetilde{\varpi}_{n+1} \leq \cdots \leq \vartheta. \tag{4.4}$$

Indeed, first we observe that f_∞ satisfies Lemma 4.1(ii). So, with similar arguments to those of [21], we show $\widetilde{\varpi}_n \leq \widetilde{\varpi}_{n+1}, \ n \geq n_0$.

To prove $\widetilde{\varpi}_n \leq \vartheta$, first we will prove that $\widetilde{\varpi}_n(0) \leq \vartheta(0)$, for all $n \in \mathbb{N}$. In fact, if $\widetilde{\varpi}_n(0) > \vartheta(0)$ for some n, then there is one $T_n > 0$ such that

$$\vartheta(r) < \widetilde{\varpi}_n(r), \quad r \in [0, T_n), \qquad \vartheta(T_n) = \widetilde{\varpi}_n(T_n) > 0, \tag{4.5}$$

because $\widetilde{\varpi}_n(n) = 0$ and $\vartheta > 0$ with $\vartheta(r) \to 0$ as $r \to \infty$.

So, using Lemma A.1 (see the appendix) with $h(r,s) = \tilde{a}(r) f_\infty(s)$, $r \in [0, T_n)$, and $s > 0$, we obtain

$$H_{\tilde{\varpi}_n,\vartheta}(r) \leq \int_0^r t^{N-1}\tilde{a}(t)\left[\frac{F_0(\vartheta(t)) + \lambda^\star F(\tau_\infty, \vartheta(t))}{\vartheta^{p-1}} - \frac{f_\infty(\tilde{\varpi}_n(t))}{\tilde{\varpi}_n^{p-1}}\right]\left(\vartheta^p - \tilde{\varpi}_n^p\right)dr \tag{4.6}$$

and from Lemma 4.1(i),

$$H_{\tilde{\varpi}_n,\vartheta}(r) \leq \int_0^r t^{N-1}\tilde{a}(t)\left[\frac{f_\infty(\vartheta(t))}{\vartheta^{p-1}} - \frac{f_\infty(\tilde{\varpi}_n(t))}{\tilde{\varpi}_n^{p-1}}\right]\left(\vartheta^p - \tilde{\varpi}_n^p\right)dr \leq 0, \quad r \in [0, T_n). \tag{4.7}$$

As a consequence of the contradiction hypothesis and the definition of $H_{\tilde{\varpi}_n,\vartheta}$, we get

$$\frac{|\tilde{\varpi}_n'(r)|^{p-2}\tilde{\varpi}_n'(r)}{\tilde{\varpi}_n^{p-1}(r)} - \frac{|\vartheta'(r)|^{p-2}\vartheta'(r)}{\vartheta^{p-1}(r)} \geq 0, \quad r \in [0, T_n). \tag{4.8}$$

Recalling that $\tilde{\varpi}'(r),\ \vartheta'(r) \leq 0,\ r \in [0, T_n)$, it follows that

$$\frac{\tilde{\varpi}_n}{\vartheta}, \quad \text{is non-decreasing in } [0, T_n]. \tag{4.9}$$

So,

$$1 < \frac{\tilde{\varpi}_n(0)}{\vartheta(0)} \leq \frac{\tilde{\varpi}_n(T_n)}{\vartheta(T_n)} = 1. \tag{4.10}$$

However, this is impossible. To end the proof of claim (4.4), we will suppose that there exist an n and $r_0 > 0$ such that $\tilde{\varpi}_n(r_0) > \vartheta(r_0)$. Hence, there are S_n, T_n with $0 < S_n < r_0 < T_n$ such that $\tilde{\varpi}_n(S_n) = \vartheta(S_n)$, $\tilde{\varpi}_n(T_n) = \vartheta(T_n)$ and $\tilde{\varpi}_n(r) > \vartheta(r)$ for all $r \in (S_n, T_n)$.

Following the same above arguments, we obtain

$$1 = \frac{\tilde{\varpi}_n(S_n)}{\vartheta(T_n)} < \frac{\tilde{\varpi}_n(r_0)}{\vartheta(r_0)} < \frac{\tilde{\varpi}_n(T_n)}{\vartheta(T_n)} = 1. \tag{4.11}$$

This is impossible again. Thus, we completed the proof of claim (4.4). Setting

$$\lim_{n\to\infty} \tilde{\varpi}_n(r) = \hat{w}(r), \quad r \geq 0, \tag{4.12}$$

it follows by claim (4.4) that

$$0 < \hat{w}(r) \leq \vartheta(r), \quad r \geq 0. \tag{4.13}$$

Moreover, making $n \to \infty$ in (4.3), we use Lebesgue's theorem that

$$\widehat{w}(r) = \widehat{w}(0) - \int_0^r \left[t^{1-N} \int_0^t s^{N-1} \widetilde{a}(s) f_\infty(\widehat{w}(s)) ds \right]^{1/(p-1)} dt, \quad r \geq 0. \tag{4.14}$$

