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Research Article

On an Inverse Scattering Problem for a Discontinuous Sturm-Liouville Equation with a Spectral Parameter in the Boundary Condition

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An inverse scattering problem is considered for a discontinuous Sturm-Liouville equation on the half-line $[0, \infty)$ with a linear spectral parameter in the boundary condition. The scattering data of the problem are defined and a new fundamental equation is derived, which is different from the classical Marchenko equation. With help of this fundamental equation, in terms of the scattering data, the potential is recovered uniquely.

1. Introduction

We consider inverse scattering problem for the equation

$$-\psi'' + q(x)\psi = \lambda^2 \rho(x)\psi \quad (0 < x < +\infty), \tag{1.1}$$

with the boundary condition

$$-\left(\alpha_1 \psi(0) - \alpha_2 \psi'(0)\right) = \lambda^2 \left(\beta_1 \psi(0) - \beta_2 \psi'(0)\right), \tag{1.2}$$

where λ is a spectral parameter, $q(x)$ is a real-valued function satisfying the condition

$$\int_0^{+\infty} (1 + x)\left|q(x)\right| dx < \infty, \tag{1.3}$$

$\rho(x)$ is a positive piecewise-constant function with a finite number of points of discontinuity, α_i, β_i $(i = 1, 2)$ are real numbers, and $\gamma = \alpha_1 \beta_2 - \alpha_2 \beta_1 > 0$.

The aim of the present paper is to investigate the direct and inverse scattering problem on the half-line $[0,\infty)$ for the boundary value problem (1.1)–(1.3). In the case $\rho(x) \equiv 1$, the inverse problem of scattering theory for (1.1) with boundary condition not containing spectral parameter was completely solved by Marchenko [1, 2], Levitan [3, 4], Aktosun [5], as well as Aktosun and Weder [6]. The discontinuous version was studied by Gasymov [7] and Darwish [8]. In these papers, solution of inverse scattering problem on the half-line $[0,\infty)$ by using the transformation operator was reduced to solution of two inverse problems on the intervals $[0,a]$ and $[a,\infty)$. In the case $\rho(x) \neq 1$, the inverse scattering problem was solved by Guseĭnov and Pashaev [9] by using the new (nontriangular) representation of Jost solution of (1.1). It turns out that in this case the discontinuity of the function $\rho(x)$ strongly influences the structure of representation of the Jost solution and the fundamental equation of the inverse problem. We note that similar cases do not arise for the system of Dirac equations with discontinuous coefficients in [10]. Uniqueness of the solution of the inverse problem and geophysical application of this problem for (1.1) when $q(x) \equiv 0$ were given by Tihonov [11] and Alimov [12]. Inverse problem for a wave equation with a piecewise-constant coefficient was solved by Lavrent'ev [13]. Direct problem of scattering theory for the boundary value problem (1.1)–(1.3) in the special case was studied in [14].

When $\rho(x) \equiv 1$ in (1.1) with the spectral parameter appearing in the boundary conditions, the inverse problem on the half-line was considered by Pocheykina-Fedotova [15] according to spectral function, by Yurko [16–18] according to Weyl function, and according to scattering data in [19, 20]. This type of boundary condition arises from a varied assortment of physical problems and other applied problems such as the study of heat conduction by Cohen [21] and wave equation by Yurko [16, 17]. Spectral analysis of the problem on the half-line was studied by Fulton [22].

Also, physical application of the problem with the linear spectral parameter appearing in the boundary conditions on the finite interval was given by Fulton [23]. We recall that inverse spectral problems in finite interval for Sturm-Liouville operators with linear or nonlinear dependence on the spectral parameter in the boundary conditions were studied by Chernozhukova and Freiling [24], Chugunova [25], Rundell and Sacks [26], Guliyev [27], and other works cited therein.

This paper is organized as follows. In Section 2, the scattering data for the boundary value problem (1.1)–(1.3) are defined. In Section 3, the fundamental equation for the inverse problem is obtained and the continuity of the scattering function is showed. Finally, the uniqueness of solution of the inverse problem is given in Section 4.

For simplicity we assume that in (1.1) the function $\rho(x)$ has a discontinuity point:

$$\rho(x) = \begin{cases} \alpha^2, & 0 \le x < a, \\ 1, & x \ge a, \end{cases} \tag{1.4}$$

where $0 < \alpha \neq 1$.

The function

$$f_0(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^-(x)}, \tag{1.5}$$

is the Jost solution of (1.1) when $q(x) \equiv 0$, where $\mu^{\pm}(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)})$.

It is well known [9] that, for all λ from the closed upper half-plane, (1.1) has a unique Jost solution $f(x,\lambda)$ which satisfies the condition

$$\lim_{x\to+\infty} f(x,\lambda)e^{-i\lambda x} = 1 \tag{1.6}$$

and it can be represented in the form

$$f(x,\lambda) = f_0(x,\lambda) + \int_{\mu^+(x)}^{+\infty} K(x,t)e^{i\lambda t}dt, \tag{1.7}$$

where the kernel $K(x,t)$ satisfies the inequality

$$\int_{\mu^+(x)}^{+\infty} |K(x,t)|dt \le C\left(\exp\left(\int_x^{+\infty} t\left|q(t)dt\right|\right)\right), \quad 0 < C = \text{const}, \tag{1.8}$$

and possesses the following properties:

$$\frac{dK(x,\mu^+(x))}{dx} = -\frac{1}{4\sqrt{\rho(x)}}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)q(x), \tag{1.9}$$

$$\frac{d}{dx}\left\{K\left(x,\mu^-(x)+0\right) - K\left(x,\mu^-(x)-0\right)\right\} = \frac{1}{4\sqrt{\rho(x)}}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)q(x). \tag{1.10}$$

