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Research Article

Existence Result for a Class of Nonlinear Elliptic Systems on Punctured Unbounded Domains

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We establish the existence of a nontrivial solution for systems with an arbitrary number of coupled Poisson equations with critical growth in punctured unbounded domains. The proof depends on a generalized linking theorem due to Krysewski and Szulkin, and on a concentration-compactness argument, proved by Frigon and the author. Applications to reaction-diffusion systems with skew gradient structure are also discussed in the last section.

1. Introduction

In this paper, we consider the following systems with an arbitrary number of coupled Poisson equations with critical growth in punctured unbounded domains Ω. More precisely, we study the following system:

$$\begin{aligned}
-\Delta u_1 &= F_{u_1}(x, \mathbf{u}, \mathbf{v}), \\
\vdots \quad & \quad \vdots \quad \vdots \\
-\Delta u_n &= F_{u_n}(x, \mathbf{u}, \mathbf{v}), \\
\Delta v_1 &= F_{v_1}(x, \mathbf{u}, \mathbf{v}), \\
\vdots \quad & \quad \vdots \quad \vdots \\
\Delta v_m &= F_{v_m}(x, \mathbf{u}, \mathbf{v}), \\
(\mathbf{u}, \mathbf{v}) &= 0, \quad \text{on } \partial\Omega,
\end{aligned} \tag{S}$$

where $N \geq 3$, $n, m \geq 1$, $x \in \Omega \subset \mathbb{R}^N$, $u_i, v_j \in D_0^{1,2}(\Omega)$, for all i, j, and where $F(x, \mathbf{u}, \mathbf{v}) \in C^1(\mathbb{R}^N \times \mathbb{R}^{n+m})$, using the following notations: $\mathbf{u} = (u_1, \ldots, u_n)$, and $\mathbf{v} = (v_1, \ldots, v_m)$. We will show in Section 3 that the corresponding variational formulation is given by

$$\varphi(\mathbf{u}, \mathbf{v}) := \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n} |\nabla u_i|^2 - \frac{1}{2} \sum_{j=1}^{m} |\nabla v_j|^2 - F(x, \mathbf{u}, \mathbf{v}) \right) dx. \tag{1.1}$$

The first difficulty to be tackled in order to prove the existence of a solution for the system (S) is due to the fact that the preceding functional has a strong indefinite quadratic part. Therefore the classical min-max results cannot be applied, unlike the generalized linking theorem presented by Kryszewski and Szulkin (cf. [1]).

Let us mention that similar types of problems for systems of two equations on bounded domains were studied in the subcritical growth case by Husholf and van der Vorst [2] using the Indefinite Functional Theorem due to Benci and Rabinowitz [3], and by Felmer and Wang [4] who obtained multiplicity results in using Galerkin type methods. The critical growth case was studied by Husholf et al. [5] where they used a dual formulation due to Clarke and Ekeland [6]. Let us also mention that Zhang and Liu in [7], and Alves et al. in [8], studied elliptic systems with two equations on unbounded domains, while Silva and Xavier [9] showed the existence of multiple solutions for similar systems on smooth bounded domains, where the Laplacian is replaced with the p-Laplacian.

In 2005, Frigon and the author studied in [10] the following system of two coupled Poisson equations with critical growth:

$$\begin{aligned} -\Delta u &= |v|^{2^*-2} v, \\ -\Delta v &= |u|^{2^*-2} u \end{aligned} \tag{1.2}$$

on unbounded punctured domains Ω of the form

$$\Omega := \mathbb{R}^N \setminus E, \tag{1.3}$$

where

$$E := \bigcup_{a \in \mathbb{Z}^N} a + \omega, \tag{1.4}$$

and where ω is a bounded domain with a C^1-boundary, such that $0 \in \omega \subseteq B(0, R)$, $R < 1/2$. Indeed, these domains are invariant under $\mathbb{Z}^N$-translations and have a C^1-boundary. Due to the latter invariance and the periodicity of the function F, the corresponding functional, besides its strong indefinite quadratic part, is also invariant under $\mathbb{Z}^N$-translations. Consequently, the Palais-Smale condition fails at every critical level, and that is a second difficulty to overcome. In the proof of the existence of a solution for the above system, the generalized linking theorem of Krysewki and Szulkin was used to obtain a Palais-Smale sequence, and a concentration-compactness lemma à la Lions also proved in [10] was invoked in order to show the nontriviality of the solution. The same method will be applied to the system (S).

In addition, we would like $F(x,\mathbf{u},\mathbf{v})$ to fulfill the following classical assumptions.

(A_1) The function $F \in C^1(\mathbb{R}^N \times \mathbb{R}^{n+m})$ must be 1-periodic in x_k for each $1 \le k \le N$, and $F(x,\mathbf{0},\mathbf{0}) = 0$, for all $x \in \mathbb{R}^N$.

(A_2) There exists a constant $C > 0$ such that

$$(|\nabla_{\mathbf{u}} F(x,\mathbf{u},\mathbf{v})| + |\nabla_{\mathbf{v}} F(x,\mathbf{u},\mathbf{v})|)(|\mathbf{u}| + |\mathbf{v}|) \le C\left(|\mathbf{u}|^{2^*} + |\mathbf{v}|^{2^*}\right). \tag{1.5}$$

(A_3) There exists $\alpha, \beta > 2$ such that, for every $\mathbf{u}, \mathbf{v} \neq 0$,

$$\frac{1}{\alpha}\nabla_{\mathbf{u}} F(x,\mathbf{u},\mathbf{v}) \cdot \mathbf{u} + \frac{1}{\beta}\nabla_{\mathbf{v}} F(x,\mathbf{u},\mathbf{v}) \cdot \mathbf{v} \ge F(x,\mathbf{u},\mathbf{v}) > 0 \tag{1.6}$$

with

$$\nabla_{\mathbf{u}} F(x,\mathbf{u},\mathbf{v}) \cdot \mathbf{u} > 0, \qquad \nabla_{\mathbf{v}} F(x,\mathbf{u},\mathbf{v}) \cdot \mathbf{v} > 0, \tag{1.7}$$

where $\nabla_{\mathbf{u}} F(x,\mathbf{u},\mathbf{v})$, (resp., $\nabla_{\mathbf{v}} F(x,\mathbf{u},\mathbf{v})$), denotes the gradient of F with respect to the variables u_i, (resp., v_j).

