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Research Article

On Coupled Klein-Gordon-Schrödinger Equations with Acoustic Boundary Conditions

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We are concerned with the existence and energy decay of solution to the initial boundary value problem for the coupled Klein-Gordon-Schrödinger equations with acoustic boundary conditions.

1. Introduction

In this paper, we are concerned with global existence and uniform decay for the energy of solutions of Klein-Gordon-Schrödinger equations:

$$\begin{gathered}
iu' + \Delta u + i|u|^2 u + i\alpha u = -uv \quad \text{in } Q = \Omega \times (0,\infty), \\
v'' - \Delta v + \mu^2 v + \beta v' = |u|^2 \quad \text{in } Q = \Omega \times (0,\infty), \\
u = 0 \quad \text{on } \Sigma = \Gamma \times (0,\infty), \\
v = 0 \quad \text{on } \Sigma_0 = \Gamma_0 \times (0,\infty), \\
\gamma v' + f(x)z' + g(x)z = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,\infty), \\
\frac{\partial v}{\partial \nu} = h(x)z' \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,\infty), \\
u(x,0) = u_0, \quad v(x,0) = v_0, \quad v'(x,0) = v_1 \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $1 \le n \le 3$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, where Γ_0 and Γ_1 are two disjoint pieces of Γ each having nonempty interior and $f, g, h : \overline{\Gamma}_1 \to \mathbb{R}$ are given functions. We will denote by ν the unit outward normal vector to Γ. Δ stands for the Laplacian with respect to the spatial variables; $'$ denotes the derivative with respect to time t. Here $z(x,t)$ is the normal displacement to the boundary at time t with the boundary point x.

The above equations describe a generalization of the classical model of the Yukawa interaction of conserved complex nucleon field with neutral real meson field. Here, u is complex scalar nucleon field while v and z are real scalar meson one.

In three dimension, [1–5] studied the global existence for the Cauchy problem to

$$\begin{aligned} iu' + \frac{1}{2}\Delta u &= -uv, \\ v'' - \Delta v + \mu^2 v &= |u|^2. \end{aligned} \tag{1.2}$$

Klein-Gordon-Schrödinger equations have been studied as many as ever by many authors (cf. [6–11], and a list of references therein). However, they did not have treated acoustic boundary conditions.

Boundary conditions of the fifth and sixth equations are called acoustic boundary conditions. Equation $(1.1)_5$ (the fifth equation of (1.1)) does not contain the second derivative z'', which physically means that the material of the surface is much more lighter than a liquid flowing along it. As far as $h \equiv 1$ in $(1.1)_6$ (the sixth equation of (1.1)) is concerned to the case of a nonporous boundary, $(1.1)_6$ simulates a porous boundary when a function h is nonnegative. When general acoustic boundary conditions, which had the presence of z'' in $(1.1)_5$, are prescribed on the whole boundary, Beale [12–14] proved the global existence and regularity of solutions in a Hilbert space of data with finite energy by means of semigroup methods. The asymptotic behavior was obtained in [13], but no decay rate was given there. Recently, the acoustic boundary conditions have been treated by many authors (cf. [15–21] and a list of references therein). However, energy decay problem with acoustic boundary conditions was studied by a few authors. For instance, Rivera and Qin [22] proved the polynomial decay for the energy of the wave motion using the Lyapunov functional technique in the case of general acoustic boundary conditions and $n = 3$. Frota and Larkin [23] considered global solvability and the exponential decay of the energy for the wave equation with acoustic boundary conditions, which eliminated the second derivative term for $1 \leq n \leq 3$. However, it is not simple to apply the semigroup theory as well as Galerkin's method in [23] because a system of corresponding ordinary equations is not normal and one cannot apply directly the Carathéodory's theorem. So they overcame this problem using the degenerated second order equation. And Park and Ha [20] studied the existence and uniqueness of solutions and uniform decay rates for the Klein-Gordon-type equation by using the multiplier technique. Moreover, [20] proved the exponential and polynomial decay rates of solutions for all $n \geq 1$.

In this paper, we prove the existence and uniqueness of solutions and uniform decay rates for the Klein-Gordon-Schrödinger equations with acoustic boundary conditions and allow to apply the method developed in [23]. However, [23] did not treat the Klein-Gordon-Schrödinger equations.

This paper is organized as follows. In Section 2, we recall the notation and hypotheses and introduce our main result. In Section 3, using Galerkin's method, we prove the existence and uniqueness of solutions to problem (1.1). In Section 4, we prove the exponential energy decay rate for the solutions obtained in Section 3.

2. Notations and Main Results

We begin this section introducing some notations and our main results. Throughout this paper we define the Hilbert space $\mathscr{H} = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$ with the norm $\|u\|_{\mathscr{H}} = (\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2)^{1/2}$, where $\|\cdot\|$ is an L^2-norm and $(u,v) = \int_\Omega u(x)\overline{v(x)}dx$; without loss of generality we denote $(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)\overline{v(x)}d\Gamma$. Moreover, $L^p(\Omega)$-norm and $L^p(\Gamma)$-norm are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{p,\Gamma}$, respectively. Denoting by $\rho_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\rho_1 : \mathscr{H} \to H^{-1/2}(\Gamma)$ the trace map of order zero and the Neumann trace map on $\mathscr{H}$, respectively, we have

$$\rho_0(u) = u_{|\Gamma}, \quad \rho_1(u) = \left(\frac{\partial u}{\partial \nu}\right)_{|\Gamma}, \quad \forall u \in D\left(\overline{\Omega}\right), \tag{2.1}$$

and the generalized Green formula

$$\int_\Omega (-\Delta u) v dx = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx - (\rho_1(u), \rho_0(v))_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \tag{2.2}$$

holds for all $u \in \mathscr{H}$ and $v \in H^1(\Omega)$. We denoted $V = \{u \in H^1(\Omega); \rho_0(u) = 0 \text{ on } \Gamma_0\}$. By the Poincare's inequality, the norm $\|u\|_V = (\sum_{i=1}^n \int_\Omega (\partial u/\partial x_i)^2 dx)^{1/2}$ is equivalent to the usual norm from $H^1(\Omega)$. Now we give the hypotheses for the main results.

(H_1) *Hypotheses on* Ω

Let Ω be a bounded domain in $\mathbb{R}^n$, $1 \le n \le 3$ with boundary Γ of class C^2. Here Γ_0 and Γ_1 are two disjoint pieces of Γ, each having nonempty interior and satisfying the following conditions:

$$\begin{gathered} m \cdot \nu \ge \sigma > 0 \quad \text{on } \Gamma_1, \qquad m \cdot \nu \le 0 \quad \text{on } \Gamma_0, \\ m(x) = x - x^0 \quad \left(x^0 \in \mathbb{R}^n\right), \qquad R = \max_{x \in \overline{\Omega}} |m(x)|, \end{gathered} \tag{2.3}$$

where ν represents the unit outward normal vector to Γ.

(H_2) *Hypotheses on* f, g, *and* h

Assume that $f, g, h : \overline{\Gamma}_1 \to \mathbb{R}$ are continuous functions such that

$$f(x), g(x) > 0, \quad h(x) = \lambda(x)(m(x) \cdot \nu(x)), \quad \forall x \in \overline{\Gamma}_1, \tag{2.4}$$

where

$$\lambda(x) \in C^1\left(\overline{\Gamma}_1\right) \text{ satisfies } 0 < \lambda_0 \le \lambda(x), \quad \forall x \in \overline{\Gamma}_1. \tag{2.5}$$

Moreover, $0 < \alpha < 1/4$, $1/2 < \beta < 4$, $\gamma > 0$, and μ is a real constant.

In physical situation, α and β are parameters representing the gratitude of diffusion and dissipation effects. Also, γ is a fluid density and μ describes the mass of a meson. Boundary condition $(1.1)_6$ (the sixth equation of (1.1)) simulates a porous boundary because of the function h.

We define the energy of system (1.1) by

$$E(t) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v'\|^2 + \frac{1}{2}\|v\|_V^2 + \frac{\mu^2}{2}\|v\|^2 + \frac{1}{2\gamma}\int_{\Gamma_1} h(x)g(x)z^2 d\Gamma. \tag{2.6}$$

Now, we are in a position to state our main result.

Theorem 2.1. *Let* $\{u_0, v_0, v_1\} \in V \cap H^2(\Omega) \times V \cap H^2(\Omega) \times V$ *satisfy the inequality*

$$\left\| \frac{\rho_1(v_0)}{h(x)} \right\| < C_0 \quad \forall x \in \Gamma_1, \tag{2.7}$$

where C_0 *is a positive constant. Assume that* (H_1) *and* (H_2) *hold. Then problem* (1.1) *has a unique strong solution verifying*

$$\begin{gathered} u \in L^\infty\left(0,\infty; V \cap H^2(\Omega)\right), \qquad u' \in L^\infty\left(0,\infty; L^2(\Omega)\right), \\ v \in L^\infty\left(0,\infty; V \cap H^2(\Omega)\right), \qquad v' \in L^\infty(0,\infty; V), \qquad v'' \in L^\infty\left(0,\infty; L^2(\Omega)\right), \\ h^{1/2}z, h^{1/2}z', h^{1/2}z'' \in L^2\left(0,\infty; L^2(\Gamma_1)\right). \end{gathered} \tag{2.8}$$

Moreover, if μ *satisfies*

$$\mu^2 < \min\left\{ \frac{2}{2R - 2(n-1) + 8\beta^2(n-1)^2 + (n-1)^2}, \frac{1}{2R^2\beta^2 + 2R^2 + R} \right\}, \tag{2.9}$$

then one has the following energy decay:

$$E(t) \le C_1 e^{-\omega t}, \quad \forall t \ge 0, \tag{2.10}$$

where C_1 *and* ω *are positive constants.*

Note

By the hypothesis in β, we have $4\beta^2(n-1)^2 \ge (n-1)$ for all $1 \le n \le 3$, so we can assume (2.9).