In addition, if $q(x)$ is differentiable, $K(x,t)$ satisfies (a.e.) the equation

$$\rho(x)\frac{\partial^2 K}{\partial t^2} - \frac{\partial^2 K}{\partial x^2} + q(x)K = 0, \quad 0 < x < +\infty,\ t > \mu^+(x). \tag{1.11}$$

Denote that

$$\varphi(\lambda) = \left(\alpha_2+\beta_2\lambda^2\right)f'(0,\lambda) - \left(\alpha_1+\beta_1\lambda^2\right)f(0,\lambda). \tag{1.12}$$

According to Lemma 2.2 in Section 2, the equation $\varphi(\lambda) = 0$ has only a finite number of simple roots in the half-plane $\operatorname{Im}\lambda > 0$; all these roots lie in the imaginary axis. The behavior of this boundary value problem (1.1)–(1.3) is expressed as a self-adjoint eigenvalue problem.

We will call the function

$$S(\lambda) = \frac{(\alpha_2+\beta_2\lambda^2)\overline{f'(0,\lambda)} - (\alpha_1+\beta_1\lambda^2)\overline{f(0,\lambda)}}{(\alpha_2+\beta_2\lambda^2)f'(0,\lambda) - (\alpha_1+\beta_1\lambda^2)f(0,\lambda)} \tag{1.13}$$

the scattering function for the boundary value problem (1.1)–(1.3), where $\overline{f(0,\lambda)}$ denotes the complex conjugate of $f(0,\lambda)$.

We denote by m_k^{-2} *the normalized numbers* for the boundary problem (1.1)–(1.3):

$$m_k^{-2} \equiv \int_0^{+\infty} \rho(x)\left|f(x,i\lambda_k)\right|^2 dx + \frac{1}{\gamma}\left|\beta_2 f'(0,i\lambda_k) - \beta_1 f(0,i\lambda_k)\right|^2, \tag{1.14}$$

where $k = 1,2,\ldots,n$. It turns out that the potential $q(x)$ in the boundary value problem (1.1)–(1.3) is uniquely determined by specifying the set of values $\{S(\lambda),\lambda_k,m_k\}$. The set of values is called the *scattering data* of the boundary value problem (1.1)–(1.3). The inverse scattering problem for boundary value problem (1.1)–(1.3) consists in recovering the coefficient $q(x)$ from the scattering data.

The potential $q(x)$ is constructed by slightly varying the method of Marchenko. Set

$$\begin{gathered} F_0(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[S_0(\lambda) - S(\lambda)]e^{-i\lambda x}d\lambda + \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x}, \\ F(x,y) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)F_0\left(y + \mu^+(x)\right) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)F_0\left(y + \mu^-(x)\right), \end{gathered} \tag{1.15}$$

where

$$S_0(\lambda) = \begin{cases} \dfrac{\overline{f_0(0,\lambda)}}{f_0(0,\lambda)} = e^{-2i\lambda a}\dfrac{1 + \tau e^{-2i\lambda a\alpha}}{e^{-2i\lambda a\alpha} + \tau}, & \text{if } \beta_2 = 0, \\ \dfrac{\overline{f_0'(0,\lambda)}}{f_0'(0,\lambda)} = -e^{-2i\lambda a}\dfrac{1 - \tau e^{-2i\lambda a\alpha}}{e^{-2i\lambda a\alpha} - \tau}, & \text{if } \beta_2 \neq 0, \end{cases} \tag{1.16}$$

and $\tau = (\alpha - 1)/(\alpha + 1)$.

We can write out the integral equation

$$F(x,y) + \int_{\mu^+(x)}^{+\infty} K(x,t)F_0(t+y)dt + K(x,y) + \frac{1 - \sqrt{\rho(x)}}{1 + \sqrt{\rho(x)}}K(x,2a - y) = 0, \tag{1.17}$$

for the unknown function $K(x,t)$. The integral equation is called *the fundamental equation* of the inverse problem of scattering theory for the boundary problem (1.1)–(1.3). The fundamental equation is different from the classic equation of Marchenko and we call the equation the *modified Marchenko equation*. The discontinuity of the function $\rho(x)$ strongly influences the structure of the fundamental equation of the boundary problem (1.1)–(1.3). By Theorem 4.1 in Section 4, the integral equation has a unique solution for every $x \geq 0$. Solving this equation, we find the kernel $K(x,y)$ of the special solution (1.7), and hence according to formula (1.10) it is constructed the potential $q(x)$.

We show that formula (1.7) is valid for (1.1). For this, let us give the algorithm of the proof in [9]. For $f(x,\lambda)$ let us consider the integral equation

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^{+\infty} \Phi(x,t,\lambda)q(t)f(t,\lambda)dt, \tag{1.18}$$

where

$$\Phi(x,t,\lambda) = s_0(t,\lambda)c_0(x,\lambda) - s_0(x,\lambda)c_0(t,\lambda), \tag{1.19}$$

while $s_0(x,\lambda)$ and $c_0(x,\lambda)$ are solutions of (1.1) when $q(x) \equiv 0$, satisfying the initial conditions $c_0(0,\lambda) = s_0'(0,\lambda) = 1$ and $c_0'(0,\lambda) = s_0(0,\lambda) = 0$.