The main result of this paper can now be stated as follows.

Theorem 1.1 (existence of a nontrivial solution). *Let Ω be a punctured domain defined in* (1.3), *and let $F(x,\mathbf{u},\mathbf{v})$ be a function satisfying assumptions* (A_1), (A_2), *and* (A_3). *Then the system* (S) *has a nontrivial solution.*

This paper is organized as follows. In the next section, the key concentration-compactness lemma, and the generalized linking of Krysewski and Szulkin are presented. In Section 3, we show that the functional φ fulfills the assumptions of the Krysewski and Szulkin theorem. The proof of the main theorem is presented in Section 4. Finally, Section 5 is devoted to an application of the main result to reaction-diffusion systems with skew-gradient structure. Existence of steady-states solutions for these systems will be established.

2. Preliminaries

In the sequel, we will study the case $N \ge 3$, and let $2^* := 2N/(N-2)$. Let us define the Hilbert space

$$D^{1,2}\left(\mathbb{R}^N\right) := \left\{u \in L^{2^*}\left(\mathbb{R}^N\right) : \nabla u \in L^2\left(\mathbb{R}^N\right)\right\}, \tag{2.1}$$

endowed with the inner product

$$\int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla v(x) dx, \tag{2.2}$$

and the associated norm noted $\|u\|$. The Sobolev Imbedding Theorem asserts that the imbedding

$$D^{1,2}\left(\mathbb{R}^N\right) \hookrightarrow L^{2^*}\left(\mathbb{R}^N\right) \tag{2.3}$$

is continuous.

For a domain $\Omega \subset \mathbb{R}^N$, we denote by $D_0^{1,2}(\Omega)$ the closure of $D(\Omega)$ in $D^{1,2}(\mathbb{R}^N)$. Obviously $D^{1,2}(\mathbb{R}^N) = D_0^{1,2}(\mathbb{R}^N)$, and $D_0^{1,2}(\Omega) = H_0^1(\Omega)$, provided that the Poincaré Inequality is satisfied.

2.1. Concentration-Compactness Lemmas

Let us recall that the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact because of the action of dilatations, but we have the following well-known result (refer to Wang and Willem [11] for a generalization).

Lemma 2.1. *If $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, then $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.*

The following lemma due to Ramos, Ramos et al. [12] gives sufficient conditions ensuring the convergence to 0 in $L^{2^*}(\mathbb{R}^N)$ of a sequence in $H^1(\mathbb{R}^N)$. This type of results was firstly established by Lions [13] for an exponent $p < 2^*$. See also Colin [14] or [15] for a similar result in a weighted space on a cylindrical domain.

Lemma 2.2. *Let $r > 0$. If (u_n) is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{x\in\mathbb{R}^N}\int_{B(x,r)} |u_n|^{2^*} dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.4}$$

then $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$.

In [10], Frigon and the author proved a similar result for punctured unbounded domains invariant by $\mathbb{Z}^N$-translations. As mentioned earlier, this lemma will be used in order to prove the nontriviality of the weak solution given by the Krysewski-Szulkin's theorem.

Lemma 2.3. *Let $(u_n) \subset D_0^{1,2}(\Omega)$ be a bounded sequence, where Ω is a punctured unbounded domain defined as in* (1.3). *If*

$$\sup_{a\in\mathbb{Z}^N}\int_{B(a,\sqrt{N})} |u_n|^{2^*} dx \longrightarrow 0, \quad \text{when } n \longrightarrow \infty, \tag{2.5}$$

then $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$.

2.2. Kryszewski-Szulkin Linking Theorem

In 1996, Krysewski and Szulkin [1] (interested readers could also refer to [16] for an elegant proof) presented a generalized linking theorem for a suitable functional defined on a Hilbert space $X = Y \oplus Z$ with Y a separable subspace of X which could be infinite dimensional, and $Z := Y^{\perp}$. Let us state a corollary of their result that will be sufficient for our purposes.

Let $P : X \to Y$, $Q : X \to Z$ be the orthogonal projections. Now, let $\rho > r > 0$ and let $z \in Z$ be such that $\|z\| = 1$. Define

$$M := \{u = y + \lambda z : \|u\| \le \rho,\ \lambda \ge 0,\ y \in Y\}, \tag{2.6}$$

$$M_0 := \{u = y + \lambda z : y \in Y,\ (\|u\| = \rho \text{ and } \lambda \ge 0) \text{ or } (\|u\| \le \rho \text{ and } \lambda = 0)\}, \tag{2.7}$$

$$N := \{u \in Z : \|u\| = r\}. \tag{2.8}$$

Theorem 2.4 (see [1]). *Let $\psi \in C^1(X, \mathbb{R})$ be weakly sequentially lower semicontinuous, bounded below and such that ψ' is weakly sequentially continuous. If*

$$\varphi(u) := \frac{\|Qu\|^2}{2} - \frac{\|Pu\|^2}{2} - \psi(u) \tag{2.9}$$

satisfies

$$b := \inf_N \varphi > 0 = \sup_{M_0} \varphi, \qquad d := \sup_M \varphi < \infty, \tag{2.10}$$

then there exists $c \in [b, d]$ and a sequence $(u_n) \subset X$ such that

$$\varphi(u_n) \longrightarrow c, \qquad \varphi'(u_n) \longrightarrow 0. \tag{2.11}$$

3. Existence of a Bounded Palais-Smale Sequence for the System (S)

For the sake of simplicity and readability, we chose to divide the present section into five subsections, each of them dedicated to a specific aspect of the assumptions that must be fulfilled by the functional in order to apply the Krysewski-Szulkin theorem.