Hence, after some calculations, we obtain $\widehat{w} \in C^2((0,\infty)) \cap C^1([0,\infty))$ and setting $w(x) = \widehat{w}(|x|)$, $x \in \mathbb{R}^N$ it follows, by DiBenedetto [23], that $w \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^{1,\mu}_{\text{loc}}(\mathbb{R}^N)$ for some $\mu \in (0,1)$. Recalling that $v(x) = \vartheta(|x|)$, $x \in \mathbb{R}^N$ and using Lemma 4.1(i), it follows that w is a lower solution of (1.1) with

$$0 < w(x) \leq v(x), \quad \forall x \in \mathbb{R}^N. \tag{4.15}$$

So, by Lemma 1.7, we conclude that problem (1.1) admits a solution. Besides this, the inequality (1.4) is a consequence of a result in [6]. This completes the proof of Theorem 1.1.

Appendix

Proof of Lemma 2.4. The proof of item (iv) is an immediate consequence of Lemma 2.3(i). The item (v) follows by Lemma 2.3(i) and (ii) using Lebesgue's Theorem. □

Proof of (i) and (iii). By Lemma 2.2(i),

$$0 \leq H_\lambda(\tau) \leq \frac{\tau}{\widetilde{F}_\lambda(\tau,\tau)} = \frac{1}{\tau} \int_0^\tau \frac{t}{F_\lambda(\tau,t)^{1/(p-1)}}, \quad \tau > 0. \tag{A.1}$$

So, using (2.2), (2.5), and Lemma 2.1(i) and (ii), we get

$$0 \leq H_\lambda(\tau) \leq \left[\frac{\tau^{p-1}}{F_0(\tau) + \lambda F(\tau,\tau)} \right]^{1/(p-1)}, \quad \tau > 0. \tag{A.2}$$

Since, by Lemma 2.1(iv),

$$\lim_{\tau \to 0} \frac{F_0(\tau) + \lambda F(\tau,\tau)}{\tau^{p-1}} = \infty, \quad \lambda \geq 0, \tag{A.3}$$

then the claim (i) of Lemma 2.4 follows from (A.2).

On the other hand, for all $\lambda > 0$, it follows from Lemma 2.1(vi) that

$$\lim_{\tau \to \infty} \frac{F_0(\tau) + \lambda F(\tau,\tau)}{\tau^{p-1}} = \lambda \lim_{\tau \to \infty} \frac{F(\tau,\tau)}{\tau^{p-1}} = \lambda \lim_{\tau \to \infty} G(\tau,\tau) = \lambda \lim_{\tau \to \infty} \frac{g(\tau)}{\tau^{p-1}} = \infty, \tag{A.4}$$

where the last equality is obtained by using (H_2)-(ii). Hence, using (A.2), the proof of Lemma 2.4(iii) is concluded. □

Proof of (ii). In this case ($\lambda = 0$),

$$\widetilde{F}_0(\tau, s) = \frac{s^2}{\int_0^s \left(t/F_0(t)^{1/(p-1)}\right)dt} := \widetilde{F}_0(s), \quad s > 0. \tag{A.5}$$

That is, $\widetilde{F}_0(\tau, s)$ does not depend on τ. So, by L'Hopital and Lemma 2.2(iv),

$$\lim_{\tau\to\infty} H_\lambda(\tau) = \lim_{\tau\to\infty} \frac{\tau}{\widetilde{F}_0(\tau)} = \infty. \tag{A.6}$$

This ends the proof of Lemma 2.4. □

The next lemma, proved in [21], was used in the proofs of Theorems 1.1 and 1.6. To enunciate it, we will consider $u, v \in C^2((0,T)) \cap C^1([0,T)) \cap C([0,T])$, for some $T > 0$, satisfying

$$\begin{gathered} -\left(r^{N-1}|\varphi'|^{p-2}\varphi'\right)' = r^{N-1}h(r,\varphi(r)) \quad \text{in } (0,T), \\ \varphi > [0,T], \qquad \varphi'(0) = 0, \end{gathered} \tag{A.7}$$

and we define the continuous function $H_{u,v} : [0,T) \to \mathbb{R}$ by

$$H_{u,v}(r) := r^{N-1}\left[\frac{|u'(r)|^{p-2}u'(r)}{u^{p-1}(r)} - \frac{|v'(r)|^{p-2}v'(r)}{v^{p-1}(r)}\right](v^p(r) - u^p(r)), \quad r \in [0,T). \tag{A.8}$$

So, we have $H_{u,v}(0) = 0$ and

Lemma A.1. *If* $0 \le s \le r < T$, *then*

$$H_{u,v}(r) - H_{u,v}(s) \le \int_s^r \left[\frac{\left(r^{N-1}|u'|^{p-2}u'\right)'}{u^{p-1}} - \frac{\left(r^{N-1}|v'|^{p-2}v'\right)'}{v^{p-1}}\right](v^p - u^p)dr. \tag{A.9}$$