It is not hard to show that the function $\Phi(x,t,\lambda)$ satisfies the formula

$$\Phi(x,t,\lambda) = \int_{-\sigma(x,t)}^{\sigma(x,t)} K_0(x,t,z)e^{i\lambda z}dz, \tag{1.20}$$

where

$$K_0(x,t,z) = \begin{cases} \dfrac{1}{2\alpha}, & |z| \le \sigma(x,t),\ x \le t \le a, \\ \dfrac{1}{4}\left(1+\dfrac{1}{\alpha}\right), & t-a-\alpha(a-x) \le |z| \le \sigma(x,t),\ x \le a \le t, \\ \dfrac{1}{2}, & |z| \le t-a-\alpha(a-x),\ x \le a \le t, \\ \dfrac{1}{2}, & |z| \le \sigma(x,t),\ t \ge x \ge a, \end{cases}$$
$$\sigma(x,t) = \int_x^t \sqrt{\rho(s)}ds = \begin{cases} \alpha(t-x), & x \le t \le a, \\ \alpha(a-x)+t-a, & x \le a \le t, \\ t-x, & a \le x \le t. \end{cases} \tag{1.21}$$

Substituting the expression (1.7) for $f(x,\lambda)$ in the integral equation (1.18) and using formula (1.20) for $\Phi(x,t,\lambda)$ after elementary operations, the following integral equations for the kernel $K(x,t)$ are obtained:

$$\begin{aligned} K(x,t) = {} & \frac{1}{4\alpha}\left(1+\frac{1}{\alpha}\right)\int_{(\alpha x+\alpha a-a+t)/2\alpha}^{a} q(z)dz + \frac{1}{4\alpha}\left(1-\frac{1}{\alpha}\right)\int_{(\alpha x+\alpha a+a-t)/2\alpha}^{a} q(z)\mathrm{d}z \\ & + \frac{1}{4}\left(1+\frac{1}{\alpha}\right)\int_a^{+\infty} q(z)dz - \frac{1}{4}\left(1-\frac{1}{\alpha}\right)\int_a^{(t-\alpha x+\alpha a+a)/2} q(z)dz \\ & + \frac{1}{2\alpha}\int_x^{(\min(t,\alpha)+\alpha a-a)/\alpha} q(z)\int_{t-\alpha(z-x)}^{t+\alpha(z-x)} K(z,s)ds\,dz \\ & - \frac{1}{4}\int_a^{(t+a+\alpha a-\alpha x)/2} q(z)\int_{t+z-a-\alpha a+\alpha x}^{t-z-a-\alpha a+\alpha x} K(z,s)ds\,dz, \end{aligned} \tag{1.22}$$

for $0 < x < a$, $\alpha x - \alpha a + a < t < -\alpha x + \alpha a + a$;

$$
\begin{aligned}
K(x,t) = {} & \frac{1}{4}\left(1+\frac{1}{\alpha}\right)\int_{(t+\alpha x-\alpha a+a)/2}^{+\infty} q(z)dz + \frac{1}{4}\left(1-\frac{1}{\alpha}\right)\int_{(t-\alpha x+\alpha a+a)/2}^{+\infty} q(z)dz \\
& + \frac{1}{2\alpha}\int_{x}^{a} q(z)\int_{t-a(z-x)}^{t+\alpha(z-x)} K(z,s)ds\,dz \\
& - \frac{1}{4}\left(1-\frac{1}{\alpha}\right)\int_{a}^{a+\alpha a-\alpha x} q(z)\int_{t+z-a-\alpha a+\alpha x}^{t-z+a+\alpha a-\alpha x} K(z,s)ds\,dz \\
& + \frac{1}{4}\left(1-\frac{1}{\alpha}\right)\int_{a+\alpha a-\alpha x}^{+\infty} q(z)\int_{t-z+a+\alpha a-\alpha x}^{t+z-a-\alpha a+\alpha x} K(z,s)ds\,dz,
\end{aligned}
\tag{1.23}
$$

for $0 < x < a, t > -\alpha x + \alpha a + a$;

$$
K(x,t) = \frac{1}{2}\int_{(x+t)/2}^{+\infty} q(z)dz + \frac{1}{2}\int_{x}^{+\infty} q(z)dz\int_{t-(z-x)}^{t+(z-x)} K(z,s)ds, \tag{1.24}
$$

for $t \geq x \geq a$.

The solvability of these integral equations is obtained through the method of successive approximations. By using integral equations (1.22)–(1.24) for $K(x,t)$, equalities (1.9), (1.10) are obtained. By substituting the expressions for the functions $f(x,\lambda)$ and $f''(x,\lambda)$ in (1.1), it can be shown that (1.11) holds.

2. The Scattering Data

For real $\lambda \neq 0$, the functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ form a fundamental system of solutions of (1.1) and their Wronskian is computed as $W\{f(x,\lambda), \overline{f(x,\lambda)}\} = 2i\lambda$. Here the Wronskian is defined as $W\{f,g\} = f'g - fg'$.

Let $\omega(x,\lambda)$ be the solution of (1.1) satisfying the initial condition

$$
\omega(0,\lambda) = \alpha_2 + \beta_2\lambda^2, \quad \omega'(0,\lambda) = \alpha_1 + \beta_1\lambda^2. \tag{2.1}
$$

The following assertion is valid.

Lemma 2.1. *The identity*

$$
\frac{2i\lambda\omega(x,\lambda)}{(\alpha_2+\beta_2\lambda^2)f'(0,\lambda) - (\alpha_1+\beta_1\lambda^2)f(0,\lambda)} = \overline{f(x,\lambda)} - S(\lambda)f(x,\lambda) \tag{2.2}
$$

holds for all real $\lambda \neq 0$, where

$$
S(\lambda) = \frac{(\alpha_2+\beta_2\lambda^2)\overline{f'(0,\lambda)} - (\alpha_1+\beta_1\lambda^2)\overline{f(0,\lambda)}}{(\alpha_2+\beta_2\lambda^2)f'(0,\lambda) - (\alpha_1+\beta_1\lambda^2)f(0,\lambda)}. \tag{2.3}
$$

with

$$S(\lambda) = \overline{S(-\lambda)} = [S(-\lambda)]^{-1}. \tag{2.4}$$

The function $S(\lambda)$ is called *the scattering function* of the boundary value problem (1.1)–(1.3).