3.1. Functional Setting

First of all, we establish some general results. Let Ω be a punctured domain in $\mathbb{R}^N$ defined as in (1.3). Denote by $X := (D_0^{1,2}(\Omega))^n \times (D_0^{1,2}(\Omega))^m$, the Hilbert space endowed with the inner product

$$((\mathbf{u}, \mathbf{v}), (\mathbf{u}_1, \mathbf{v}_1)) := \int_\Omega (\langle \nabla \mathbf{u}, \nabla \mathbf{u}_1 \rangle + \langle \nabla \mathbf{v}, \nabla \mathbf{v}_1 \rangle) dx, \tag{3.1}$$

where

$$\langle \nabla \mathbf{u}, \nabla \mathbf{u}_1 \rangle := \sum_{i=1}^{n} \nabla u_i \cdot \nabla u_{1,i}, \qquad \langle \nabla \mathbf{v}, \nabla \mathbf{v}_1 \rangle := \sum_{i=1}^{m} \nabla v_i \cdot \nabla v_{1,i}, \tag{3.2}$$

and where ∇ stands for the gradient operator with respect to x. In the sequel, the norm induced by the preceding inner product on the space X will be denoted by $\| \ \|$. If we set

$$Y := \{(\mathbf{0},\mathbf{v}) \in X\}, \qquad Z := \{(\mathbf{u},\mathbf{0}) \in X\}, \tag{3.3}$$

then $X = Y \oplus Z$. Let us denote by P (resp., Q) the projection of X onto Y (resp., Z) and let us define the functional $\varphi : X \to \mathbb{R}$ by

$$\begin{aligned}\varphi(\mathbf{u},\mathbf{v}) &:= \int_\Omega \left(\frac{1}{2}(\langle \nabla\mathbf{u},\nabla\mathbf{u}\rangle - \langle \nabla\mathbf{v},\nabla\mathbf{v}\rangle) - F(x,\mathbf{u},\mathbf{v})\right)dx \\ &= \frac{\|Q(\mathbf{u},\mathbf{v})\|^2}{2} - \frac{\|P(\mathbf{u},\mathbf{v})\|^2}{2} - \psi(\mathbf{u},\mathbf{v}),\end{aligned} \tag{3.4}$$

where

$$\psi(\mathbf{u},\mathbf{v}) := \int_\Omega F(x,\mathbf{u},\mathbf{v})dx, \tag{3.5}$$

and where $F(x,\mathbf{u},\mathbf{v})$ satisfies the assumptions (A_1), (A_2), and (A_3). We will see in Section 3.3 (Lemma 3.3) that the system (S) allows a variational formulation since its solutions will correspond to critical points of φ in X.

3.2. *Growth Conditions on the Function* $F(x,\mathbf{u},\mathbf{v})$

The assumptions (A_1), (A_2), and (A_3) have important consequences on the growth of $F(x,\mathbf{u},\mathbf{v})$, that are summarized in the next two lemmas.

Lemma 3.1. *Under assumption (A_2), there exists $C > 0$ such that*

$$|F(x,\mathbf{u},\mathbf{v})| \le C\left(|\mathbf{u}|^{2^*} + |\mathbf{v}|^{2^*}\right). \tag{3.6}$$

Proof. We have

$$\begin{aligned}|F(x,\mathbf{u},\mathbf{v})| &\le \int_0^1 \left|\frac{d}{dt}F(x,t\mathbf{u},t\mathbf{v})\right|dt \\ &\le \int_0^1 |\nabla_\mathbf{u} F(x,t\mathbf{u},t\mathbf{v})\cdot\mathbf{u} + \nabla_\mathbf{v} F(x,t\mathbf{u},t\mathbf{v})\cdot\mathbf{v}|dt \\ &\le \int_0^1 (|\nabla_\mathbf{u} F(x,t\mathbf{u},t\mathbf{v})| + |\nabla_\mathbf{v} F(x,t\mathbf{u},t\mathbf{v})|)(|\mathbf{u}|+|\mathbf{v}|)dt \\ &\le C\int_0^1 \left(|\mathbf{u}|^{2^*} + |\mathbf{v}|^{2^*}\right)t^{2^*-1}\,dt \quad \text{by assumption } (A_2) \\ &\le C\left(|\mathbf{u}|^{2^*} + |\mathbf{v}|^{2^*}\right).\end{aligned} \tag{3.7}$$

□

The next growth condition is almost identical to the one considered by several authors (see e.g., [17, 18] or [19]).