3. Existence of Solutions

In this section, we prove the existence and uniqueness of solutions to problem (1.1). Let $\{w_j\}_{j\in\mathbb{N}}$ and $\{\xi_j\}_{j\in\mathbb{N}}$ be orthonormal bases of $V \cap H^2(\Omega)$ and $L^2(\Gamma)$, respectively, and define

$V_m = \text{span}\{\xi_1, \xi_2, \ldots, \xi_m\}$ and $W_m = \text{span}\{w_1, w_2, \ldots, w_m\}$. Let u_{0m}, v_{0m}, and v_{1m} be sequences of W_m such that $u_{0m} \to u_0$, $v_{0m} \to v_0$ strongly in $V \cap H^2(\Omega)$, and $v_{1m} \to v_1$ strongly in V. For each $\eta \in (0,1)$ and $m \in \mathbb{N}$, we consider

$$\begin{aligned} u_{\eta m}(x,t) &= \sum_{j=1}^{m} a_{jm}(t)\omega_j(x), \quad x \in \Omega,\ t \in [0,T_m], \\ v_{\eta m}(x,t) &= \sum_{j=1}^{m} b_{jm}(t)\omega_j(x), \quad x \in \Omega,\ t \in [0,T_m], \\ z_{\eta m}(x,t) &= \sum_{j=1}^{m} c_{jm}(t)\xi_j(x), \quad x \in \Gamma_1,\ t \in [0,T_m], \end{aligned} \tag{3.1}$$

satisfying the approximate perturbed equations

$$\begin{gathered} \left(u'_{\eta m}, w\right) + i(\nabla u_{\eta m}, \nabla w) + \left(\left|u_{\eta m}\right|^2 u_{\eta m}, w\right) + \alpha(u_{\eta m}, w) = i(u_{\eta m} v_{\eta m}, w), \\ \left(v''_{\eta m}, y\right) + (\nabla v_{\eta m}, \nabla y) + \mu^2(v_{\eta m}, y) + \beta\left(v'_{\eta m}, y\right) - \left(h z'_{\eta m}, \rho_0(y)\right)_{\Gamma_1} = \left(\left|u_{\eta m}\right|^2, y\right), \\ \eta\left(z''_{\eta m}, \xi\right)_{\Gamma_1} + \left(h\left[\gamma\rho_0\left(v'_{\eta m}\right) + f z'_{\eta m} + g z_{\eta m}\right], \xi\right)_{\Gamma_1} = 0, \\ u_{\eta m}(0) = u_{0m}, \qquad v_{\eta m}(0) = v_{0m}, \qquad v'_{\eta m}(0) = v_{1m}, \\ z_{\eta m}(0) = z_{0m} = -\left(\frac{\gamma v_{1m} + f z'_{\eta m}(0)}{g}\right), \quad z'_{\eta m}(0) = z_{1m} = \frac{1}{h}\rho_1(v_{0m}), \end{gathered} \tag{3.2}$$

where $z_{0m} \in L^2(\Gamma_1)$ and for all $w, y \in W_m$, $\xi \in V_m$, $0 < T_m \le T$. The local existence of regular functions a_{jm}, b_{jm}, and c_{jm} is standard, because (3.2) is a normal system of ordinary differential equation. A solution (u, v, z) to the problem (1.1) on some interval $[0, T_m)$ will be obtained as the limit of $(u_{\eta m}, z_{\eta m})$ as $m \to \infty$ and $\eta \to 0$. Then, this solution can be extended to the whole interval $[0, T]$, for all T > 0, as a consequence of the a priori estimates that will be proved in the next step.

3.1. The First Estimate

Replacing w, y, and ξ by $u_{\eta m}$, $v'_{\eta m}$, and $z'_{\eta m}$ in (3.2), respectively, we obtain

$$\left(u'_{\eta m}, u_{\eta m}\right) + i(\nabla u_{\eta m}, \nabla u_{\eta m}) + \left(\left|u_{\eta m}\right|^2 u_{\eta m}, u_{\eta m}\right) + \alpha(u_{\eta m}, u_{\eta m}) = i(u_{\eta m} v_{\eta m}, u_{\eta m}), \tag{3.3}$$

$$\begin{aligned} &\left(v''_{\eta m}, v'_{\eta m}\right) + \left(\nabla v_{\eta m}, \nabla v'_{\eta m}\right) + \mu^2\left(v_{\eta m}, v'_{\eta m}\right) \\ &\quad + \beta\left(v'_{\eta m}, v'_{\eta m}\right) - \left(h z'_{\eta m}, v'_{\eta m}\right)_{\Gamma_1} = \left(\left|u_{\eta m}\right|^2, v'_{\eta m}\right), \end{aligned} \tag{3.4}$$

$$\eta\left(z''_{\eta m}, z'_{\eta m}\right)_{\Gamma_1} + \left(h\left[\gamma v'_{\eta m} + f z'_{\eta m} + g z_{\eta m}\right], z'_{\eta m}\right)_{\Gamma_1} = 0. \tag{3.5}$$

Taking the real part in (3.3), we get

$$\frac{1}{2}\frac{d}{dt}\left[\|u_{\eta m}\|^2+\left\|v'_{\eta m}\right\|^2+\|v_{\eta m}\|_V^2+\mu^2\|v_{\eta m}\|^2+\frac{\eta}{\gamma}\left\|z'_{\eta m}\right\|_{2,\Gamma_1}^2+\frac{1}{\gamma}\int_{\Gamma_1}hg(z_{\eta m})^2d\Gamma\right]$$
$$+\|u_{\eta m}\|_4^4+\alpha\|u_{\eta m}\|^2+\beta\left\|v'_{\eta m}\right\|^2+\frac{1}{\gamma}\int_{\Gamma_1}hf\left(z'_{\eta m}\right)^2d\Gamma=\int_\Omega|u_{\eta m}|^2v'_{\eta m}dx. \tag{3.6}$$

On the other hand, by Young's inequality we have

$$\int_\Omega|u_{\eta m}|^2v'_{\eta m}dx\le\frac{1}{2}\|u_{\eta m}\|_4^4+\frac{1}{2}\left\|v'_{\eta m}\right\|^2. \tag{3.7}$$

Substituting the above inequality in (3.6), and then integrating (3.6) over $(0,t)$ with $t\in(0,T_m)$, we get

$$\|u_{\eta m}\|^2+\left\|v'_{\eta m}\right\|^2+\|v_{\eta m}\|_V^2+\mu^2\|v_{\eta m}\|^2+\frac{\eta}{\gamma}\left\|z'_{\eta m}\right\|_{2,\Gamma_1}^2+\frac{1}{\gamma}\int_{\Gamma_1}hg(z_{\eta m})^2d\Gamma$$
$$+\int_0^t\|u_{\eta m}(s)\|_4^4ds+2\alpha\int_0^t\|u_{\eta m}(s)\|^2ds+2\beta\int_0^t\left\|v'_{\eta m}(s)\right\|^2ds+\frac{2}{\gamma}\int_0^t\int_{\Gamma_1}hf\left(z'_{\eta m}(s)\right)^2d\Gamma\,ds$$
$$\le\|u_{0m}\|^2+\|v_{1m}\|^2+\|v_{0m}\|_V^2+\mu^2\|v_{0m}\|^2+\frac{\eta}{\gamma}\|z_{1m}\|_{2,\Gamma_1}^2+\frac{1}{\gamma}\left(\max_{x\in\overline{\Gamma}_1}|h(x)g(x)|\right)\|z_{0m}\|_{2,\Gamma_1}^2$$
$$+\int_0^t\left\|v'_{\eta m}(s)\right\|^2ds. \tag{3.8}$$

Using the fact that $0<\min_{x\in\overline{\Gamma}_1}f(x):=f_0$, $0<\min_{x\in\overline{\Gamma}_1}g(x):=g_0$, (2.7) and Gronwall's lemma, we obtain

$$\|u_{\eta m}\|^2+\left\|v'_{\eta m}\right\|^2+\|v_{\eta m}\|_V^2+\|v_{\eta m}\|^2+\left\|z'_{\eta m}\right\|_{2,\Gamma_1}^2+\left\|h^{1/2}z_{\eta m}\right\|_{2,\Gamma_1}^2$$
$$+\int_0^t\|u_{\eta m}(s)\|_4^4ds+\int_0^t\|u_{\eta m}(s)\|^2ds+\int_0^t\left\|v'_{\eta m}(s)\right\|^2ds+\int_0^t\left\|h^{1/2}z'_{\eta m}\right\|_{2,\Gamma_1}^2ds\le C_2, \tag{3.9}$$

where C_2 is a positive constant which is independent of m, η, and t.