Lemma 2.2. *The function $\varphi(\lambda)$ may have only a finite number of zeros in the half-plane* $\operatorname{Im}\lambda > 0$. *Moreover, all these zeros are simple and lie in the imaginary axis.*

Proof. Since $\varphi(\lambda) \neq 0$ for all real $\lambda \neq 0$, the point $\lambda = 0$ is the possible real zero of the function $\varphi(\lambda)$. Using the analyticity of the function $\varphi(\lambda)$ in upper half-plane and the properties of solution (1.7) are obtained that zeros of $\varphi(\lambda)$ form at most countable and bounded set having 0 as the only possible limit point.

Now let us show that all zeros of the function $\varphi(\lambda)$ lie on the imaginary axis. Suppose that μ_1 and μ_2 are arbitrary zeros of the function $\varphi(\lambda)$. We consider the following relations:

$$\begin{gathered} -f''(x,\mu_1) + q(x)f(x,\mu_1) = \mu_1^2\rho(x)f(x,\mu_1), \\ -\overline{f''(x,\mu_2)} + q(x)\overline{f(x,\mu_2)} = \overline{\mu_2}^2\rho(x)\overline{f(x,\mu_2)}. \end{gathered} \tag{2.5}$$

Multiplying the first of these relations by $\overline{f(x,\mu_2)}$ and the second by $f(x,\mu_1)$, subtracting the second resulting relation from the first, and integrating the resulting difference from zero to infinity, we obtain

$$\left(\mu_1^2 - \overline{\mu_2}^2\right)\int_0^{+\infty}\rho(x)f(x,\mu_1)\overline{f(x,\mu_2)}dx - W\left\{f(x,\mu_1),\overline{f(x,\mu_2)}\right\}_{x=0} = 0. \tag{2.6}$$

On the other hand, according to the definition of the function $\varphi(\lambda)$, the following relation holds:

$$\varphi(\mu_j) = \left(\alpha_2 + \beta_2\mu_j^2\right)f'(0,\mu_j) - \left(\alpha_1 + \beta_1\mu_j^2\right)f(0,\mu_j) = 0, \quad j = 1,2. \tag{2.7}$$

Therefore,

$$f(x,\mu_j) = \frac{1}{\gamma}[\beta_2 f'(0,\mu_j) - \beta_1 f(0,\mu_j)]\omega(x,\mu_j), \quad j = 1,2. \tag{2.8}$$

This formula yields

$$\begin{aligned} W\left\{f(x,\mu_1),\overline{f(x,\mu_2)}\right\}_{x=0} &= \frac{1}{\gamma}[\beta_2 f'(0,\mu_1) - \beta_1 f(0,\mu_1)] \\ &\quad\times\left[\beta_2\overline{f'(0,\mu_2)} - \beta_1\overline{f(0,\mu_2)}\right]\left(\overline{\mu_2}^2 - \mu_1^2\right). \end{aligned} \tag{2.9}$$

Thus, using (2.6) and (2.9) we have

$$\left(\mu_1^2-\overline{\mu_2}^2\right)\left\{\int_0^{+\infty}\rho(x)f(x,\mu_1)\overline{f(x,\mu_2)}dx+\frac{1}{\gamma}[\beta_2 f'(0,\mu_1)-\beta_1 f(0,\mu_1)]\right.$$
$$\left.\times\left[\overline{\beta_2 f'(0,\mu_2)-\beta_1 f(0,\mu_2)}\right]\right\}=0. \tag{2.10}$$

Here $\rho(x)>0$, $\gamma>0$. In particular, the choice $\mu_2=\mu_1$ at (2.10) implies that $\mu_1^2-\overline{\mu_1}^2=0$, or $\mu_1=i\lambda_1$, where $\lambda_1\geq 0$. Therefore, zeros of the function $\varphi(\lambda)$ can lie only on the imaginary axis. Now, let us now prove that function $\varphi(\lambda)$ has zeros in finite numbers. This is obvious if $\varphi(0)\neq 0$, because, under this assumption, the set of zeros cannot have limit points. In the general case, since we can give an estimate for the distance between the neighboring zeros of the function $\varphi(\lambda)$, it follows that the number of zeros is finite (see [2, page 186]). □

Let

$$m_k^{-2}\equiv\int_0^{+\infty}\rho(x)\left|f(x,i\lambda_k)\right|^2dx+\frac{1}{\gamma}\left|\beta_2 f'(0,i\lambda_k)-\beta_1 f(0,i\lambda_k)\right|^2$$
$$=\frac{1}{2i\mu_k\gamma}\varphi'(i\lambda_k)\left[\beta_2 f'(0,i\lambda_k)-\beta_1 f(0,i\lambda_k)\right],\quad k=1,2,\ldots,n. \tag{2.11}$$

These numbers are called *the normalized numbers* for the boundary problem (1.1)–(1.3).

The collections $\{S(\lambda),\ (-\infty<\lambda<+\infty);\lambda_k;m_k\ (k=1,2,\ldots,n)\}$ are called *the scattering data* of the boundary value problem (1.1)–(1.3). The inverse scattering problem consists in recovering the coefficient $q(x)$ from the scattering data.