Lemma 3.2. *If* $F(x,\mathbf{u},\mathbf{v})$ *satisfies* (A_1) *and* (A_3) *then there exist constants* $c_1, c_2 > 0$ *such that*

$$F(x,\mathbf{u},\mathbf{v}) \;\geq\; c_1\left(|\mathbf{u}|^{\alpha} + |\mathbf{v}|^{\beta}\right) - c_2, \quad \forall(\mathbf{u},\mathbf{v}) \in \mathbb{R}^{n+m},\ \forall x \in \mathbb{R}^N. \tag{3.8}$$

Proof. Let S be the set given by $S := \{(\mathbf{u},\mathbf{v}) \in \mathbb{R}^{n+m} \mid |\mathbf{u}|^{\alpha} + |\mathbf{v}|^{\beta} = 1\}$, and let us define, for every $(\mathbf{u},\mathbf{v}) \in S$, the function $g : [1,+\infty) \to \mathbb{R}$ as

$$g(s) := F\left(x, s^{1/\alpha}\mathbf{u}, s^{1/\beta}\mathbf{v}\right). \tag{3.9}$$

Next,

$$\begin{aligned}
\frac{dg}{ds}(s) &= \frac{1}{s\alpha}\nabla_{\mathbf{u}}F\left(x, s^{1/\alpha}\mathbf{u}, s^{1/\beta}\mathbf{v}\right)\cdot s^{1/\alpha}\mathbf{u} + \frac{1}{s\beta}\nabla_{\mathbf{v}}F\left(x, s^{1/\alpha}\mathbf{u}, s^{1/\beta}\mathbf{v}\right)\cdot s^{1/\beta}\mathbf{v} \\
&\geq \frac{1}{s}F\left(x, s^{1/\alpha}\mathbf{u}, s^{1/\beta}\mathbf{v}\right) \quad \text{using assumption } (A_3) \\
&= \frac{1}{s}g(s).
\end{aligned} \tag{3.10}$$

Letting $c = \inf\{F(x,\mathbf{u},\mathbf{v}) \mid (u,v) \in S,\ x \in \mathbb{R}^N\}$, we get from the periodicity of F (assumption (A_1)), and from (A_3) that $c > 0$. An integration of inequality (3.10) results in

$$g(s) \;\geq\; cs. \tag{3.11}$$

Now, let us consider $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^{n+m}$ such that $|\mathbf{u}|^{\alpha} + |\mathbf{v}|^{\beta} > 1$. Next, let $s := |\mathbf{u}|^{\alpha} + |\mathbf{v}|^{\beta}$, $\tilde{\mathbf{u}} := s^{-1/\alpha}\mathbf{u}$, and $\tilde{\mathbf{v}} := s^{-1/\beta}\mathbf{v}$, so we have $(\tilde{\mathbf{u}},\tilde{\mathbf{v}}) \in S$. Inequality (3.11) implies that

$$F(x,\mathbf{u},\mathbf{v}) = g(s) \geq c\left(|\mathbf{u}|^{\alpha} + |\mathbf{v}|^{\beta}\right). \tag{3.12}$$

Since by the continuity of F (assumption (A_1)) we have

$$c_2 = \inf\left\{F(x,\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in S,\ x \in [0,1]^N\right\} > 0, \tag{3.13}$$

then there exists $c_1 > 0$ such that

$$F(x,\mathbf{u},\mathbf{v}) \geq c_1\left(|\mathbf{u}|^{\alpha} + |\mathbf{v}|^{\beta}\right) - c_2, \quad \forall x \in [0,1]^N. \tag{3.14}$$

Finally, the result holds for all $x \in \mathbb{R}^N$ because of the periodicity of the function F. □

3.3. Regularity of the Functional φ

Lemma 3.3. *The function* φ *is* C^1. *Moreover, for every* $(\mathbf{u},\mathbf{v}),(\mathbf{w},\mathbf{z})\in X$,

$$\langle \varphi'(\mathbf{u},\mathbf{v}),(\mathbf{w},\mathbf{z})\rangle = \int_\Omega (\langle \nabla\mathbf{u},\nabla\mathbf{w}\rangle - \langle \nabla\mathbf{v},\nabla\mathbf{z}\rangle - (\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{w}+\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{z}))dx. \tag{3.15}$$

Proof. Let $(\mathbf{u},\mathbf{v}),(\mathbf{w},\mathbf{z})\in X$. For $x\in\Omega$ and $|t|\in\,]0,1[$, there exists $\lambda\in\,]0,1[$ such that

$$\begin{aligned}\frac{F(x,\mathbf{u}+t\mathbf{w},\mathbf{v}+t\mathbf{z}) - F(x,\mathbf{u},\mathbf{v})}{t} &= \nabla_{\mathbf{u}}F(x,\mathbf{u}+\lambda t\mathbf{w},\mathbf{v}+\lambda t\mathbf{z})\cdot\mathbf{w}\\ &\quad+\nabla_{\mathbf{v}}F(x,\mathbf{u}+\lambda t\mathbf{w},\mathbf{v}+\lambda t\mathbf{z})\cdot\mathbf{z}.\end{aligned} \tag{3.16}$$

On the other hand, assumption (A_2) implies specific growth conditions on $\nabla_{\mathbf{u}}F$ and $\nabla_{\mathbf{v}}F$ that lead to

$$\begin{aligned}\frac{|F(x,\mathbf{u}+t\mathbf{w},\mathbf{v}+t\mathbf{z}) - F(x,\mathbf{u},\mathbf{v})|}{|t|} &\le C\left(|\mathbf{u}+\lambda t\mathbf{w}|^{2^*-1}+|\mathbf{v}+\lambda t\mathbf{z}|^{2^*-1}\right)(|\mathbf{w}|+|\mathbf{z}|)\\ &\le C(|\mathbf{u}|+|\mathbf{w}|+|\mathbf{v}|+|\mathbf{z}|)^{2^*-1}(|\mathbf{w}|+|\mathbf{z}|).\end{aligned} \tag{3.17}$$