3.2. The Second Estimate

First of all, we are going to estimate $z''_{\eta m}(0)$. By taking $t=0$ in $(3.2)_3$ (the third equation of (3.2)), we get

$$\eta\left(z''_{\eta m}(0),\xi\right)_{\Gamma_1}+\left(h[\gamma\rho_0(v_{1m})+fz_{1m}+gz_{0m}],\xi\right)_{\Gamma_1}=0. \tag{3.10}$$

By considering $\xi = z''_{\eta m}(0)$ and hypotheses on the initial data, for all $m \in \mathbb{N}$ and $\eta \in (0,1)$, we obtain

$$\left\| z''_{\eta m}(0) \right\|_{2,\Gamma_1} = 0. \tag{3.11}$$

Now, by replacing w and y by $-\Delta u_{\eta m}$ and $-\Delta v'_{\eta m}$ in (3.2), respectively, also differentiating $(3.2)_3$ with respect to t, and then substituting $\xi = z''_{\eta m}$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{\eta m}\|_V^2 + i\|\Delta u_{\eta m}\|^2 + \alpha\|u_{\eta m}\|_V^2 + \left(|u_{\eta m}|^2 u_{\eta m}, -\Delta u_{\eta m}\right) = i(u_{\eta m}v_{\eta m}, -\Delta u_{\eta m}), \tag{3.12}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\left\|v'_{\eta m}\right\|_V^2 + \|\Delta v_{\eta m}\|^2 + \mu^2\|v_{\eta m}\|_V^2\right] + \beta\left\|v'_{\eta m}\right\|_V^2 - \left(v''_{\eta m}, hz''_{\eta m}\right)_{\Gamma_1} \\ &\quad - \mu^2\left(v_{\eta m}, hz''_{\eta m}\right)_{\Gamma_1} - \beta\left(v'_{\eta m}, hz''_{\eta m}\right)_{\Gamma_1} = \left(\nabla u_{\eta m}\overline{u_{\eta m}} + u_{\eta m}\nabla\overline{u_{\eta m}}, \nabla v'_{\eta m}\right), \end{aligned} \tag{3.13}$$

$$\frac{1}{2}\frac{d}{dt}\left[\eta\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2 + \int_{\Gamma_1} h(x)g(x)\left(z'_{\eta m}\right)^2 d\Gamma\right] + \gamma\left(hv''_{\eta m}, z''_{\eta m}\right)_{\Gamma_1} + \int_{\Gamma_1} h(x)f(x)\left(z''_{\eta m}\right)^2 d\Gamma = 0. \tag{3.14}$$

We now estimate the last term on the left-hand side of (3.12) and the term on the right-hand side of (3.12). Applying Green's formula, we deduce

$$\left(|u_{\eta m}|^2 u_{\eta m}, -\Delta u_{\eta m}\right) = 2\int_\Omega |u_{\eta m}|^2|\nabla u_{\eta m}|^2 dx + \int_\Omega \left(u_{\eta m}\nabla\overline{u_{\eta m}}\right)^2 dx. \tag{3.15}$$

Considering the equality

$$2[\mathrm{Re}(z_1\overline{z_2})]^2 = |z_1|^2|z_2|^2 + \mathrm{Re}\left[(z_1\overline{z_2})^2\right] \tag{3.16}$$

for all $z_1, z_2 \in \mathbb{C}$, we have

$$\mathrm{Re}\left[\left(u_{\eta m}\nabla\overline{u_{\eta m}}\right)^2\right] = 2\left[\mathrm{Re}\left(u_{\eta m}\nabla\overline{u_{\eta m}}\right)\right]^2 - |u_{\eta m}|^2|\nabla u_{\eta m}|^2. \tag{3.17}$$

Hence,

$$\mathrm{Re}\left[\left(|u_{\eta m}|^2 u_{\eta m}, -\Delta u_{\eta m}\right)\right] = \int_\Omega |u_{\eta m}|^2|\nabla u_{\eta m}|^2 dx + 2\int_\Omega \left[\mathrm{Re}\left(u_{\eta m}\nabla\overline{u_{\eta m}}\right)\right]^2 dx. \tag{3.18}$$

Also,

$$i(u_{\eta m}v_{\eta m}, -\Delta u_{\eta m}) = i\int_\Omega v_{\eta m}|\nabla u_{\eta m}|^2 dx + i\int_\Omega \nabla v_{\eta m}u_{\eta m}\nabla\overline{u_{\eta m}}dx. \tag{3.19}$$

Hence,

$$\operatorname{Re}\left[i\left(u_{\eta m} v_{\eta m},-\Delta u_{\eta m}\right)\right] \leq \int_{\Omega}\left|\nabla v_{\eta m}\right|\left|u_{\eta m} \nabla \overline{u_{\eta m}}\right| d x \leq \frac{1}{2}\left\|v_{\eta m}\right\|_{V}^{2}+\frac{1}{2} \int_{\Omega}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x. \tag{3.20}$$

Replacing the above calculations in (3.12) and then taking the real part, we obtain

$$\frac{1}{2} \frac{d}{d t}\left\|u_{\eta m}\right\|_{V}^{2}+\alpha\left\|u_{\eta m}\right\|_{V}^{2}+\frac{1}{2} \int_{\Omega}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x+2 \int_{\Omega}\left[\operatorname{Re}\left(u_{\eta m} \nabla \overline{u_{\eta m}}\right)\right]^{2} d x \leq \frac{1}{2}\left\|v_{\eta m}\right\|_{V}^{2}. \tag{3.21}$$

On the other hand, we can easily check that

$$\nabla u_{\eta m} \overline{u_{\eta m}}+u_{\eta m} \nabla \overline{u_{\eta m}}=\overline{\nabla \overline{u_{\eta m}} u_{\eta m}}+u_{\eta m} \nabla \overline{u_{\eta m}}=2 \operatorname{Re}\left(u_{\eta m} \nabla \overline{u_{\eta m}}\right). \tag{3.22}$$

Therefore,

$$\begin{aligned}
\left(\nabla u_{\eta m} \overline{u_{\eta m}}+u_{\eta m} \nabla \overline{u_{\eta m}}, \nabla v_{\eta m}^{\prime}\right) & \leq 2 \int_{\Omega}\left|\operatorname{Re}\left(u_{\eta m} \nabla \overline{u_{\eta m}}\right)\right|\left|\nabla v_{\eta m}^{\prime}\right| d x \\
& \leq 2 \int_{\Omega}\left|u_{\eta m}\right|\left|\nabla u_{\eta m}\right|\left|\nabla v_{\eta m}^{\prime}\right| d x \\
& \leq \frac{1}{4} \int_{\Omega}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x+4\left\|v_{\eta m}^{\prime}\right\|_{V}^{2} \\
& \leq \frac{1}{4} \int_{\left|u_{\eta m}\right| \leq 1}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x \\
& \quad+\frac{1}{4} \int_{\left|u_{\eta m}\right|>1}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x+4\left\|v_{\eta m}^{\prime}\right\|_{V}^{2} \\
& \leq \frac{1}{4}\left\|u_{\eta m}\right\|_{V}^{2}+\frac{1}{4} \int_{\Omega}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x+4\left\|v_{\eta m}^{\prime}\right\|_{V}^{2}.
\end{aligned} \tag{3.23}$$

Replacing (3.23) in (3.13) and using the imbedding $V \hookrightarrow L^{2}(\Gamma)$, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{d t} & {\left[\left\|v_{\eta m}^{\prime}\right\|_{V}^{2}+\left\|\Delta v_{\eta m}\right\|^{2}+\mu^{2}\left\|v_{\eta m}\right\|_{V}^{2}\right]+\beta\left\|v_{\eta m}^{\prime}\right\|_{V}^{2}-\left(v_{\eta m}^{\prime \prime}, h z_{\eta m}^{\prime \prime}\right)_{\Gamma_{1}} } \\
& \leq \frac{1}{4}\left\|u_{\eta m}\right\|_{V}^{2}+4\left\|v_{\eta m}^{\prime}\right\|_{V}^{2}+\frac{1}{4} \int_{\Omega}\left|u_{\eta m}\right|^{2}\left|\nabla u_{\eta m}\right|^{2} d x \\
& \quad+\frac{\mu^{4} \widetilde{c}^{2}}{2 \epsilon}\left\|v_{\eta m}\right\|_{V}^{2}+\frac{\beta^{2} \widetilde{c}^{2}}{2 \epsilon}\left\|v_{\eta m}^{\prime}\right\|_{V}^{2}+\epsilon\left\|h^{1 / 2}\right\|_{C\left(\overline{\Gamma}_{1}\right)}\left\|h^{1 / 2} z_{\eta m}^{\prime \prime}\right\|_{2, \Gamma_{1}}^{2},
\end{aligned} \tag{3.24}$$

where $\widetilde{c}$ is an imbedding constant.

Adding (3.14), (3.21), and (3.24), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|u_{\eta m}\|_V^2+\left\|v'_{\eta m}\right\|_V^2+\|\Delta v_{\eta m}\|^2+\mu^2\|v_{\eta m}\|_V^2+\frac{\eta}{\gamma}\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2+\frac{1}{\gamma}\int_{\Gamma_1}h(x)g(x)\left(z''_{\eta m}\right)^2d\Gamma\right]\\
&\quad+\frac{1}{4}\int_{\Omega}|u_{\eta m}|^2|\nabla u_{\eta m}|^2dx+2\int_{\Omega}\left[\operatorname{Re}\left(u_{\eta m}\nabla\overline{u_{\eta m}}\right)\right]^2dx+\frac{1}{\gamma}\int_{\Gamma_1}h(x)f(x)\left(z''_{\eta m}\right)^2d\Gamma\\
&\quad\le\left(\frac{1}{4}-\alpha\right)\|u_{\eta m}\|_V^2+(4-\beta)\left\|v'_{\eta m}\right\|_V^2+\frac{\mu^4\tilde{c}^2}{2\epsilon}\|v_{\eta m}\|_V^2\\
&\quad+\frac{\beta^2\tilde{c}^2}{2\epsilon}\left\|v'_{\eta m}\right\|_V^2+\epsilon\left\|h^{1/2}\right\|_{C(\overline{\Gamma}_1)}\left\|h^{1/2}z''_{\eta m}\right\|_{2,\Gamma_1}^2.
\end{aligned}\tag{3.25}$$

By choosing $\epsilon=\min_{x}\in\overline{\Gamma}_1/\gamma\|h^{1/2}\|_{C(\overline{\Gamma}_1)}$ and integrating (3.25) from 0 to t, we have

$$\begin{aligned}
&\|u_{\eta m}\|_V^2+\left\|v'_{\eta m}\right\|_V^2+\|\Delta v_{\eta m}\|^2+\mu^2\|v_{\eta m}\|_V^2+\frac{\eta}{\gamma}\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2+\int_{\Gamma_1}h(x)g(x)\left(z'_{\eta m}\right)^2d\Gamma\\
&\quad\le\|u_{0m}\|_V^2+\|v_{1m}\|_V^2+\left\|\Delta v_{\eta m}(0)\right\|^2+\mu^2\|v_{0m}\|_V^2+\frac{\eta}{\gamma}\left\|z''_{\eta m}(0)\right\|_{2,\Gamma_1}^2\\
&\quad+\int_{\Gamma_1}h(x)g(x)(z_{1m})^2d\Gamma+2\left(\frac{1}{4}-\alpha\right)\int_0^t\|u_{\eta m}(s)\|_V^2ds+2(4-\beta)\int_0^t\left\|v'_{\eta m}(s)\right\|_V^2ds\\
&\quad+C_3\int_0^t\left(\|v_{\eta m}(s)\|_V^2+\left\|v'_{\eta m}(s)\right\|_V^2\right)ds,
\end{aligned}\tag{3.26}$$

where C_3 is a positive constant. Using the hypotheses on α and β, (2.7), (3.11), and Gronwall's lemma, we obtain

$$\|u_{\eta m}\|_V^2+\|v_{\eta m}\|_V^2+\left\|v'_{\eta m}\right\|_V^2+\|\Delta v_{\eta m}\|^2+\eta\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2+\left\|h^{1/2}z'_{\eta m}\right\|_{2,\Gamma_1}^2\le C_4,\tag{3.27}$$

where C_4 is a positive constant which is independent of m, η, and t.