3. Fundamental Equation or Modified Marchenko Equation

From (1.9), (1.10), it is clear that in order to determine $q(x)$ it is sufficient to know $K(x,t)$. To derive the fundamental equation for the kernel $K(x,t)$ of the solution (1.7), we use equality (2.2), which was obtained in Lemma 2.1. Substituting expression (1.7) for $f(x,\lambda)$ into this equality, we get

$$\frac{2i\lambda\omega(x,\lambda)}{\varphi(\lambda)}-\overline{f_0(x,\lambda)}+S_0(\lambda)f_0(x,\lambda)$$
$$=\int_{\mu^+(x)}^{+\infty}K(x,t)e^{-i\lambda t}dt+[S_0(\lambda)-S(\lambda)]f_0(x,\lambda) \tag{3.1}$$
$$+\int_{\mu^+(x)}^{+\infty}K(x,t)[S_0(\lambda)-S(\lambda)]e^{i\lambda t}dt-S_0(\lambda)\int_{\mu^+(x)}^{+\infty}K(x,t)e^{-i\lambda t}dt.$$

Multiplying both sides of relation (3.1) by $(1/2\pi)e^{i\lambda y}$ and integrating over λ from $-\infty$ to $+\infty$, for $y > \mu^{+}(x)$ at the right-hand side we get

$$\begin{aligned} &K(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}[S_0(\lambda) - S(\lambda)]f_0(x,\lambda)e^{i\lambda y}d\lambda \\ &\quad + \int_{\mu^{+}(x)}^{+\infty} K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}[S_0(\lambda) - S(\lambda)]e^{i\lambda(t+y)}d\lambda\right\}dt \\ &\quad - \int_{\mu^{+}(x)}^{+\infty} K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_0(\lambda)e^{i\lambda(t+y)}d\lambda\right\}dt. \end{aligned} \tag{3.2}$$

Now we will compute the integral $(1/2\pi)\int_{-\infty}^{+\infty} S_0(\lambda)e^{i\lambda(t+y)}d\lambda$. By elementary transforms we obtain

$$\begin{aligned} S_0(\lambda) &= e^{-2i\lambda a}\frac{(1-\tau^2)e^{2i\lambda a\alpha}}{1+\tau e^{2i\lambda a\alpha}} + \tau e^{-2i\lambda a} \\ &= e^{-2i\lambda a(1-\alpha)}\left(1-\tau^2\right)\sum_{k=0}^{\infty}(-1)^k\tau^k e^{2i\lambda a\alpha k} + \tau e^{-2i\lambda a}, \end{aligned} \tag{3.3}$$

where $\beta_2 = 0$. Thus we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_0(\lambda)e^{i\lambda(t+y)}d\lambda = \left(1-\tau^2\right)\sum_{k=0}^{\infty}(-1)^k\tau^k\delta(t+y-2a(1-\alpha)+2a\alpha k) + \tau\delta(t+y-2a), \tag{3.4}$$

where $\delta(t)$ is the Dirac delta function.

For $\beta_2 \neq 0$, similarly we get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_0(\lambda)e^{i\lambda(t+y)}d\lambda = \left(\tau^2-1\right)\sum_{k=0}^{\infty}(-1)^k\tau^k\delta(t+y-2a(1-\alpha)+2a\alpha k) + \tau\delta(t+y-2a). \tag{3.5}$$

Consequently, (3.2) can be written as

$$\begin{aligned} &K(x,y) + F_S(x,y) + \int_{\mu^{+}(x)}^{+\infty} K(x,t)F_{0S}(t+y)dt - \tau K(x,2a-y) \\ &\quad - \left(1-\tau^2\right)\sum_{k=0}^{\infty}(-1)^k\tau^k K(x,2a(1-\alpha)-2a\alpha k-y), \end{aligned} \tag{3.6}$$

where

$$
\begin{aligned}
F_{0S}(x) &\equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty}[S_0(\lambda)-S(\lambda)]e^{i\lambda x}d\lambda, \\
F_S(x,y) &\equiv \frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)F_{0S}(\mu^+(x)+y) \\
&\quad+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)F_{0S}(\mu^-(x)+y).
\end{aligned}
\tag{3.7}
$$

Let us show that for $y > \mu^+(x)$ the last expression in the sum equals zero. We note that $K(x,z)=0$ for $z<x$. For $y>\mu^+(x)$ we have

$$
2a(1-\alpha)-2a\alpha k-y<\mu^+(x), \quad k=0,1,2,\dots. \tag{3.8}
$$

If $0<x<a$, then $\mu^+(x)=\alpha x-\alpha a+a$, and hence

$$
\begin{aligned}
2a(1-\alpha)-2a\alpha k-y &< 2a-2a\alpha(k+1)-\alpha x+\alpha a-a \\
&= a-a\alpha-2a\alpha k-\alpha x<a(1-\alpha)\le\mu^+(x).
\end{aligned}
\tag{3.9}
$$

If $x\ge a$, then $\mu^+(x)=x$, and hence, for this case, the inequality holds.

Therefore, for $y>\mu^+(x)$ (3.2) takes the form

$$
K(x,y)+F_S(x,y)+\int_{\mu^+(x)}^{+\infty}K(x,t)F_{0S}(t+y)dt+\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}}K(x,2a-y). \tag{3.10}
$$

On the left-hand side of (3.1) with help of Jordan's lemma and the residue theorem and by taking Lemma 2.2 into account for $y>\mu^+(x)$, we obtain

$$
-\sum_{k=1}^{n}\frac{2i\lambda_k\omega(x,i\lambda_k)}{\varphi'(i\lambda_k)}e^{-\lambda_k y}. \tag{3.11}
$$

From the definition of normalized numbers m_k $(k=1,2,\dots,n)$ in (2.11) we have

$$
\begin{aligned}
-\sum_{k=1}^{n}\frac{2i\lambda_k\omega(x,i\lambda_k)e^{-\lambda_k y}}{\varphi'(i\lambda_k)} &= -\sum_{k=1}^{n}\frac{2i\lambda_k e^{-\lambda_k y}f(x,i\lambda_k)}{[\beta_2 f'(0,i\lambda_k)-\beta_1 f(0,i\lambda_k)]\varphi'(i\lambda_k)} \\
&= -\sum_{k=1}^{n}m_k^2 f(x,i\lambda_k)e^{-\lambda_k y} \\
&= -\sum_{k=1}^{n}m_k^2\left\{f_0(x,i\lambda_k)e^{-\lambda_k(x+y)}+\int_{\mu^+(x)}^{+\infty}K(x,t)e^{-\lambda_k(t+y)}dt\right\}.
\end{aligned}
\tag{3.12}
$$