Finally, we will sketch the proof of claim (H_3), implies $(H_3)'$, if $N \ge 3$.
Below, $C_1, C_2, \ldots$ will denote several positive constants and I, the function

$$I(r) = \int_0^r \left[t^{1-N}\int_0^t (\widetilde{a}(s) + \widetilde{b}(s))ds\right]^{1/(p-1)} dt, \quad r \ge 0. \tag{A.10}$$

If $1 < p \le 2$, by estimating the integral in (A.10), we obtain

$$I(r) \le C_1 + C_2\int_1^r t^{(1-N)/(p-1)}\left[\int_0^t s^{N-1}\widetilde{a}(s)ds\right]^{1/(p-1)} dt + \int_1^r t^{(1-N)/(p-1)}\left[\int_0^t s^{N-1}\widetilde{b}(s)ds\right]^{1/(p-1)} dt. \tag{A.11}$$

Using the assumption $N \geq 3$ in the computation of the first integral above and Jensen's inequality to estimate the last one, we have

$$\int_1^r t^{(1-N)/(p-1)} \left[\int_0^t s^{N-1}\tilde{a}(s)ds\right]^{1/(p-1)} dt \leq C_3 + C_4\int_1^r t^{(3-N-p)/(p-1)} \left[\int_1^t s^{(N-1)/(p-1)}\tilde{a}(s)^{1/(p-1)}ds\right]dt. \tag{A.12}$$

Computing the above integral, we obtain

$$\int_1^r t^{(1-N)/(p-1)} \left[\int_0^t s^{N-1}\tilde{a}(s)ds\right]^{1/(p-1)} dt \leq C_3 + C_5\int_1^r t^{1/(p-1)}\tilde{a}(t)^{1/(p-1)}\ dt. \tag{A.13}$$

Similar calculations show that

$$\int_1^r t^{(1-N)/(p-1)} \left[\int_0^t s^{N-1}\tilde{b}(s)ds\right]^{1/(p-1)} dt \leq C_6 + C_7\int_1^r t^{1/(p-1)}\tilde{b}(t)^{1/(p-1)}dt. \tag{A.14}$$

So, by (H_3),

$$0 < \alpha = \lim_{r\to\infty} I(r) \leq C_8 + C_5\int_1^\infty t^{1/(p-1)}\tilde{a}(t)^{1/(p-1)}dt + C_7\int_1^\infty t^{1/(p-1)}\tilde{b}(t)^{1/(p-1)}dt < \infty. \tag{A.15}$$

On the other hand, if $p \geq 2$, set

$$H(r) := \int_0^r s^{N-1}\left(\tilde{a}(s) + \tilde{b}(s)\right)ds, \quad r \geq 0, \tag{A.16}$$

and note that either $H(r) \leq 1$ for all $r \geq 0$ or $H(r_0) = 1$ for some $r_0 > 0$. In the first case, $H(r)^{1/(p-1)} \leq 1$, for all $r \geq r_0$. Hence

$$I(r) = \int_0^r t^{(1-N)/(p-1)}H(t)^{1/(p-1)}dt \leq C_8 + \int_1^r t^{(1-N)/(p-1)}dt, \quad \forall r \geq 0. \tag{A.17}$$

So $I(r)$ has a finite limit as $r \to \infty$, because $p < N$. In the second case, $H(r)^{1/(p-1)} \leq H(r)$ for $r \geq r_0$ and hence,

$$I(r) \leq C_9 + \int_1^r \left[t^{(1-N)/(p-1)}\int_0^t s^{N-1}\left(\tilde{a}(s) + \tilde{b}(s)\right)ds\right]dt, \quad r \geq 0. \tag{A.18}$$

Integrating by parts and estimating using $p < N$, we obtain

$$
\begin{aligned}
I(r) &\le C_9 + C_{10}\int_1^r t^{(1-N)/(p-1)}dt \\
&\quad + \frac{p-1}{N-p}\left[\int_1^r t^{((p-2)N+1)/(p-1)}\left(\tilde{a}(t)+\tilde{b}(t)\right)dt - r^{(p-N)/(p-1)}\int_0^r t^{N-1}\left(\tilde{a}(t)+\tilde{b}(t)\right)dt\right] \\
&\le C_{11} + C_{12}\int_1^r t^{((p-2)N+1)/(p-1)}\tilde{a}(t)dt + \int_1^r t^{((p-2)N+1)/(p-1)}\tilde{b}(t)dt, \quad r \ge 0.
\end{aligned}
\tag{A.19}
$$

Again by (H_3), we obtain that $\alpha = \lim_{r\to\infty} I(r)$ is a finite number. This shows the claim.

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