Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 236560, 15 pages
doi:10.1155/2010/236560

Research Article

Unbounded Solutions of Second-Order Multipoint Boundary Value Problem on the Half-Line

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Received 14 May 2010; Revised 4 September 2010; Accepted 11 October 2010

Academic Editor: Vicentiu Radulescu

This paper investigates the second-order multipoint boundary value problem on the half-line $u''(t) + f(t,u(t),u'(t)) = 0$, $t \in \mathbb{R}^+$, $\alpha u(0) - \beta u'(0) - \sum_{i=1}^{n} k_i u(\xi_i) = a \geq 0$, $\lim_{t\to+\infty} u'(t) = b > 0$, where $\alpha > 0$, $\beta > 0$, $k_i \geq 0$, $0 \leq \xi_i < \infty$ $(i = 1,2,\ldots,n)$, and $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. We establish sufficient conditions to guarantee the existence of unbounded solution in a special function space by using nonlinear alternative of Leray-Schauder type. Under the condition that f is nonnegative, the existence and uniqueness of unbounded positive solution are obtained based upon the fixed point index theory and Banach contraction mapping principle. Examples are also given to illustrate the main results.

1. Introduction

In this paper, we consider the following second-order multipoint boundary value problem on the half-line

$$u''(t) + f(t,u(t),u'(t)) = 0, \quad t \in \mathbb{R}^+,$$
$$\alpha u(0) - \beta u'(0) - \sum_{i=1}^{n} k_i u(\xi_i) = a \geq 0, \qquad \lim_{t\to+\infty} u'(t) = b > 0, \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$, $k_i \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \infty$, and $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, in which $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{R} = (-\infty,+\infty)$.

The study of multipoint boundary value problems (BVPs) for second-order differential equations was initiated by Bicadze and Samarskiĭ [1] and later continued by Il'in and

Moiseev [2, 3] and Gupta [4]. Since then, great efforts have been devoted to nonlinear multi-point BVPs due to their theoretical challenge and great application potential. Many results on the existence of (positive) solutions for multi-point BVPs have been obtained, and for more details the reader is referred to [5–10] and the references therein. The BVPs on the half-line arise naturally in the study of radial solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium [11–13] and have been also widely studied [14–27]. When $n = 1$, $\beta = 0$, $a = b = 0$, BVP (1.1) reduces to the following three-point BVP on the half-line:

$$\begin{gathered} u''(t) + f(t, u(t), u'(t)) = 0, \quad t \in (0, +\infty), \\ u(0) = \alpha u(\eta), \qquad \lim_{t \to +\infty} u'(t) = 0, \end{gathered} \tag{1.2}$$

where $\alpha \neq 1, \eta \in (0, +\infty)$. Lian and Ge [16] only studied the solvability of BVP (1.2) by the Leray-Schauder continuation theorem. When $k_i = 0$, $i = 1, 2, \ldots, n$, and nonlinearity f is variable separable, BVP (1.1) reduces to the second order two-point BVP on the half-line

$$\begin{gathered} u'' + \Phi(t) f(t, u, u') = 0, \quad t \in (0, +\infty), \\ au(0) - bu'(0) = u_0 \geq 0, \qquad \lim_{t \to +\infty} u'(t) = k > 0. \end{gathered} \tag{1.3}$$

Yan et al. [17] established the results of existence and multiplicity of positive solutions to the BVP (1.3) by using lower and upper solutions technique.

Motivated by the above works, we will study the existence results of unbounded (positive) solution for second order multi-point BVP (1.1). Our main features are as follows. Firstly, BVP (1.1) depends on derivative, and the boundary conditions are more general. Secondly, we will study multi-point BVP on infinite intervals. Thirdly, we will obtain the unbounded (positive) solution to BVP (1.1). Obviously, with the boundary condition in (1.1), if the solution exists, it is unbounded. Hence, we extend and generalize the results of [16, 17] to some degree. The main tools used in this paper are Leray-Schauder nonlinear alternative and the fixed point index theory.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and lemmas. In Section 3, the existence of unbounded solution is established. In Section 4, the existence and uniqueness of positive solution are obtained. Finally, we formulate two examples to illustrate the main results.