Using the Hölder inequality, we conclude that the term on the right-hand side is in $L^1(\Omega)$, since $|\mathbf{w}|,|\mathbf{z}|\in L^{2^*}(\Omega)$. Hence, the Lebesgue dominated convergence theorem implies that

$$\lim_{t\to 0}\frac{1}{t}\int_\Omega F(x,\mathbf{u}+t\mathbf{w},\mathbf{v}+t\mathbf{z}) - F(x,\mathbf{u},\mathbf{v})dx = \int_\Omega (\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{w}+\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{z})dx. \tag{3.18}$$

Now, assume that $(\mathbf{u}_n,\mathbf{v}_n)\to(\mathbf{u},\mathbf{v})$ in X. From Lemma 2.1 and assumption (A_1), we deduce

$$\begin{aligned}\nabla_{\mathbf{u}}F(x,\mathbf{u}_n(x),\mathbf{v}_n(x)) - \nabla_{\mathbf{u}}F(x,\mathbf{u}(x),\mathbf{v}(x)) &\longrightarrow 0,\\ \nabla_{\mathbf{v}}F(x,\mathbf{u}_n(x),\mathbf{v}_n(x)) - \nabla_{\mathbf{v}}F(x,\mathbf{u}(x),\mathbf{v}(x)) &\longrightarrow 0\end{aligned} \tag{3.19}$$

a.e. in Ω. The continuous embedding $D_0^{1,2}(\Omega)\hookrightarrow L^{2^*}(\Omega)$ and assumption (A_2) imply that

$$\begin{aligned}&\nabla_{\mathbf{u}}F(x,\mathbf{u}_n(x),\mathbf{v}_n(x)) - \nabla_{\mathbf{u}}F(x,\mathbf{u}(x),\mathbf{v}(x)),\\ &\nabla_{\mathbf{v}}F(x,\mathbf{u}_n(x),\mathbf{v}_n(x)) - \nabla_{\mathbf{v}}F(x,\mathbf{u}(x),\mathbf{v}(x))\end{aligned} \tag{3.20}$$

are, respectively, bounded in the spaces $(L^{2^*/(2^*-1)}(\Omega))^n$ and $(L^{2^*/(2^*-1)}(\Omega))^m$

$$\begin{aligned}
&\left|\int_\Omega (\nabla_{\mathbf{u}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{w}+\nabla_{\mathbf{v}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{z}-\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{w}-\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{z})dx\right| \\
&\quad\le \|\mathbf{w}\|_{L^{2^*}}\left(\int_\Omega |\nabla_{\mathbf{u}}F(x,\mathbf{u}_n,\mathbf{v}_n)-\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})|^{2^*/(2^*-1)}dx\right)^{(2^*-1)/2^*} \\
&\qquad+\|\mathbf{z}\|_{L^{2^*}}\left(\int_\Omega |\nabla_{\mathbf{v}}F(x,\mathbf{u}_n,\mathbf{v}_n)-\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})|^{2^*/(2^*-1)}dx\right)^{(2^*-1)/2^*} \\
&\quad\le C\Bigg(\left(\int_\Omega |\nabla_{\mathbf{u}}F(x,\mathbf{u}_n,\mathbf{v}_n)-\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})|^{2^*/(2^*-1)}dx\right)^{(2^*-1)/2^*} \\
&\qquad+\left(\int_\Omega |\nabla_{\mathbf{v}}F(x,\mathbf{u}_n,\mathbf{v}_n)-\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})|^{2^*/(2^*-1)}dx\right)^{(2^*-1)/2^*}\Bigg),
\end{aligned} \tag{3.21}$$

for every $(\mathbf{w},\mathbf{z})\in X$ such that $\|\mathbf{w},\mathbf{z}\|\le 1$ because of the continuous embedding $X\hookrightarrow (L^{2^*}(\Omega))^{n+m}$. Now a direct application of the Lebesgue's Dominated Convergence Theorem gives the continuity of the Gâteau derivative of ψ and hence ψ is C^1.

On the other hand,

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\int_\Omega \frac{1}{2}(\langle\nabla(\mathbf{u}+t\mathbf{w}),\nabla(\mathbf{u}+t\mathbf{w})\rangle-\langle\nabla(\mathbf{v}+t\mathbf{z}),\nabla(\mathbf{v}+t\mathbf{z})\rangle \\
&\qquad -\langle\nabla(\mathbf{u}),\nabla(\mathbf{u})\rangle+\langle\nabla(\mathbf{v}),\nabla(\mathbf{v})\rangle)dx \\
&\quad=((\mathbf{u},0),(\mathbf{w},0))-((0,\mathbf{v}),(0,\mathbf{z})).
\end{aligned} \tag{3.22}$$

This Gâteau derivative is obviously continuous; so φ is C^1 and

$$\begin{aligned}
\langle\varphi'(\mathbf{u},\mathbf{v}),(\mathbf{w},\mathbf{z})\rangle=\int_\Omega\Bigg(&\frac{1}{2}(\langle\nabla\mathbf{u},\nabla\mathbf{w}\rangle-\langle\nabla\mathbf{v},\nabla\mathbf{z}\rangle) \\
&-\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{w}-\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{z}\Bigg)dx.
\end{aligned} \tag{3.23}$$

□

Lemma 3.4. *Under assumptions* (A_1) *and* (A_2)*, the map* ψ *is weakly sequentially lower semicontinuous, while the map* ψ' *is weakly sequentially continuous.*