Thus, for $y > \mu^{+}(x)$, by taking (3.10) and (3.12) into account, from (3.2) we derive the relation

$$-\sum_{k=1}^{n} m_k^2 \left[f_0(x, i\lambda_k) e^{-\lambda_k (x+y)} + \int_{\mu^+(x)}^{+\infty} K(x,t) e^{-\lambda_k (t+y)} dt \right]$$
$$= F_S(x,y) + \int_{\mu^+(x)}^{+\infty} K(x,t) F_{0S}(t+y) dt + K(x,y) + \frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} K(x, 2a-y). \tag{3.13}$$

Consequently, we obtain for $y > \mu^{+}(x)$

$$F(x,y) + \int_{\mu^+(x)}^{+\infty} K(x,t) F_0(t+y) dt + K(x,y) + \frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} K(x, 2a-y) = 0, \tag{3.14}$$

where

$$F_0(x) = F_{0S}(x) + \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [S_0(\lambda) - S(\lambda)] e^{-\lambda_k x} d\lambda + \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x},$$
$$F(x,y) = F_S(x,y) + \sum_{k=1}^{n} m_k^2 f_0(x, i\lambda_k) e^{-\lambda_k (x+y)}$$
$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right) F_0(y + \mu^{+}(x)) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right) F_0(y + \mu^{-}(x)). \tag{3.15}$$

Equation (3.14) is called *the fundamental equation* of the inverse problem of the scattering theory for the boundary problem (1.1)–(1.3). The fundamental equation is different from the classic equation of Marchenko and we call equation (3.14) the *modified Marchenko equation*. The discontinuity of the function $\rho(x)$ strongly influences the structure of the fundamental equation of the boundary problem (1.1)–(1.3).

Thus, we have proved the following theorem.

Theorem 3.1. *For each $x \geq 0$, the kernel $K(x,y)$ of the special solution* (1.7) *satisfies the fundamental equation* (3.14).

By using the fundamental equation it is shown that the scattering function $S(\lambda)$ is continuous at all real points λ and

$$S(0) = \begin{cases} 1, & \text{if } \varphi(0) \neq 0, \\ -1, & \text{if } \varphi(0) = 0. \end{cases} \tag{3.16}$$

It can be shown that $S_0(\lambda) - S(\lambda)$ tends to zero as $|\lambda| \to \infty$ and is the Fourier transform of some function in $L_2(-\infty, +\infty)$.

4. Solvability of the Fundamental Equation

Substituting scattering data into (3.15), we construct $F_0(x)$ and $F(x,y)$. The fundamental equation (3.14) can be written in the more convenient form

$$\begin{aligned} &K\left(x,t+\mu^+(x)\right)+qK\left(x,2a-t-\mu^+(x)\right)+F\left(x,t+\mu^+(x)\right)\\ &\quad+\int_0^{+\infty}K\left(x,\xi+\mu^+(x)\right)F_0\left(\xi+t+2\mu^+(x)\right)d\xi=0,\quad (t>0). \end{aligned} \tag{4.1}$$

We will seek the solution $K(x,t+\mu^+(x))$ of (4.1) for every $x\geq 0$ in the same space $L_1(0,\infty)$.

We consider the operators $\mathbf{F}_{0x}^S, \mathbf{F}_{0x}$ acting in the spaces $L_i(0,\infty)$ $(i=1,2)$, respectively, by the rules

$$\begin{aligned} \mathbf{F}_{0x}^S f&=\int_0^{+\infty}F_{0S}\left(\xi+t+2\mu^+(x)\right)f(\xi)d\xi,\\ \mathbf{F}_{0x}f&=\int_0^{+\infty}F_0\left(\xi+t+2\mu^+(x)\right)f(\xi)d\xi \end{aligned} \tag{4.2}$$

which appear in the fundamental equation.

The operators $\mathbf{F}_{0x}^S, \mathbf{F}_{0x}$ are compact in each space $L_i(0,\infty)$ $(i=1,2)$ for every choice of $\mu^+(x)\geq 0$. The proof of this fact completely repeats the proof of Lemma 3.3.1 which can be found in [2].

Substituting $f(t)\equiv K(x,t+2\mu^+(x))$ into (4.1), we obtain

$$f(t)+\tau\mathbf{T}f(t)+\mathbf{F}_{0x}f(t)+F\left(x,t+\mu^+(x)\right)=0, \tag{4.3}$$

where

$$\mathbf{T}f(t)=f(2a-t). \tag{4.4}$$

In order to prove the solvability of the given fundamental equation, it suffices to verify that the homogenous equation

$$f(t)+\tau\mathbf{T}f(t)+\mathbf{F}_{0x}f(t)=0 \tag{4.5}$$

has no nontrivial solutions in the corresponding space.

From the homogenous equation (4.5) we obtain

$$\tau\mathbf{T}\left(f(t)+\tau\mathbf{T}f(t)\right)+\tau\mathbf{T}\mathbf{F}_{0x}f(t)=0, \tag{4.6}$$

and, since $\mathbf{T}^2=\mathbf{I}$, we have

$$\tau\mathbf{T}f(t)=-\tau^2f(t)-\tau\mathbf{T}\mathbf{F}_{0x}f(t). \tag{4.7}$$

Using this equality in (4.5), we have

$$f(t) + \tau \mathbf{T} f(t) + \mathbf{F}_{0x} f(t) = \left(1 - \tau^2\right) f(t) + (\mathbf{I} - \tau \mathbf{T}) \mathbf{F}_{0x} f(t) = 0, \tag{4.8}$$

or taking $f(t) = \mathbf{f}$, we obtain the equation

$$\mathbf{f} - \frac{1}{1-\tau^2} (\mathbf{I} - \tau \mathbf{T}) \mathbf{F}_{0x} \mathbf{f} = \mathbf{0} \tag{4.9}$$

from which (4.5) is obtained.