2. Preliminaries and Lemmas

Denote $v_0(t) = t + (a/b + \delta)/\Delta$, where $\Delta = \alpha - \sum_{i=1}^{n} k_i \neq 0, \delta = \beta + \sum_{i=1}^{n} k_i \xi_i$. Let

$$E = C^1_\infty(\mathbb{R}^+, \mathbb{R}) = \left\{ x \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to +\infty} \frac{x(t)}{1 + v_0(t)} \text{exists}, \ \lim_{t \to +\infty} x'(t) \text{ exists} \right\}. \tag{2.1}$$

For any $x \in E$, define

$$\|x\|_\infty = \max\left\{ \sup_{t\in\mathbb{R}^+}\left|\frac{x(t)}{1+v_0(t)}\right|,\ \sup_{t\in\mathbb{R}^+}\left|x'(t)\right| \right\}, \tag{2.2}$$

then $E = C^1_\infty(\mathbb{R}^+,\mathbb{R})$ is a Banach space with the norm $\|\cdot\|_\infty$ (see [17]).

The Arzela-Ascoli theorem fails to work in the Banach space E due to the fact that the infinite interval $[0,+\infty)$ is noncompact. The following compactness criterion will help us to resolve this problem.

Lemma 2.1 (see [17]). *Let $M \subset E = C^1_\infty(\mathbb{R}^+,\mathbb{R})$. Then, M is relatively compact in E if the following conditions hold:*

(a) *M is bounded in E;*

(b) *the functions belonging to $\{y : y(t) = x(t)/(1+v_0(t)),\ x \in M\}$ and $\{z : z(t) = x'(t), x \in M\}$ are locally equicontinuous on $\mathbb{R}^+$;*

(c) *the functions from $\{y : y(t) = x(t)/(1+v_0(t)),\ x \in M\}$ and $\{z : z(t) = x'(t),\ x \in M\}$ are equiconvergent, at ∞.*

Throughout the paper we assume the following.

(H_1) Suppose that $f(t,0,0)\not\equiv 0$, $t \in \mathbb{R}^+$, and there exist nonnegative functions $p(t)$, $q(t)$, $r(t) \in L^1[0,+\infty)$ with $tp(t)$, $tq(t)$, $tr(t) \in L^1[0,+\infty)$ such that

$$\left|f(t,(1+v_0(t))u,v)\right| \le p(t)|u| + q(t)|v| + r(t), \quad \text{a.e. } (t,u,v) \in \mathbb{R}^+\times\mathbb{R}\times\mathbb{R}. \tag{2.3}$$

(H_2) $\Delta = \alpha - \sum_{i=1}^n k_i > 0$.

(H_3) $P_1 + Q_1 < 1$, where

$$P_1 = \int_0^{+\infty} p(t)dt, \qquad Q_1 = \int_0^{+\infty} q(t)dt. \tag{2.4}$$

Denote

$$\begin{gathered} P_2 = \int_0^{+\infty}(1+v_0(t))p(t)dt, \qquad Q_2 = \int_0^{+\infty}(1+v_0(t))q(t)dt, \\ R_1 = \int_0^{+\infty} r(t)dt, \qquad R_2 = \int_0^{+\infty}(1+v_0(t))r(t)dt. \end{gathered} \tag{2.5}$$