3.3. *The Third Estimate*

First of all, we are going to estimate $u'_{\eta m}(0)$ and $v''_{\eta m}(0)$. By taking $t=0$ in (3.2), we get

$$\begin{aligned}
&\left(u'_{\eta m}(0),w\right)-i(\Delta u_{0m},w)+\left(|u_{0m}|^2u_{0m},w\right)+\alpha(u_{0m},w)=i(u_{0m}v_{0m},w),\\
&\left(v''_{\eta m}(0),y\right)-(\Delta v_{0m},y)+\mu^2(v_{0m},y)+\beta(v_{1m},y)=\left(|u_{0m}|^2,y\right).
\end{aligned}\tag{3.28}$$

By considering $w = u'_{\eta m}(0)$ and $y = v''_{\eta m}(0)$ and hypotheses on the initial data, for all $m \in \mathbb{N}$ and $\eta \in (0,1)$, we obtain

$$\left\| u'_{\eta m}(0) \right\| \leq \|\Delta u_{0m}\| + \|u_{0m}\|_6^3 + \alpha\|u_{0m}\| + \|u_{0m}\|_4\|v_{0m}\|_4 \leq C_5, \tag{3.29}$$

$$\left\| v''_{\eta m}(0) \right\| \leq \|\Delta v_{0m}\| + \mu^2\|v_{0m}\| + \beta\|v_{1m}\| + \|u_{0m}\|_4^2 \leq C_6, \tag{3.30}$$

where C_5 and C_6 are positive constants.

Now by differentiating (3.2) with respect to t and substituting $w = u'_{\eta m}$, $y = v''_{\eta m}$, and $\xi = z''_{\eta m}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\| u'_{\eta m} \right\|^2 + i\left\| u'_{\eta m} \right\|_V^2 + \left(2\left| u_{\eta m} \right|^2 u'_{\eta m} + \left[u_{\eta m} \right]^2 \overline{u'_{\eta m}}, u'_{\eta m} \right) + \alpha\left\| u'_{\eta m} \right\|^2 \\ &\quad = i\left(u'_{\eta m} v_{\eta m} + u_{\eta m} v'_{\eta m}, u'_{\eta m} \right), \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\left\| v''_{\eta m} \right\|^2 + \left\| v'_{\eta m} \right\|_V^2 + \mu^2\left\| v'_{\eta m} \right\|^2 \right] + \beta\left\| v''_{\eta m} \right\|^2 - \left(h z''_{\eta m}, v''_{\eta m} \right)_{\Gamma_1} \\ &\quad = \left(u'_{\eta m}\overline{u_{\eta m}} + u_{\eta m}\overline{u'_{\eta m}}, v''_{\eta m} \right), \end{aligned} \tag{3.32}$$

$$\frac{1}{2}\frac{d}{dt}\left[\eta\left\| z''_{\eta m} \right\|_{2,\Gamma_1}^2 + \int_{\Gamma_1} h(x)g(x)\left(z'_{\eta m} \right)^2 d\Gamma \right] + \gamma\left(h v''_{\eta m}, z''_{\eta m} \right)_{\Gamma_1} + \int_{\Gamma_1} h(x)f(x)\left(z''_{\eta m} \right)^2 d\Gamma = 0. \tag{3.33}$$

Taking the real part in (3.31), we infer

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\| u'_{\eta m} \right\|^2 + 2\int_\Omega \left| u_{\eta m} \right|^2 \left| u'_{\eta m} \right|^2 dx + \operatorname{Re}\int_\Omega \left[u_{\eta m}\overline{u'_{\eta m}} \right]^2 dx + \alpha\left\| u'_{\eta m} \right\|^2 \\ &\quad = \operatorname{Re}\left[i\int_\Omega v'_{\eta m} u_{\eta m}\overline{u'_{\eta m}} dx \right]. \end{aligned} \tag{3.34}$$

Considering the equality

$$2\left[\operatorname{Re}(z_1\overline{z_2})\right]^2 = |z_1|^2|z_2|^2 + \operatorname{Re}\left[(z_1\overline{z_2})^2 \right] \tag{3.35}$$

for all $z_1, z_2 \in \mathbb{C}$, we have

$$\operatorname{Re}\left[\left(u_{\eta m}\overline{u'_{\eta m}} \right)^2 \right] = 2\left[\operatorname{Re}\left(u_{\eta m}\overline{u'_{\eta m}} \right) \right]^2 - \left| u_{\eta m} \right|^2 \left| u'_{\eta m} \right|^2. \tag{3.36}$$

Also,

$$\operatorname{Re}\left[i\int_\Omega v'_{\eta m} u_{\eta m}\overline{u'_{\eta m}} dx \right] \leq \int_\Omega \left| v'_{\eta m} \right| \left| u_{\eta m} \right| \left| u'_{\eta m} \right| dx \leq \frac{1}{2}\left\| v'_{\eta m} \right\|^2 + \frac{1}{2}\int_\Omega \left| u_{\eta m} \right|^2 \left| u'_{\eta m} \right|^2 dx. \tag{3.37}$$

From (3.34)–(3.37), we conclude

$$\frac{1}{2}\frac{d}{dt}\left\|u'_{\eta m}\right\|^2+\frac{1}{2}\int_\Omega |u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx+2\int_\Omega\left[\operatorname{Re}\left(u_{\eta m}\overline{u'_{\eta m}}\right)\right]^2dx+\alpha\left\|u'_{\eta m}\right\|^2\le\frac{1}{2}\left\|v'_{\eta m}\right\|^2. \tag{3.38}$$

On the other hand, we can easily check that

$$u'_{\eta m}\overline{u_{\eta m}}+u_{\eta m}\overline{u'_{\eta m}}=u'_{\eta m}\overline{u_{\eta m}}+\overline{\overline{u_{\eta m}}u'_{\eta m}}=2\operatorname{Re}\left(u_{\eta m}\overline{u'_{\eta m}}\right). \tag{3.39}$$

Therefore,

$$\begin{aligned}\int_\Omega\left(u'_{\eta m}\overline{u_{\eta m}}+u_{\eta m}\overline{u'_{\eta m}}\right)v''_{\eta m}dx &\le 2\int_\Omega\left|\operatorname{Re}\left(u_{\eta m}\overline{u'_{\eta m}}\right)\right|\left|v''_{\eta m}\right|dx\\ &\le 2\int_\Omega|u_{\eta m}|\left|u'_{\eta m}\right|\left|v''_{\eta m}\right|dx\\ &\le \frac{1}{4}\int_\Omega|u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx+4\|v''_{\eta m}\|^2\\ &\le \frac{1}{4}\int_{|u_{\eta m}|\le 1}|u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx+\frac{1}{4}\int_{|u_{\eta m}|>1}|u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx\\ &\quad+4\left\|v''_{\eta m}\right\|^2\\ &\le\frac{1}{4}\left\|u'_{\eta m}\right\|^2+\frac{1}{4}\int_\Omega|u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx+4\left\|v''_{\eta m}\right\|^2.\end{aligned} \tag{3.40}$$

Replacing (3.40) in (3.32), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|v''_{\eta m}\right\|^2+\left\|v'_{\eta m}\right\|_V^2+\mu^2\left\|v'_{\eta m}\right\|^2\right)+\beta\left\|v''_{\eta m}\right\|^2-\left(hz''_{\eta m},v''_{\eta m}\right)_{\Gamma_1}\\ &\quad\le\frac{1}{4}\left\|u'_{\eta m}\right\|^2+\frac{1}{4}\int_\Omega|u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx+4\left\|v''_{\eta m}\right\|^2.\end{aligned} \tag{3.41}$$

Combining (3.33), (3.38), and (3.41), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left\|u'_{\eta m}\right\|^2+\left\|v''_{\eta m}\right\|^2+\left\|v'_{\eta m}\right\|_V^2+\mu^2\left\|v'_{\eta m}\right\|^2+\frac{\eta}{\gamma}\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2+\frac{1}{\gamma}\int_{\Gamma_1}h(x)g(x)\left(z'_{\eta m}\right)^2d\Gamma\right]\\ &\quad+\frac{1}{4}\int_\Omega|u_{\eta m}|^2\left|u'_{\eta m}\right|^2dx+2\int_\Omega\left[\operatorname{Re}\left(u_{\eta m}\overline{u'_{\eta m}}\right)\right]^2dx+\frac{1}{\gamma}\int_{\Gamma_1}h(x)f(x)\left(z''_{\eta m}\right)^2d\Gamma\\ &\quad\le\left(\frac{1}{4}-\alpha\right)\left\|u'_{\eta m}\right\|^2+\frac{1}{2}\left\|v'_{\eta m}\right\|^2+(4-\beta)\left\|v''_{\eta m}\right\|^2.\end{aligned} \tag{3.42}$$