Theorem 4.1. *Equation* (4.5) *has a unique solution* $K(x,\cdot) \in L_1(\mu^+(x), \infty)$ *for each fixed* $x \geq 0$.

To prove this theorem we need some of auxiliary lemmas.

Lemma 4.2. *If* $f(t) \in L_1(0,\infty)$ *is a solution of the homogenous equation* (4.5)*, then* $f(t) \in L_\infty(0,\infty)$.

Proof. In fact, the kernel $F_0(\xi + t + 2\mu^+(x))$ of $\mathbf{F}_{0x}$ can be approximated by a bounded function $\Phi(\xi + t + 2\mu^+(x))$, so that $\int_0^{+\infty} |F_0(t) - \Phi(t)| dt < 1$. By rewriting (4.5) in the form

$$\mathbf{f} - \frac{1}{1-\tau^2} (\mathbf{I} - \tau \mathbf{T})(\mathbf{\Phi} - \mathbf{F}_{0x}) \mathbf{f} = -\frac{1}{1-\tau^2} (\mathbf{I} - \tau \mathbf{T}) \mathbf{\Phi} \mathbf{f}, \tag{4.10}$$

we obtain an equation with a bounded function on the right-hand side, where

$$\mathbf{\Phi f} \equiv \mathbf{\Phi} f(t) = \int_0^{+\infty} f(\xi) \Phi\left(\xi + t + 2\mu^+(x)\right) d\xi. \tag{4.11}$$

In the space $L_\infty = L_\infty(0,\infty)$ we get

$$\begin{aligned} \left\| (\mathbf{\Phi} - \mathbf{F}_{0x}) f \right\|_{L_\infty} &\leq \|\mathbf{f}\|_{L_\infty} \int_0^{+\infty} \left| \Phi\left(\xi + t + 2\mu^+(x)\right) - F_0\left(\xi + t + 2\mu^+(x)\right) d\xi \right| \\ &= \|\mathbf{f}\|_{L_\infty} \int_{t+2\mu^+(\xi)}^{+\infty} |\Phi(\xi) - F_0(\xi)| d\xi \leq \|\mathbf{f}\|_{L_\infty}. \end{aligned} \tag{4.12}$$

Hence

$$\left\| \frac{1}{1-\tau^2} (\mathbf{I} - \tau \mathbf{T})(\mathbf{\Phi} - \mathbf{F}_{0x}) \right\|_{L_\infty \to L_\infty} \leq \frac{1+\tau}{1-\tau^2} \|\mathbf{\Phi} - \mathbf{F}_{0x}\|_{L_\infty \to L_\infty} \leq 1. \tag{4.13}$$

Thus, the function on the right-hand side of (4.10) is bounded. Consequently, we have $\mathbf{f} = \phi + \sum_{n=1}^{\infty} \mathbf{B}^n \phi$, where

$$\phi = -\frac{1}{1-\tau^2} (\mathbf{I} - \tau \mathbf{T}) \mathbf{\Phi f}, \qquad \mathbf{B} = \frac{1}{1-\tau^2} (\mathbf{I} - \tau \mathbf{T})(\mathbf{\Phi} - \mathbf{F}_{0x}), \tag{4.14}$$

and the series converges in $L_1(0,\infty)$ as well as in $L_\infty(0,\infty)$; that is, the solution of the homogenous equation (4.5) is bounded. □

Corollary 4.3. *If $f(t) \in L_1(0,\infty)$ is a solution of the homogenous equation* (4.5)*, then $f(t) \in L_2(0,\infty)$.*

Proof. In fact, $f(t) \in L_1(0,\infty) \cap L_\infty(0,\infty) \subset L_2(0,\infty)$. □

Thus, it suffices to investigate (4.5) in the space $L_2(0,\infty)$.

Lemma 4.4. *The operators $\mathbf{I} + \tau\mathbf{T} + \mathbf{F}_{0x}$ acting in $L_2(0,\infty)$ are nonnegative for every $\mu^+(x) \geq 0$:*

$$((\mathbf{I} + \tau\mathbf{T} + \mathbf{F}_{0x})\mathbf{f}, \mathbf{f}) \geq 0, \tag{4.15}$$

and equality is attained if and only if

$$\begin{gathered} \tilde{f}(\lambda) - S(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda) = 0, \quad (-\infty < \lambda < +\infty), \\ \tilde{f}(-i\lambda_k) = 0, \quad (k = 1, 2, \ldots, n), \end{gathered} \tag{4.16}$$

where $\tilde{f}(\lambda)$ is Fourier transform of the function $f(t)$.