Lemma 2.2. *Supposing that $\sigma(t) \in L^1[0,+\infty)$ with $t\sigma(t) \in L^1[0,+\infty)$, then BVP*

$$\begin{gathered} u''(t) + \sigma(t) = 0, \quad t \in \mathbb{R}^+, \\ \alpha u(0) - \beta u'(0) - \sum_{i=1}^n k_i u(\xi_i) = a \ge 0, \qquad \lim_{t\to+\infty} u'(t) = b > 0 \end{gathered} \tag{2.6}$$

has a unique solution

$$u(t)=\int_0^{+\infty}G(t,s)\sigma(s)ds+\frac{a+b\delta}{\Delta}+bt,\quad t\in\mathbb{R}^+,\tag{2.7}$$

where

$$G(t,s)=\begin{cases}\dfrac{\beta+\Sigma_{i=1}^{j}k_i\xi_i+\Sigma_{i=j+1}^{n}k_is}{\Delta}+t,\\ \quad t\in\mathbb{R}^+,\ \max\{t,\xi_j\}\le s\le\xi_{j+1},\ j=0,1,2,\dots,n,\\ \dfrac{\beta+\Sigma_{i=1}^{j}k_i\xi_i+\Sigma_{i=j+1}^{n}k_is}{\Delta}+s,\\ \quad t\in\mathbb{R}^+,\ \xi_j\le s\le\min\{t,\xi_{j+1}\},\ j=0,1,2,\dots,n,\end{cases}\tag{2.8}$$

in which $\xi_0=0,\xi_{n+1}=+\infty$, *and* $\sum_{i=m_1}^{m_2}f(i)=0$ *for* $m_2<m_1$.

Proof. Integrating the differential equation from t to $+\infty$, one has

$$u'(t)=b+\int_t^{+\infty}\sigma(s)ds,\quad t\in\mathbb{R}^+\tag{2.9}$$

Then, integrating the above integral equation from 0 to t, noticing that $\sigma(t)\in L^1[0,+\infty)$ and $t\sigma(t)\in L^1[0,+\infty)$, we have

$$u(t)=u(0)+bt+\int_0^t\int_s^{+\infty}\sigma(\tau)d\tau\,ds.\tag{2.10}$$

Since $\alpha u(0)-\beta u'(0)-\sum_{i=1}^{n}k_iu(\xi_i)=a$, it holds that

$$\begin{aligned}u(t)&=\frac{1}{\Delta}\left[a+b\delta+\beta\int_0^{+\infty}\sigma(s)ds+\Sigma_{i=1}^{n}k_i\int_0^{\xi_i}\int_s^{+\infty}\sigma(\tau)\,d\tau ds\right]+bt+\int_0^t\int_s^{+\infty}\sigma(\tau)\,d\tau ds\\&=\frac{1}{\Delta}\left[\beta\int_0^{+\infty}\sigma(s)ds+\Sigma_{i=1}^{n}k_i\int_0^{\xi_i}s\sigma(s)ds+\Sigma_{i=1}^{n}k_i\int_{\xi_i}^{+\infty}\xi_i\sigma(s)\,ds\right]\\&\quad+\int_0^t s\sigma(s)ds+\int_t^{+\infty}t\sigma(s)ds+bt+\frac{a+b\delta}{\Delta}.\end{aligned}\tag{2.11}$$

By using arguments similar to those used to prove Lemma 2.2 in [9], we conclude that (2.7) holds. This completes the proof. □

Now, BVP (1.1) is equivalent to

$$u(t)=\int_0^{+\infty}G(t,s)f\left(s,u(s),u'(s)\right)ds+\frac{a+b\delta}{\Delta}+bt,\qquad t\in\mathbb{R}^+.\tag{2.12}$$

Letting $v(t)=u(t)-bt-((a+b\delta)/\Delta)$, $t\in\mathbb{R}^+$, (2.12) becomes

$$v(t)=\int_0^{+\infty}G(t,s)f\left(s,v(s)+\frac{a+b\delta}{\Delta}+bs,v'(s)+b\right)ds,\quad t\in\mathbb{R}^+.\tag{2.13}$$