Integrating (3.42) from 0 to t, we have

$$\begin{aligned}
&\left\|u'_{\eta m}\right\|^2+\left\|v''_{\eta m}\right\|^2+\left\|v'_{\eta m}\right\|_V^2+\mu^2\left\|v'_{\eta m}\right\|^2+\frac{\eta}{\gamma}\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2\\
&\quad+\frac{1}{\gamma}\int_{\Gamma_1}h(x)g(x)\left(z'_{\eta m}\right)^2d\Gamma+\frac{2}{\gamma}\int_0^t\int_{\Gamma_1}h(x)f(x)\left(z''_{\eta m}\right)^2d\Gamma ds\\
&\qquad\leq\left\|u'_{\eta m}(0)\right\|^2+\left\|v''_{\eta m}(0)\right\|^2+\|v_{1m}\|_V^2+\mu^2\|v_{1m}\|^2+\frac{\eta}{\gamma}\left\|z''_{\eta m}(0)\right\|_{2,\Gamma_1}^2+\frac{1}{\gamma}\int_{\Gamma_1}h(x)g(x)(z_{1m})^2d\Gamma\\
&\quad+2\left(\frac{1}{4}-\alpha\right)\int_0^t\left\|u'_{\eta m}(s)\right\|^2ds+\int_0^t\left\|v'_{\eta m}(s)\right\|^2ds+2(4-\beta)\int_0^t\left\|v''_{\eta m}(s)\right\|^2ds.
\end{aligned}\tag{3.43}$$

Therefore, using the hypotheses on α and β, (2.7), (3.11), (3.29), (3.30), and Gronwall's lemma, we get

$$\begin{aligned}
&\left\|u'_{\eta m}\right\|^2+\left\|v''_{\eta m}\right\|^2+\left\|v'_{\eta m}\right\|_V^2+\left\|v'_{\eta m}\right\|^2+\eta\left\|z''_{\eta m}\right\|_{2,\Gamma_1}^2\\
&\quad+\left\|h^{1/2}z'_{\eta m}\right\|_{2,\Gamma_1}^2+\int_0^t\int_{\Gamma_1}h(x)f(x)\left(z''_{\eta m}\right)^2d\Gamma ds\leq C_7,
\end{aligned}\tag{3.44}$$

where C_7 is a positive constant which is independent of m, η, and t.

According to (3.9), (3.27), and (3.44), we obtain that

$$\{u_{\eta m}\}\text{ is bounded in }L^\infty(0,T;V),\tag{3.45}$$

$$\left\{u'_{\eta m}\right\}\text{ is bounded in }L^\infty\left(0,T;L^2(\Omega)\right),\tag{3.46}$$

$$\{v_{\eta m}\}\text{ is bounded in }L^\infty\left(0,T;V\cap H^2(\Omega)\right),\tag{3.47}$$

$$\left\{v'_{\eta m}\right\}\text{ is bounded in }L^\infty(0,T;V),\tag{3.48}$$

$$\left\{v''_{\eta m}\right\}\text{ is bounded in }L^\infty\left(0,T;L^2(\Omega)\right),\tag{3.49}$$

$$\left\{h^{1/2}z_{\eta m}\right\}\text{ is bounded in }L^\infty\left(0,T;L^2(\Gamma_1)\right),\tag{3.50}$$

$$\left\{h^{1/2}z'_{\eta m}\right\}\text{ is bounded in }L^\infty\left(0,T;L^2(\Gamma_1)\right),\tag{3.51}$$

$$\left\{h^{1/2}z''_{\eta m}\right\}\text{ is bounded in }L^2\left(0,T;L^2(\Gamma_1)\right),\tag{3.52}$$

$$\lim_{\substack{m\to\infty\\ \eta\to 0}}\eta\left\|z''_{\eta m}(t)\right\|_{L^2(\Gamma_1)}=0\quad\text{a.e. in }[0,T].\tag{3.53}$$

From (3.45)–(3.52), there exist subsequences $\{u_{\eta m}\}$, $\{v_{\eta m}\}$, and $\{z_{\eta m}\}$, which we still denote by $\{u_{\eta m}\}$, $\{v_{\eta m}\}$, and $\{z_{\eta m}\}$, respectively, such that

$$u_{\eta m} \longrightarrow u_\eta \text{ weak star in } L^\infty(0,T;V), \tag{3.54}$$

$$u'_{\eta m} \longrightarrow u'_\eta \text{ weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \tag{3.55}$$

$$v_{\eta m} \longrightarrow v_\eta \text{ weak star in } L^\infty\Big(0,T;V\cap H^2(\Omega)\Big), \tag{3.56}$$

$$v'_{\eta m} \longrightarrow v'_\eta \text{ weak star in } L^\infty(0,T;V), \tag{3.57}$$

$$v''_{\eta m} \longrightarrow v''_\eta \text{ weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \tag{3.58}$$

$$h^{1/2}z_{\eta m} \longrightarrow h^{1/2}z_\eta \text{ weak star in } L^\infty\Big(0,T;L^2(\Gamma_1)\Big), \tag{3.59}$$

$$h^{1/2}z'_{\eta m} \longrightarrow h^{1/2}z'_\eta \text{ weak star in } L^\infty\Big(0,T;L^2(\Gamma_1)\Big), \tag{3.60}$$

$$h^{1/2}z''_{\eta m} \longrightarrow h^{1/2}z''_\eta \text{ weak star in } L^2\Big(0,T;L^2(\Gamma_1)\Big). \tag{3.61}$$

We can see that (3.9), (3.27), and (3.44) are also independent of η. Therefore, by the same argument as (3.45)–(3.61) used to obtain u_η, v_η, and z_η from $u_{\eta m}$, $v_{\eta m}$, and $z_{\eta m}$, respectively, we can pass to the limit when $\eta \to 0$ in u_η, v_η, and z_η, obtaining functions u, v, and z such that

$$\begin{gathered}
u_\eta \longrightarrow u \text{ weak star in } L^\infty(0,T;V), \\
u'_\eta \longrightarrow u' \text{ weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \\
v_\eta \longrightarrow v \text{ weak star in } L^\infty\Big(0,T;V\cap H^2(\Omega)\Big), \\
v'_\eta \longrightarrow v' \text{ weak star in } L^\infty(0,T;V), \\
v''_\eta \longrightarrow v'' \text{ weak star in } L^\infty\Big(0,T;L^2(\Omega)\Big), \\
h^{1/2}z_\eta \longrightarrow h^{1/2}z \text{ weak star in } L^\infty\Big(0,T;L^2(\Gamma_1)\Big), \\
h^{1/2}z'_\eta \longrightarrow h^{1/2}z' \text{ weak star in } L^\infty\Big(0,T;L^2(\Gamma_1)\Big), \\
h^{1/2}z''_\eta \longrightarrow h^{1/2}z'' \text{ weak star in } L^2\Big(0,T;L^2(\Gamma_1)\Big).
\end{gathered} \tag{3.62}$$

Thus, by the above convergences and (3.53), we can prove the existence of solutions to (1.1) satisfying (2.8).

3.4. Uniqueness

Let (u_1, v_1, z_1) and (u_2, v_2, z_2) be two-solution pair to problem (1.1). Then we put

$$\tilde{u} := u_1 - u_2, \qquad \tilde{v} := v_1 - v_2, \qquad \tilde{z} := z_1 - z_2. \tag{3.63}$$

From (3.2), we have

$$\begin{gathered}(\widetilde{u}', w) + i(\nabla\widetilde{u}, \nabla w) + \left(|u_1|^2 u_1 - |u_2|^2 u_2, w\right) + \alpha(\widetilde{u}, w) = i(u_1 v_1 - u_2 v_2, w), \\ (\widetilde{v}'', y) + (\nabla\widetilde{v}, \nabla y) + \mu^2(\widetilde{v}, y) + \beta(\widetilde{v}', y) - (h\widetilde{z}', \rho_0(\widetilde{v}'))_{\Gamma_1} = \left(|u_1|^2 - |u_2|^2, y\right), \\ \left(h\left[\gamma\rho_0(\widetilde{v}') + f\widetilde{z}' + g\widetilde{z}\right], \xi\right)_{\Gamma_1} = 0,\end{gathered} \tag{3.64}$$

for all $w, y \in V \cap H^2(\Omega)$ and $\xi \in L^2(\Gamma)$. By replacing $w = \widetilde{u}$, $y = \widetilde{v}'$, and $\xi = \widetilde{z}'$ in (3.64), it holds that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|^2 + i\|\widetilde{u}\|_V^2 + \alpha\|\widetilde{u}\|^2 + \left(|u_1|^2 u_1 - |u_2|^2 u_2, \widetilde{u}\right) = i(u_1 v_1 - u_2 v_2, \widetilde{u}), \tag{3.65}$$

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\widetilde{v}'\right\|^2 + \|\widetilde{v}\|_V^2 + \mu^2\|\widetilde{v}\|^2\right] + \beta\left\|\widetilde{v}'\right\|^2 - (h\widetilde{z}', \widetilde{v}')_{\Gamma_1} = \left(|u_1|^2 - |u_2|^2, \widetilde{v}'\right), \tag{3.66}$$

$$\gamma\left(h\widetilde{v}', \widetilde{z}'\right)_{\Gamma_1} + \left(hf\widetilde{z}', \widetilde{z}'\right)_{\Gamma_1} + \left(hg\widetilde{z}, \widetilde{z}'\right)_{\Gamma_1} = 0. \tag{3.67}$$

Taking the real part in (3.65), we get

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|^2 + \alpha\|\widetilde{u}\|^2 + \operatorname{Re}\left[\left(|u_1|^2 u_1 - |u_2|^2 u_2, \widetilde{u}\right)\right] = \operatorname{Re}[i(u_1 v_1 - u_2 v_2, \widetilde{u})]. \tag{3.68}$$