Proof. Suppose that $(\mathbf{u}_n,\mathbf{v}_n)\rightharpoonup(\mathbf{u},\mathbf{v})$ in X. So, $\{|\mathbf{u}_n|\}$ and $\{|\mathbf{v}_n|\}$ are bounded in $D_0^{1,2}(\Omega)$, and consequently in $L^{2^*}(\Omega)$. Lemma 2.1 implies that $|\mathbf{u}_n|\to|\mathbf{u}|$ and $|\mathbf{v}_n|\to|\mathbf{v}|$ in $L^2_{\text{loc}}(\Omega)$. Going, if necessary to a subsequence, $\mathbf{u}_n\to\mathbf{u}$ and $\mathbf{v}_n\to\mathbf{v}$ a.e. on Ω, thus $F(x,\mathbf{u}_n,\mathbf{v}_n)\to F(x,\mathbf{u},\mathbf{v})$ a.e. on Ω, by the continuity of the function F. Now, the growth condition (3.6) together with the Fatou's lemma implies that

$$\psi(\mathbf{u},\mathbf{v})\le\liminf_{n\to\infty}\psi(\mathbf{u}_n,\mathbf{v}_n). \tag{3.24}$$

On the other hand, reusing the arguments presented in the preceding proof, we have for every $\mathbf{w},\mathbf{z}\in D(\Omega)$,

$$\int_\Omega (\nabla_{\mathbf{u}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{w}+\nabla_{\mathbf{v}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{z})dx \longrightarrow \int_\Omega (\nabla_{\mathbf{u}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{w}+\nabla_{\mathbf{v}}F(x,\mathbf{u},\mathbf{v})\cdot\mathbf{z})dx, \tag{3.25}$$

that is, $\langle\psi'(\mathbf{u}_n,\mathbf{v}_n),(\mathbf{w},\mathbf{z})\rangle \to \langle\psi'(\mathbf{u},\mathbf{v}),(\mathbf{w},\mathbf{z})\rangle$. Moreover, $\{\psi'(\mathbf{u}_n,\mathbf{v}_n)\}$ is bounded in X, so $\psi'(\mathbf{u}_n,\mathbf{v}_n)\rightharpoonup\psi'(\mathbf{u},\mathbf{v})$. □

3.4. The Functional φ Has a Linking Geometry

Choose $(\mathbf{z},\mathbf{0})\in Z$ such that $\|(\mathbf{z},0)\|=1$, and $\mathbf{z}\in(D(B_1(0)))^n$, where $B_1(0)$ is the ball with center at the origin, and radius 1. Let M and M_0 be defined, respectively, by (2.6) and (2.7).

Lemma 3.5. *There exists $r>0$ such that*

$$b:=\inf_{\substack{(\mathbf{u},\mathbf{0})\in Z\\ \|(\mathbf{u},\mathbf{0})\|=r}}\varphi(\mathbf{u},\mathbf{0})>0=\min_{\substack{(\mathbf{u},\mathbf{0})\in Z\\ \|(\mathbf{u},\mathbf{0})\|\le r}}\varphi(\mathbf{u},\mathbf{0}). \tag{3.26}$$

Moreover, there exists $\rho>r$ such that

$$\max_{M_0}\varphi=0,\qquad d:=\sup_{M}\varphi<\infty. \tag{3.27}$$

Proof. The Sobolev Imbedding Theorem of $D_0^{1,2}(\Omega)\hookrightarrow L^{2^*}(\Omega)$ implies directly (3.26) since for $(\mathbf{u},\mathbf{0})\in Z$,

$$\varphi(\mathbf{u},\mathbf{0})\ge\frac{\|(\mathbf{u},\mathbf{0})\|^2}{2}-C\|(\mathbf{u},\mathbf{0})\|^{2^*}. \tag{3.28}$$

Next, observe that on Y we have

$$\varphi(\mathbf{0},\mathbf{v})=\frac{-\|(\mathbf{0},\mathbf{v})\|^2}{2}-\int_{\mathbb{R}^N}F(x,\mathbf{0},\mathbf{v})dx\le 0. \tag{3.29}$$

On the other hand, invoking Lemma 3.2, we have on $Y\oplus\mathbb{R}(\mathbf{z},\mathbf{0})$

$$\begin{aligned}\varphi((\mathbf{0},\mathbf{v})+\lambda(\mathbf{z},\mathbf{0}))&=-\frac{1}{2}\|(\mathbf{0},\mathbf{v})\|^2+\frac{\lambda^2}{2}\|(\mathbf{z},\mathbf{0})\|^2-\int_{\mathbb{R}^N}F(x,\lambda\mathbf{z},\mathbf{v})dx\\ &\le-\frac{1}{2}\|(\mathbf{0},\mathbf{v})\|^2+\frac{\lambda^2}{2}\|(\mathbf{z},\mathbf{0})\|^2-\int_{B_1(0)}F(x,\lambda\mathbf{z},\mathbf{v})dx\\ &\le-\frac{1}{2}\|(\mathbf{0},\mathbf{v})\|^2+\frac{\lambda^2}{2}-\int_{B_1(0)}\left(c_1\left(|\lambda\mathbf{z}|^\alpha+|\mathbf{v}|^\beta\right)-c_2\right)dx.\end{aligned} \tag{3.30}$$

Since the function $\mathbf{z}$ has a compact support in $B_1(0)$, it follows that

$$\varphi(\lambda\mathbf{z},\mathbf{v}) \leq -\frac{1}{2}\|(\mathbf{0},\mathbf{v})\|^2 + \frac{\lambda^2}{2} - K_1\lambda^\alpha - K_2, \tag{3.31}$$

for some positive constants K_1 and K_2. Therefore we have, for $\mathbf{w} \in Y \oplus \mathbb{R}(z,0)$

$$\varphi(\mathbf{w}) \longrightarrow -\infty \quad \text{whenever } \|\mathbf{w}\| \longrightarrow \infty, \tag{3.32}$$

thanks to the inequality $\alpha > 2$. Thus, for some $\rho > r$, $\max_{M_0}\varphi = 0$.