Proof. According to definitions of the operators $\mathbf{F}_{0x}$ and $\mathbf{T}$ we get

$$\begin{aligned} ((\mathbf{I} + \tau\mathbf{T} + \mathbf{F}_{0x})\mathbf{f}, \mathbf{f}) &= \|\mathbf{f}\|^2 + \tau\int_0^{+\infty} f(t) f\left(2a - t + 2\mu^+(x)\right)dt \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty}[S_0(\lambda) - S(\lambda)]e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)\overline{\tilde{f}(\lambda)}d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\tilde{f}(\lambda)\right|^2 d\lambda - \frac{1}{2\pi}\int_{-\infty}^{+\infty}S_0(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)\overline{\tilde{f}(\lambda)}d\lambda \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty}[S_0(\lambda) - S(\lambda)]e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)\overline{\tilde{f}(\lambda)}d\lambda + \sum_{k=1}^{n} m_k^2\left|\tilde{f}(-i\lambda_k)\right|^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\tilde{f}(\lambda) - S(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)\right]\overline{\tilde{f}(\lambda)}d\lambda + \sum_{k=1}^{n} m_k^2\left|\tilde{f}(-i\lambda_k)\right|^2. \end{aligned} \tag{4.17}$$

Since $|S(\lambda)e^{2i\lambda\mu^+(x)}| = 1$,

$$\left|\int_{-\infty}^{+\infty} S(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)\overline{\tilde{f}(\lambda)}d\lambda\right|^2 \leq \int_{-\infty}^{+\infty}\left|\tilde{f}(-\lambda)\right|^2 d\lambda \int_{-\infty}^{+\infty}\left|\tilde{f}(\lambda)\right|^2 d\lambda \tag{4.18}$$

by the Cauchy-Bunyakovskii inequality, or, equivalently,

$$\left|\int_{-\infty}^{+\infty} S(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)\overline{\tilde{f}(\lambda)}d\lambda\right| \leq \int_{-\infty}^{+\infty}\left|\tilde{f}(\lambda)\right|^2 d\lambda. \tag{4.19}$$

Therefore, the first term on the right-hand side of formula (4.17) is nonnegative. Since the second term is obviously nonnegative. Inequality (4.16) holds, with equality, if and only if

$$\tilde{f}(-i\lambda_k) = 0 \quad (k = 1, 2, \dots, n),$$
$$\int_{-\infty}^{+\infty} \left\{ \tilde{f}(\lambda) - S(\lambda) e^{2i\lambda\mu^+(x)} \tilde{f}(-\lambda) \right\} \overline{\tilde{f}(\lambda)} d\lambda = 0. \tag{4.20}$$

This shows that the function $z(\lambda) = \tilde{f}(\lambda) - S(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda)$ is orthogonal to $\tilde{f}(\lambda)$ in $L_2(-\infty, +\infty)$. But then

$$\left\| \tilde{f}(\lambda) \right\|^2 = \left\| S(\lambda) e^{2i\lambda\mu^+(x)} \tilde{f}(-\lambda) \right\|^2 = \left\| \tilde{f}(\lambda) - z(\lambda) \right\|^2 = \left\| \tilde{f}(\lambda) \right\|^2 + \|z(\lambda)\|^2, \tag{4.21}$$

which is possible if and only if $z(\lambda) = 0$. Thus, inequality (4.15) holds, with equality for those functions $f(t)$ whose Fourier transform $\tilde{f}(\lambda)$ satisfies conditions (4.16). The lemma is proved. □

With the help of Lemmas 4.2 and 4.4, we obtain the proof of Theorem 4.1. It remains to show that the homogenous equation (4.5) has only the null solution in $L_2(0, \infty)$. But, by Lemma 4.4 the Fourier transform $\tilde{f}(\lambda)$ of any solution $f(t)$ of (4.5) satisfies the identity $\tilde{f}(\lambda) - S(\lambda)e^{2i\lambda\mu^+(x)}\tilde{f}(-\lambda) = 0$. Hence, upon setting $\tilde{\varphi}_h(\lambda) = \tilde{f}(\lambda)e^{-i\lambda x\sqrt{\rho(x)}} \cos \lambda h$, $0 < h < x\sqrt{\rho(x)}$, we get

$$\tilde{\varphi}_h(\lambda) - S(\lambda) e^{2i\lambda a(1-\alpha)} \tilde{\varphi}_h(-\lambda) = 0. \tag{4.22}$$

Since $\tilde{\varphi}_h(\lambda)$ is the Fourier transform of the function

$$\varphi_h(t) = \frac{1}{2} \left[f\left(t + h - x\sqrt{\rho(x)} \right) + f\left(t - h - x\sqrt{\rho(x)} \right) \right], \tag{4.23}$$

which vanishes for $t < x\sqrt{\rho(x)} - h$, identity (4.22) yields

$$\varphi_h(t) + \tau\varphi_h(2a - t) + \int_0^{+\infty} \varphi_h(\xi) F_{os}(\xi + t) d\xi = 0 \tag{4.24}$$

for all $h \in (0, x\sqrt{\rho(x)})$. Therefore, if (4.5) has nonzero solution, (4.24) has infinitely many linear independent solutions $\varphi_h(t)$, which in turn contradicts the compactness of the operator $\mathbf{F}_{0x}$. Hence, $f(t) = 0$.

According to Theorems 3.1 and 4.1 the following result holds.

Theorem 4.5. *The scattering data uniquely determine the boundary value problem* (1.1)–(1.3).

Proof. To form the fundamental equation (3.14), it suffices to know the functions $F_0(x)$ and $F(x, y)$. In turn, to find the functions $F_0(x), F(x, y)$, it suffices to know only the scattering data $\{S(\lambda)\ (-\infty < \lambda < +\infty); \lambda_k, m_k\ (k = 1, 2, \dots, n)\}$. Given the scattering data, we can use formulas (3.15) to construct the functions $F_0(x), F(x,y)$ and write out the fundamental equation (3.14) for the unknown function $K(x,y)$. According to Theorem 4.1, the fundamental equation has a unique solution. Solving this equation, we find the kernel $K(x,y)$ of the special solution (1.7), and hence, according to formulas (1.9)-(1.10), it is constructed the potential $q(x)$. □

Remark 4.6. In the case when $\rho(x)$ is a positive piecewise-constant with a finite number of points of discontinuity, similar results can be obtained.

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