We now estimate the last term on the left-hand side of (3.68) and the term on the right-hand side of (3.68). We can easily check that

$$\int_\Omega \left(|u_1|^2 u_1 - |u_2|^2 u_2\right)(\overline{u_1} - \overline{u_2})dx = \int_\Omega |u_1|^4 + |u_2|^4 - |u_1|^2 u_1\overline{u_2} - |u_2|^2 u_2\overline{u_1}dx. \tag{3.69}$$

By using the fact that $\operatorname{Re}(z_1\overline{z_2}) = \operatorname{Re}(\overline{z_1}z_2)$ for all $z_1, z_2 \in \mathbb{C}$, we obtain

$$\begin{aligned}\operatorname{Re}\left[\left(|u_1|^2 u_1 - |u_2|^2 u_2, \widetilde{u}\right)\right] &= \operatorname{Re}\left[\int_\Omega |u_1|^4 + |u_2|^4 - |u_1|^2 u_1\overline{u_2} - |u_2|^2 u_2\overline{u_1}dx\right] \\ &= \int_\Omega |u_1|^4 + |u_2|^4 - \left(|u_1|^2 + |u_2|^2\right)\operatorname{Re}(u_1\overline{u_2})dx \\ &\geq \int_\Omega \left(|u_1|^3 - |u_2|^3\right)(|u_1| - |u_2|)dx \\ &\geq 0.\end{aligned} \tag{3.70}$$

Also,

$$\int_\Omega (u_1 v_1 - u_2 v_2)\overline{\widetilde{u}}dx = \int_\Omega (u_1(v_1 - v_2) + v_2(u_1 - u_2))\overline{\widetilde{u}}dx = \int_\Omega u_1\overline{\widetilde{u}}\widetilde{v} + |\widetilde{u}|^2 v_2 dx. \tag{3.71}$$

Hence by Hölder's inequality, (3.45), and (3.47), we deduce

$$\begin{aligned}
&\operatorname{Re}[i(u_1v_1 - u_2v_2, \widetilde{u})] \\
&\quad = \operatorname{Re}\left[i\int_\Omega u_1\overline{\widetilde{u}}\widetilde{v} + |\widetilde{u}|^2 v_2 dx\right] = \operatorname{Re}\left[i\int_\Omega u_1\overline{\widetilde{u}}\widetilde{v}dx\right] \leq \|u_1\|_4\|\widetilde{v}\|_4\|\widetilde{u}\| \leq C_8\|\widetilde{v}\|_V^2 + \frac{1}{4}\|\widetilde{u}\|^2,
\end{aligned} \tag{3.72}$$

where C_8 is a positive constant. Replacing (3.70) and (3.72) in (3.68), we have

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|^2 \leq C_8\|\widetilde{v}\|_V^2 + \left(\frac{1}{4} - \alpha\right)\|\widetilde{u}\|^2. \tag{3.73}$$

On the other hand, we can easily check that

$$\begin{aligned}
\left(|u_1|^2 - |u_2|^2, \widetilde{v}'\right) &= \int_\Omega (u_1\overline{u_1} - u_2\overline{u_1} + u_2\overline{u_1} - u_2\overline{u_2})\widetilde{v}'dx \\
&= \int_\Omega \left(\widetilde{u}\overline{u_1} + \overline{\widetilde{u}}u_2\right)\widetilde{v}'dx \\
&\leq \int_\Omega (|u_1| + |u_2|)|\widetilde{u}|\left|\widetilde{v}'\right|dx \\
&\leq C_9\|\widetilde{u}\|^2 + 4\left\|\widetilde{v}'\right\|^2,
\end{aligned} \tag{3.74}$$

where C_9 is a positive constant. Therefore, we can rewrite (3.66) as

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\widetilde{v}'\right\|^2 + \|\widetilde{v}\|_V^2 + \mu^2\|\widetilde{v}\|^2\right] - \left(h\widetilde{z}', \widetilde{v}'\right)_{\Gamma_1} \leq C_9\|\widetilde{u}\|^2 + (4 - \beta)\left\|\widetilde{v}'\right\|^2. \tag{3.75}$$

Adding (3.67), (3.73), and (3.75), we get

$$\frac{1}{2}\frac{d}{dt}\left[\|\widetilde{u}\|^2 + \left\|\widetilde{v}'\right\|^2 + \|\widetilde{v}\|_V^2 + \mu^2\|\widetilde{v}\|^2 + \frac{1}{\gamma}\int_{\Gamma_1} h(x)g(x)(\widetilde{z})^2 d\Gamma\right] \leq C_{10}\left(\|\widetilde{u}\|^2 + \|\widetilde{v}\|_V^2 + \left\|\widetilde{v}'\right\|^2\right). \tag{3.76}$$

Applying Gronwall's lemma, we conclude that $\widetilde{u} = \widetilde{v} = \widetilde{z} = 0$. This completes the proof of existence and uniqueness of solutions for problem (1.1).

4. Uniform Decay

Multiplying the first equation of (1.1) by $\overline{u}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + i\|u\|_V^2 + \|u\|_4^4 + \alpha\|u\|^2 = i\int_\Omega v|u|^2 dx. \tag{4.1}$$

Taking the real part in the above equality, it follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\|u\|_4^4 - \alpha\|u\|^2. \tag{4.2}$$

Now, multiplying the second equation of (1.1) by v' and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left[\|v'\|^2 + \mu^2\|v\|^2\right] - (\Delta v, v') = \int_\Omega v'|u|^2 dx - \beta\|v'\|^2. \tag{4.3}$$

Taking into account $(1.1)_5$ and $(1.1)_6$ (the fifth and sixth equations of (1.1)), we can see that

$$-(\Delta v, v') = \frac{1}{2}\frac{d}{dt}\left[\|v\|_V^2 + \frac{1}{\gamma}\int_{\Gamma_1} h(x)g(x)z^2 d\Gamma\right] + \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma. \tag{4.4}$$

Therefore (4.3) can be rewritten as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\|v'\|^2 + \|v\|_V^2 + \mu^2\|v\|^2 + \frac{1}{\gamma}\int_{\Gamma_1} h(x)g(x)z^2 d\Gamma\right] \\ &= \int_\Omega v'|u|^2 dx - \beta\|v'\|^2 - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma.\end{aligned} \tag{4.5}$$

Adding (4.2) and (4.5), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\|u\|^2 + \|v'\|^2 + \|v\|_V^2 + \mu^2\|v\|^2 + \frac{1}{\gamma}\int_{\Gamma_1} h(x)g(x)z^2 d\Gamma\right] \\ &= \int_\Omega v'|u|^2 dx - \|u\|_4^4 - \alpha\|u\|^2 - \beta\|v'\|^2 - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma \\ &\le -(1-\epsilon)\|u\|_4^4 - \alpha\|u\|^2 - \left(\beta - \frac{1}{4\epsilon}\right)\|v'\|^2 - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma.\end{aligned} \tag{4.6}$$

By choosing $\epsilon = 1/2\beta$ and the hypotheses on β, we get

$$E'(t) \le -\frac{2\beta-1}{2\beta}\|u\|_4^4 - \alpha\|u\|^2 - \frac{\beta}{2}\|v'\|^2 - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma \le 0. \tag{4.7}$$

So we conclude that $E(t)$ is a nonincreasing function.

Now we consider a perturbation of $E(t)$. For each $\epsilon > 0$, we define

$$E_\epsilon(t) = E(t) + \epsilon\rho(t), \tag{4.8}$$

where

$$\rho(t) = \int_\Omega \left(u\overline{u} + 2(m\cdot\nabla v)v' + (n-1)vv'\right)dx \\ + \frac{1}{2\gamma^2}\int_{\Gamma_1}(4g(x)(m\cdot\nu)+\gamma h(x))\left(\gamma vz + \frac{f(x)}{2}z^2\right)d\Gamma. \tag{4.9}$$

By definition of the function $\rho(t)$, Poincare's inequality, and imbedding theorem, we have

$$\begin{aligned}
\left|\rho(t)\right| &\le \|u\|^2 + 2\int_\Omega \left|(m\cdot\nabla v)v'\right|dx + (n-1)\int_\Omega \left|vv'\right|dx + \frac{2}{\gamma}\int_{\Gamma_1}\left|g(x)(m\cdot\nu)vz\right|d\Gamma \\
&\quad + \frac{1}{2}\int_{\Gamma_1}|h(x)vz|d\Gamma + \frac{1}{4\gamma^2}\int_{\Gamma_1}\left|(4g(x)(m\cdot\nu)+\gamma h(x))f(x)z^2\right|d\Gamma \\
&\le \|u\|^2 + R\|v\|_V^2 + R\left\|v'\right\|^2 + \frac{\widehat{c}(n-1)}{2}\|v\|_V^2 + \frac{(n-1)}{2}\left\|v'\right\|^2 \\
&\quad + \frac{\widetilde{c}}{\gamma\lambda_0}\|g\|_{C(\overline{\Gamma}_1)}\|h\|_{C(\overline{\Gamma}_1)}\|v\|_V^2 + \frac{1}{\gamma\lambda_0}\|g\|_{C(\overline{\Gamma}_1)}\int_{\Gamma_1}h(x)z^2d\Gamma + \frac{\widetilde{c}}{4}\|h\|_{C(\overline{\Gamma}_1)}\|v\|_V^2 \\
&\quad + \frac{1}{4}\int_{\Gamma_1}h(x)z^2d\Gamma + \frac{1}{\gamma^2\lambda_0}\|f\|_{C(\overline{\Gamma}_1)}\int_{\Gamma_1}h(x)g(x)z^2d\Gamma + \frac{1}{4\gamma}\|f\|_{C(\overline{\Gamma}_1)}\int_{\Gamma_1}h(x)z^2d\Gamma \\
&\le \Bigg(n+1+4R+\widehat{c}(n-1) + \frac{2\widetilde{c}}{\gamma\lambda_0}\|g\|_{C(\overline{\Gamma}_1)}\|h\|_{C(\overline{\Gamma}_1)} + \frac{2}{\lambda_0 g_0}\|g\|_{C(\overline{\Gamma}_1)} \\
&\quad + \frac{\widetilde{c}}{2}\|h\|_{C(\overline{\Gamma}_1)} + \frac{\gamma}{2g_0} + \frac{2}{\gamma\lambda_0}\|f\|_{C(\overline{\Gamma}_1)} + \frac{1}{2g_0}\|f\|_{C(\overline{\Gamma}_1)}\Bigg)E(t),
\end{aligned} \tag{4.10}$$

where $\widehat{c}$ is a Poincare constant. Hence, from (4.8) and (4.10), there exists a positive constant C_{10} such that

$$|E_\epsilon(t) - E(t)| \le \epsilon C_{10}E(t), \tag{4.11}$$

for all $\epsilon > 0$ and $t \ge 0$. This means that there exist positive constants C_{11} and C_{12} such that

$$C_{11}E(t) \le E_\epsilon(t) \le C_{12}E(t), \quad \forall t \ge 0. \tag{4.12}$$