Finally, the Cauchy-Schwarz inequality and the Sobolev inequality imply that φ maps bounded sets into bounded sets, hence $\sup_M \varphi < \infty$. □

3.5. Boundness of the Palais-Smale Sequence

Boundness of the Palais-Smale sequence implies the existence of a limit for a convenient subsequence, with respect to the weak topology.

Lemma 3.6. *There exists $c \in [b,d]$ and a bounded sequence $\{(\mathbf{u}_n,\mathbf{v}_n)\}$ in X such that*

$$\varphi(\mathbf{u}_n,\mathbf{v}_n) \longrightarrow c > 0, \qquad \varphi'(\mathbf{u}_n,\mathbf{v}_n) \longrightarrow 0. \tag{3.33}$$

Proof. It follows from Theorem 2.4 and Lemmas 3.3–3.5 that there exist $c \in [b,d]$ and a sequence $\{(\mathbf{u}_n,\mathbf{v}_n)\}$ in X satisfying (3.33).

Let $\varepsilon > 0$. Observe that for n large enough, assumption (A_3) and (3.33) lead to

$$\begin{aligned} 2c + 1 + \varepsilon\|\mathbf{u}_n,\mathbf{v}_n\| &\geq 2\varphi(\mathbf{u}_n,\mathbf{v}_n) - \langle \varphi'(\mathbf{u}_n,\mathbf{v}_n),(\mathbf{u}_n,\mathbf{v}_n)\rangle \\ &= -2\int_\Omega F(x,\mathbf{u}_n,\mathbf{v}_n)dx + \int_\Omega (\nabla_\mathbf{u}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{u}_n + \nabla_\mathbf{v}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{v}_n)dx \\ &\geq \int_\Omega \left[\left(\frac{\alpha-2}{\alpha}\right)\nabla_\mathbf{u}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{u}_n + \left(\frac{\beta-2}{\beta}\right)\nabla_\mathbf{v}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{v}_n\right]dx. \end{aligned} \tag{3.34}$$

Since $((\alpha-2)/\alpha), ((\beta-2)/\beta) > 0$, there exists $C > 0$ such that

$$2c + 1 + \varepsilon\|(\mathbf{u}_n,\mathbf{v}_n)\| \geq C\int_\Omega (\nabla_\mathbf{u}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{u}_n + \nabla_\mathbf{v}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{v}_n)dx. \tag{3.35}$$

On the other hand,

$$\begin{aligned} \|(\mathbf{u}_n,\mathbf{0})\|^2 - \varepsilon\|(\mathbf{u}_n,\mathbf{0})\| &\leq \|(\mathbf{u}_n,\mathbf{0})\|^2 - \langle \varphi'(\mathbf{u}_n,\mathbf{v}_n),(\mathbf{u}_n,\mathbf{0})\rangle \\ &= \int_\Omega \nabla_\mathbf{u}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{u}_n\,dx. \end{aligned} \tag{3.36}$$

Similarly,

$$\begin{aligned}\|(\mathbf{0},\mathbf{v}_n)\|^2-\varepsilon\|(\mathbf{0},v_n)\| &\le \left|-\|(\mathbf{0},\mathbf{v}_n)\|^2-\langle\varphi'(\mathbf{u}_n,\mathbf{v}_n),(\mathbf{0},\mathbf{v}_n)\rangle\right| \\ &= \int_\Omega \nabla_{\mathbf{v}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{v}_n\,dx.\end{aligned} \tag{3.37}$$

A direct combination of (3.35), (3.36), and (3.37) results in

$$2c+1+(2C+1)\varepsilon\|(\mathbf{u}_n,\mathbf{v}_n)\|\ge C\|(\mathbf{u}_n,\mathbf{v}_n)\|^2, \tag{3.38}$$

so, $\{(\mathbf{u}_n,\mathbf{v}_n)\}$ must be bounded in X. □

4. Proof of the Main Result

We are now ready to establish the existence of a solution to the problem (S).

Proof of the Theorem 1.1. By Lemma 3.6, there exists a bounded sequence $(\mathbf{u}_n,\mathbf{v}_n)\subset X$ satisfying (3.33)

Now, let us assume that

$$\begin{aligned}\delta_1 &:= \limsup_{n\to\infty}\sup_{a\in\mathbb{Z}^N}\int_{B(a,\sqrt{N})}|\mathbf{u}_n|^{2^*}dx=0, \\ \delta_2 &:= \limsup_{n\to\infty}\sup_{a\in\mathbb{Z}^N}\int_{B(a,\sqrt{N})}|\mathbf{v}_n|^{2^*}dx=0,\end{aligned} \tag{4.1}$$

then Lemma 2.3 implies that $|\mathbf{u}_n|,|\mathbf{v}_n|\to 0$ in $L^{2^*}(\mathbb{R}^N)$. On the other hand, for $\epsilon:=\min\{c/3,c/M\}$, where $M>0$ is an upper-bound for $\|(\mathbf{u}_n,\mathbf{v}_n)\|$, and for n large enough, assumption (A_2) implies that

$$\begin{aligned}c-\varepsilon-\frac{\varepsilon}{2}\|(\mathbf{u}_n,\mathbf{v}_n)\| &\le \varphi(\mathbf{u}_n,\mathbf{v}_n)-\frac{1}{2}\langle\varphi'(\mathbf{u}_n,\mathbf{v}_n),(\mathbf{u}_n,\mathbf{v}_n)\rangle \\ &\le \frac{1}{2}\int_\Omega(\nabla_{\mathbf{u}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{u}_n+\nabla_{\mathbf{v}}F(x,\mathbf{u}_n,\mathbf{v}_n)\cdot\mathbf{v}_n)dx \\ &\le C\int_\Omega\left(|\mathbf{u}_n|^{2^*}+|\mathbf{v}_n|^{2^*}\right)dx.\end{aligned} \tag{4.2}$$