On the other hand, differentiating $E_\epsilon(t)$, we have

$$E_\epsilon'(t) = E'(t) + \epsilon\rho'(t), \tag{4.13}$$

where

$$\rho'(t) = 2\,\mathrm{Re}\big[(u', u)\big] + 2\int_\Omega (m\cdot\nabla v')v'\,dx + 2\int_\Omega (m\cdot\nabla v)v''\,dx$$
$$+ (n-1)\int_\Omega vv''\,dx + (n-1)\|v'\|^2 + \frac{1}{2\gamma^2}\frac{d}{dt}$$
$$\times\left[\int_{\Gamma_1}(4g(x)(m\cdot\nu) + \gamma h(x))\left(\gamma vz + \frac{f(x)}{2}z^2\right)d\Gamma\right]. \tag{4.14}$$

Now, we will estimate the terms on the right-hand side of (4.14).

Estimates for $I_1 := 2\,\mathrm{Re}[(u', u)]$

Using the first equation of (1.1), we can easily check that

$$2\,\mathrm{Re}\big[(u', u)\big] = 2\,\mathrm{Re}\Big[\Big(i\Delta u - |u|^2u - \alpha u + iuv, u\Big)\Big]$$
$$= 2\,\mathrm{Re}\Big[-i\|u\|_V^2 - \|u\|_4^4 - \alpha\|u\|^2 + i(uv, u)\Big] \tag{4.15}$$
$$= -2\|u\|_4^4 - 2\alpha\|u\|^2.$$

Estimates for $I_2 := 2\int_\Omega (m\cdot\nabla v')v'\,dx$

Applying Green's formula, we deduce

$$2\int_\Omega (m\cdot\nabla v')v'\,dx = \int_{\Gamma_1}(m\cdot\nu)(v')^2\,d\Gamma - n\|v'\|^2. \tag{4.16}$$

Estimates for $I_3 := 2\int_\Omega (m\cdot\nabla v)v''\,dx$

Using the second equation of (1.1) and Young's inequality, we have

$$I_3 = 2\int_\Omega (m\cdot\nabla v)\Big(\Delta v - \mu^2 v - \beta v' + |u|^2\Big)dx$$
$$\le 2\int_\Omega (m\cdot\nabla v)\Delta v\,dx + R\Big(2R\beta^2 + 2R + 1\Big)\mu^2\|v\|_V^2 + R\mu^2\|v\|^2 + \frac{1}{2\mu^2}\|v'\|^2 + \frac{1}{2\mu^2}\|u\|_4^4$$
$$\le \Big(n - 2 + 2R^2\beta^2\mu^2 + 2R^2\mu^2 + R\mu^2\Big)\|v\|_V^2 + R\mu^2\|v\|^2 + \frac{1}{2\mu^2}\|v'\|^2 + \frac{1}{2\mu^2}\|u\|_4^4$$
$$+ 2\int_{\Gamma_1}(m\cdot\nabla v)h(x)z'\,d\Gamma - \int_{\Gamma_1}(m\cdot\nu)|\nabla v|^2\,d\Gamma. \tag{4.17}$$

Estimates for $I_4 := (n-1)\int_\Omega vv''dx$

Similar to estimates for I_3, we have

$$(n-1)\int_\Omega vv''dx \le (n-1)\int_{\Gamma_1} h(x)vz'd\Gamma + (n-1)\left(\frac{\beta^2}{4\eta_1}+\frac{1}{4\eta_2}-1\right)\mu^2\|v\|^2 - (n-1)\|v\|_V^2 + \frac{(n-1)\eta_1}{\mu^2}\|v'\|^2 + \frac{(n-1)\eta_2}{\mu^2}\|u\|_4^4. \tag{4.18}$$

By replacing (4.15)–(4.18) in (4.14) and choosing $\eta_1 = 1/16(n-1)$ and $\eta_2 = 1/2(n-1)$, we conclude that

$$\begin{aligned}
E'_\epsilon(t) &= E'(t) + \epsilon\rho'(t)\\
&\le -\frac{2\beta-1}{2\beta}\|u\|_4^4 - \alpha\|u\|^2 - \frac{\beta}{2}\|v'\|^2 - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma\\
&\quad + \epsilon\Bigg\{\left(-2+\frac{1}{2\mu^2}+\frac{(n-1)\eta_2}{\mu^2}\right)\|u\|_4^4 - 2\alpha\|u\|^2 + \left(-1+\frac{1}{2\mu^2}+\frac{(n-1)\eta_2}{\mu^2}\right)\|v'\|^2\\
&\quad + \left(R\mu^2 - (n-1)\mu^2 + \frac{\beta^2(n-1)\mu^2}{4\eta_1} + \frac{(n-1)\mu^2}{4\eta_2}\right)\|v\|^2\\
&\quad + \left(-1 + 2R^2\beta^2\mu^2 + 2R^2\mu^2 + R\mu^2\right)\|v\|_V^2\\
&\quad + \int_{\Gamma_1}\left[(m\cdot\nu)(v')^2 + 2(m\cdot\nabla v)h(x)z' - (m\cdot\nu)|\nabla v|^2 + (n-1)h(x)vz'\right]d\Gamma\\
&\quad + \frac{1}{2\gamma^2}\frac{d}{dt}\left[\int_{\Gamma_1}\left(4g(x)(m\cdot\nu)+\gamma h(x)\right)\left(\gamma vz + \frac{f(x)}{2}z^2\right)d\Gamma\right]\Bigg\}\\
&\le -\left(\frac{2\beta-1}{2\beta}-\frac{9\epsilon}{16\mu^2}\right)\|u\|_4^4 - \alpha\|u\|^2 - \left(\frac{\beta}{2}-\frac{\epsilon}{\mu^2}\right)\|v'\|^2 + \epsilon\|v\|^2 - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma\\
&\quad + \epsilon\Bigg\{-2\|u\|_4^4 - 2\alpha\|u\|^2 - \|v'\|^2 + \left(-1 + \left(R-(n-1)+4\beta^2(n-1)^2 + \frac{(n-1)^2}{2}\right)\mu^2\right)\\
&\quad \times\|v\|^2 + \left(-1 + \left(2R^2\beta^2 + 2R^2 + R\right)\mu^2\right)\|v\|_V^2\\
&\quad + \int_{\Gamma_1}\left[(m\cdot\nu)(v')^2 + 2(m\cdot\nabla v)h(x)z' - (m\cdot\nu)|\nabla v|^2 + (n-1)h(x)vz'\right]d\Gamma\\
&\quad + \frac{1}{2\gamma^2}\frac{d}{dt}\left[\int_{\Gamma_1}\left(4g(x)(m\cdot\nu)+\gamma h(x)\right)\left(\gamma vz + \frac{f(x)}{2}z^2\right)d\Gamma\right]\Bigg\}.
\end{aligned} \tag{4.19}$$

From (2.9) and (4.19), we obtain

$$
\begin{aligned}
E_{\epsilon}'(t) \leq{}& -\epsilon C_{13}E(t) - \left(\frac{2\beta-1}{2\beta} - \frac{9\epsilon}{16\mu^2}\right)\|u\|_4^4 - \alpha\|u\|^2 - \left(\frac{\beta}{2} - \frac{\epsilon}{\mu^2}\right)\|v'\|^2 + \epsilon\|v\|^2 \\
& - \frac{1}{\gamma}\int_{\Gamma_1} h(x)f(x)(z')^2 d\Gamma + \epsilon\left\{\int_{\Gamma_1}\left[(m\cdot\nu)(v')^2 + \frac{1}{2\gamma}h(x)g(x)z^2 + 2(m\cdot\nabla v)h(x)z'\right.\right. \\
& \left. + (n-1)h(x)vz' - (m\cdot\nu)|\nabla v|^2\right] d\Gamma + \frac{1}{2\gamma^2}\frac{d}{dt} \\
& \left.\times\left[\int_{\Gamma_1}(4g(x)(m\cdot\nu) + \gamma h(x))\left(\gamma vz + \frac{f(x)}{2}z^2\right)d\Gamma\right]\right\}
\end{aligned}
\tag{4.20}
$$

We now estimate the last term on the right-hand side of (4.20).