But we are now facing a contradiction, since $c>0$. Therefore, we must have $\delta:=\max\{\delta_1,\delta_2\}>0$. Going to a subsequence if needed, we can assume the existence of $a_n\in\mathbb{Z}^N$ such that

$$\int_{B(a_n,\sqrt{N})}\left(|\mathbf{u}_n|^{2^*}+|\mathbf{v}_n|^{2^*}\right)dx>\frac{\delta}{2}. \tag{4.3}$$

The sequence $(\widehat{\mathbf{u}}_n, \widehat{\mathbf{v}}_n)$ defined by $\widehat{\mathbf{u}}_n(x) := \mathbf{u}_n(x + a_n)$ and $\widehat{\mathbf{v}}_n(x) := \mathbf{v}_n(x + a_n)$ is such that

$$\int_{B(0,\sqrt{N})} \left(|\widehat{\mathbf{u}}_n|^{2^*} + |\widehat{\mathbf{v}}_n|^{2^*}\right) dx > \frac{\delta}{2} \tag{4.4}$$

and satisfies (3.33) by $\mathbb{Z}^N$ invariance. Extracting again a subsequence, if needed, we may assume that

$$(\widehat{\mathbf{u}}_n, \widehat{\mathbf{v}}_n) \rightharpoonup (\mathbf{u}, \mathbf{v}) \quad \text{in } X. \tag{4.5}$$

Since $\widehat{\mathbf{u}}_n \to \mathbf{u}$, $\widehat{\mathbf{v}}_n \to \mathbf{v}$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, then $(\mathbf{u}, \mathbf{v}) \neq \mathbf{0}$. Finally, the weakly sequentially continuity of φ' gives

$$\|\varphi'(\mathbf{u}, \mathbf{v})\| \leq \liminf_{n\to\infty} \|\varphi'(\widehat{\mathbf{u}}_n, \widehat{\mathbf{v}}_n)\| = 0. \tag{4.6}$$

Consequently $(\mathbf{u}, \mathbf{v})$ is a nontrivial solution of the system (S). □

Example 4.1. Let $\Omega \subset \mathbb{R}^N$, $N = 3$ *or* 4, *be a punctured domain defined as in* (1.3). *Then there exists a nontrivial solution for the following system of two coupled Poisson equations:*

$$\begin{aligned} -\Delta u &= 2\lambda|v|^{2^*-2}u + (2^* - 2)\gamma|u|^{2^*-4}uv^2, \\ \Delta v &= 2\gamma|u|^{2^*-2}v + (2^* - 2)\lambda|v|^{2^*-4}vu^2, \end{aligned} \qquad u, v \in D_0^{1,2}(\Omega) \tag{4.7}$$

for every $\lambda, \gamma > 0$.

Remark 4.2. In the assumption (A_1), the 1-periodicity condition on F for each variable x_k, could be replaced with any other periodicity condition on every variable. The punctured unbounded domain given by (1.3) has then to be defined accordingly, in order to remain invariant under the corresponding translations. Let us mention that, since the assumptions on the function F are quite general, some components of the solution could be equal to 0.

5. Applications to Reaction-Diffusion Systems with Skew-Gradient Structure

Let us consider the following $(n + m)$-component reaction-diffusion system:

$$\begin{aligned} \mathbf{u}_t &= \Delta\mathbf{u} + f(\mathbf{u}, \mathbf{v}) \quad \text{in } \Omega, \\ \mathbf{v}_t &= \Delta\mathbf{v} + g(\mathbf{u}, \mathbf{v}) \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{v} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

We say that system (5.1) has a *skew-gradient structure* if there exists a C^3-function $H(\mathbf{u}, \mathbf{v})$ such that

$$f(\mathbf{u}, \mathbf{v}) = \nabla_{\mathbf{u}} H(\mathbf{u}, \mathbf{v}), \qquad g(\mathbf{u}, \mathbf{v}) = -\nabla_{\mathbf{v}} H(\mathbf{u}, \mathbf{v}). \tag{5.2}$$

(For more information about reaction-diffusion systems with skew-gradient structure, see for instance [20] (or [21]) and references therein.) Consequently, any steady state solution $(\mathbf{u}, \mathbf{v})$ of (5.1) satisfies the system of Poisson equations below

$$\begin{aligned} \Delta \mathbf{u} + f(\mathbf{u}, \mathbf{v}) &= 0 \quad \text{in } \Omega, \\ \Delta \mathbf{v} + g(\mathbf{u}, \mathbf{v}) &= 0 \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{v} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.3}$$

Corollary 5.1. *Let Ω be an unbounded domain defined as in* (1.3). *Under assumptions* (A_1), (A_2), *and* (A_3), *system* (5.3) *admits a nontrivial solution* $(\mathbf{u}, \mathbf{v}) \in X$*; in addition,* $(\mathbf{u}, \mathbf{v})$ *is a steady-state solution for the reaction-diffusion system* (5.1) *with skew-gradient structure.*

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