Estimates for $I_5 := \int_{\Gamma_1}\left[(m\cdot\nu)(v')^2 + \frac{1}{2\gamma}h(x)g(x)z^2\right] d\Gamma$

From the fifth equation of (1.1), we have

$$
\begin{aligned}
I_5 &= \int_{\Gamma_1}\left[\frac{1}{\gamma^2}(m\cdot\nu)(f(x)z' + g(x)z)^2 + \frac{1}{2\gamma}h(x)g(x)z^2\right]d\Gamma \\
&\leq \frac{2}{\gamma^2}\int_{\Gamma_1}(m\cdot\nu)f^2(x)(z')^2 d\Gamma + \frac{1}{2\gamma^2}\int_{\Gamma_1}(4g(x)(m\cdot\nu) + \gamma h(x))g(x)z^2 d\Gamma \\
&= \frac{2}{\gamma^2}\int_{\Gamma_1}(m\cdot\nu)f^2(x)(z')^2 d\Gamma + \frac{1}{2\gamma}\int_{\Gamma_1}(4g(x)(m\cdot\nu) + \gamma h(x))vz^2 d\Gamma \\
&\quad - \frac{1}{2\gamma^2}\frac{d}{dt}\left[\int_{\Gamma_1}(4g(x)(m\cdot\nu) + \gamma h(x))\left(\gamma vz + \frac{f(x)}{2}z^2\right)d\Gamma\right].
\end{aligned}
\tag{4.21}
$$

Estimates for $I_6 := 2\int_{\Gamma_1}(m\cdot\nabla v)h(x)z' d\Gamma$

By Young's inequality, we have

$$
I_6 \leq R\int_{\Gamma_1}\left(\eta_3 h(x)|\nabla v|^2 + \frac{h(x)}{\eta_3}|z'|^2\right)d\Gamma, \tag{4.22}
$$

where η_3 is an arbitrary positive constant. By replacing (4.21) and (4.22) in (4.20), we get

$$
\begin{aligned}
E'_\epsilon(t) \le& -\epsilon C_{13} E(t) - \left(\frac{2\beta-1}{2\beta} - \frac{9\epsilon}{16\mu^2}\right)\|u\|_4^4 - \alpha\|u\|^2 - \left(\frac{\beta}{2} - \frac{\epsilon}{\mu^2}\right)\left\|v'\right\|^2 + \epsilon\|v\|^2 \\
&- \frac{1}{\gamma}\int_{\Gamma_1}\left(f(x) - \frac{R\gamma\epsilon}{\eta_3}\right)h(x)\left(z'\right)^2 d\Gamma \\
&+ \epsilon\left\{\frac{2}{\gamma^2}\int_{\Gamma_1}(m\cdot\nu)f^2(x)\left(z'\right)^2 d\Gamma \right.\\
&\left. + \frac{1}{2\gamma}\int_{\Gamma_1}(4g(x)(m\cdot\nu) + \gamma(2n-1)h(x))vz'd\Gamma - \int_{\Gamma_1}((m\cdot\nu) - R\eta_3 h(x))|\nabla v|^2 d\Gamma\right\}.
\end{aligned}
\tag{4.23}
$$

We note that

$$
\begin{aligned}
\int_{\Gamma_1}(4g(x)(m\cdot\nu)+\gamma(2n-1)h(x))vz'd\Gamma \le& 2M\tilde{c}\eta_3 E(t) \\
&+ \frac{1}{4\eta_3}\int_{\Gamma_1}(m\cdot\nu)(4g(x) + \gamma(2n-1)\lambda(x))\left(z'\right)^2 d\Gamma,
\end{aligned}
\tag{4.24}
$$

where $M = \max_{x\in\overline{\Gamma}_1}\{4g(x)(m\cdot\nu) + \gamma(2n-1)h(x)\}$. By the above inequality and choosing η_3 such that

$$
C_{13} - \frac{M\tilde{c}\eta_3}{\gamma} > 0, \qquad 1 - R\lambda\eta_3 > 0,
\tag{4.25}
$$

we conclude that

$$
\begin{aligned}
E'_\epsilon(t) \le& -\epsilon C_{14} E(t) - \left(\frac{2\beta-1}{2\beta} - \frac{9\epsilon}{16\mu^2}\right)\|u\|_4^4 - \alpha\|u\|^2 - \left(\frac{\beta}{2} - \frac{\epsilon}{\mu^2}\right)\left\|v'\right\|^2 + \epsilon\|v\|^2 \\
&- \int_{\Gamma_1}(m\cdot\nu)\left[\frac{\lambda(x)f(x)}{\gamma} - \epsilon\left(\frac{8\gamma^2 R + 16\eta_3 f^2(x) + 4\gamma g(x) + \gamma^2(2n-1)\lambda(x)}{8\gamma^2\eta_3}\right)\right]\left(z'\right)^2 d\Gamma,
\end{aligned}
\tag{4.26}
$$

where C_{14} is a positive constant. Now choosing ϵ sufficiently small, we obtain

$$
E'_\epsilon(t) \le -C_{15}E(t) \le -\frac{C_{15}}{C_{12}}E_\epsilon(t),
\tag{4.27}
$$

where C_{15} is a positive constant. Therefore,

$$
E_\epsilon(t) \le C_{12}E(0)\exp\left(-\frac{C_{15}}{C_{12}}t\right).
\tag{4.28}
$$

From (4.12), we have

$$E(t) \leq \frac{C_{12}}{C_{11}} E(0) \exp\left(-\frac{C_{15}}{C_{12}} t\right). \quad (4.29)$$

This implies the proof of Theorem 2.1 is completed.

References

[1] A. Bachelot, "Problème de Cauchy pour des systèmes hyperboliques semilinéaires," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 1, no. 6, pp. 453–478, 1984.

[2] J.-B. Baillon and J. M. Chadam, "The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations," in *Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Jane*, Medeiros, Ed., vol. 30 of *North-Holland Mathematics Studies*, pp. 37–44, North-Holland, Amsterdam, The Netherlands, 1978.

[3] I. Fukuda and M. Tsutsumi, "On coupled Klein-Gordon-Schrödinger equations. I," *Bulletin of Science and Engineering Research Laboratory. Waseda University*, no. 69, pp. 51–62, 1975.

[4] I. Fukuda and M. Tsutsumi, "On coupled Klein-Gordon-Schrödinger equations. II," *Journal of Mathematical Analysis and Applications*, vol. 66, no. 2, pp. 358–378, 1978.

[5] I. Fukuda and M. Tsutsumi, "On the Yukawa-coupled Klein-Gordon-Schrödinger equations in three space dimensions," *Proceedings of the Japan Academy*, vol. 51, no. 6, pp. 402–405, 1975.

[6] V. Bisognin, M. M. Cavalcanti, V. N. D. Cavalcanti, and J. Soriano, "Uniform decay for the coupled Klein-Gordon-Schrödinger equations with locally distributed damping," *Nonlinear Differential Equations and Applications*, vol. 15, no. 1-2, pp. 91–113, 2008.

[7] M. M. Cavalcanti and V. N. D. Cavalcanti, "Global existence and uniform decay for the coupled Klein-Gordon-Schrödinger equations," *Nonlinear Differential Equations and Applications*, vol. 7, no. 3, pp. 285–307, 2000.

[8] J. Colliander, J. Holmer, and N. Tzirakis, "Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems," *Transactions of the American Mathematical Society*, vol. 360, no. 9, pp. 4619–4638, 2008.

[9] C. Miao and G. Xu, "Global solutions of the Klein-Gordon-Schrödinger system with rough data in $\mathbb{R}^{2+1}$," *Journal of Differential Equations*, vol. 227, no. 2, pp. 365–405, 2006.

[10] J. Y. Park and J. A. Kim, "Global existence and uniform decay for the coupled Klein-Gordon-Schrödinger equations with non-linear boundary damping and memory term," *Mathematical Methods in the Applied Sciences*, vol. 29, no. 9, pp. 947–964, 2006.

[11] N. Tzirakis, "The Cauchy problem for the Klein-Gordon-Schrödinger system in low dimensions below the energy space," *Communications in Partial Differential Equations*, vol. 30, no. 4–6, pp. 605–641, 2005.

[12] J. T. Beale, "Spectral properties of an acoustic boundary condition," *Indiana University Mathematics Journal*, vol. 25, no. 9, pp. 895–917, 1976.

[13] J. T. Beale, "Acoustic scattering from locally reacting surfaces," *Indiana University Mathematics Journal*, vol. 26, no. 2, pp. 199–222, 1977.

[14] J. T. Beale and S. I. Rosencrans, "Acoustic boundary conditions," *Bulletin of the American Mathematical Society*, vol. 80, pp. 1276–1278, 1974.

[15] A. T. Cousin, C. L. Frota, and N. A. Larkin, "Global solvability and asymptotic behaviour of a hyperbolic problem with acoustic boundary conditions," *Funkcialaj Ekvacioj*, vol. 44, no. 3, pp. 471–485, 2001.

[16] A. T. Cousin, C. L. Frota, and N. A. Larkin, "On a system of Klein-Gordon type equations with acoustic boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 293, no. 1, pp. 293–309, 2004.

[17] C. L. Frota and J. A. Goldstein, "Some nonlinear wave equations with acoustic boundary conditions," *Journal of Differential Equations*, vol. 164, no. 1, pp. 92–109, 2000.

[18] T. G. Ha and J. Y. Park, "Existence of solutions for the Kirchhoff-type wave equation with memory term and acoustic boundary conditions," *Numerical Functional Analysis and Optimization*, vol. 31, no. 8, pp. 921–935, 2010.

[19] D. Mugnolo, "Abstract wave equations with acoustic boundary conditions," *Mathematische Nachrichten*, vol. 279, no. 3, pp. 299–318, 2006.
[20] J. Y. Park and T. G. Ha, "Well-posedness and uniform decay rates for the Klein-Gordon equation with damping term and acoustic boundary conditions," *Journal of Mathematical Physics*, vol. 50, no. 1, Article ID 013506, 18 pages, 2009.
[21] A. Vicente, "Wave equation with acoustic/memory boundary conditions," *Boletim da Sociedade Paranaense de Matemática*, vol. 27, no. 1, pp. 29–39, 2009.
[22] J. E. Muñoz Rivera and Y. Qin, "Polynomial decay for the energy with an acoustic boundary condition," *Applied Mathematics Letters*, vol. 16, no. 2, pp. 249–256, 2003.
[23] C. L. Frota and N. A. Larkin, "Uniform stabilization for a hyperbolic equation with acoustic boundary conditions in simple connected domains," in *Contributions to Nonlinear Analysis*, vol. 66 of *Progress in Nonlinear Differential Equations and Their Applications*, pp. 297–312, Birkhäuser, Basel, Switzerland, 2006.