For $v\in E$, define operator $A:E\to E$ by

$$Av(t)=\int_0^{+\infty}G(t,s)f\left(s,v(s)+\frac{a+b\delta}{\Delta}+bs,v'(s)+b\right)ds,\quad t\in\mathbb{R}^+.\tag{2.14}$$

Then,

$$(Av)'(t)=\int_t^{+\infty}f\left(s,v(s)+\frac{a+b\delta}{\Delta}+bs,v'(s)+b\right)ds,\quad t\in\mathbb{R}^+.\tag{2.15}$$

Set

$$\gamma(t)=\begin{cases}t+\dfrac{\delta}{\Delta}, & t\in[0,1],\\[2mm] 1+\dfrac{\delta}{\Delta}, & t\in(1,+\infty).\end{cases}\tag{2.16}$$

Remark 2.3. $G(t,s)$ is the Green function for the following associated homogeneous BVP on the half-line:

$$\begin{gathered}u''(t)+f\left(t,u(t),u'(t)\right)=0,\quad t\in\mathbb{R}^+,\\ \alpha u(0)-\beta u'(0)-\sum_{i=1}^{n}k_iu(\xi_i)=0,\qquad \lim_{t\to+\infty}u'(t)=0.\end{gathered}\tag{2.17}$$

It is not difficult to testify that

$$\begin{gathered}\frac{G(t,s)}{\gamma(t)}\geq\frac{G(\tau,s)}{1+v_0(\tau)},\quad \forall t,s,\tau\in\mathbb{R}^+,\\ G(t,s)\leq G(s,s),\quad \frac{G(t,s)}{1+v_0(t)}\leq 1,\quad \forall t,s\in\mathbb{R}^+.\end{gathered}\tag{2.18}$$

Let us first give the following result of completely continuous operator.

Lemma 2.4. *Supposing that* (H_1) *and* (H_2) *hold, then* $A:E\to E$ *is completely continuous.*

Proof. (1) First, we show that $A : E \to E$ is well defined.

For any $v \in E$, there exists $d_1 > 0$ such that $\|v\|_\infty \le d_1$. Then,

$$\begin{aligned}\frac{|Av(t)|}{1+v_0(t)} &\le \int_0^{+\infty} \frac{G(t,s)}{1+v_0(t)}\left|f\left(s, v(s)+\frac{a+b\delta}{\Delta}+bs, v'(s)+b\right)\right|ds \\ &\le \int_0^{+\infty} p(s)\left(\frac{|v(s)|}{1+v_0(s)}+b\right)ds + \int_0^{+\infty} q(s)\left(\left|v'(s)\right|+b\right)ds + \int_0^{\infty} r(s)ds \\ &\le (d_1+b)(P_1+Q_1)+R_1, \quad t\in\mathbb{R}^+,\end{aligned} \tag{2.19}$$

so

$$\sup_{t\in\mathbb{R}^+}\frac{|Av(t)|}{1+v_0(t)} \le (d_1+b)(P_1+Q_1)+R_1. \tag{2.20}$$

Similarly,

$$\begin{aligned}\left|(Av)'(t)\right| &= \left|\int_t^{+\infty} f\left(s, v(s)+\frac{a+b\delta}{\Delta}+bs, v'(s)+b\right)ds\right| \\ &\le \int_t^{+\infty}\left[p(s)\left(\frac{|v(s)|}{1+v_0(s)}+b\right)+q(s)\left|v'(s)+b\right|+r(s)\right]ds, \quad t\in\mathbb{R}^+,\end{aligned} \tag{2.21}$$

$$\begin{aligned}\sup_{t\in\mathbb{R}^+}\left|(Av)'(t)\right| &\le \int_0^{+\infty}\left[p(s)\left(\frac{|v(s)|}{1+v_0(s)}+b\right)+q(s)\left|v'(s)+b\right|+r(s)\right]ds \\ &\le (d_1+b)(P_1+Q_1)+R_1.\end{aligned} \tag{2.22}$$

Further,

$$\begin{aligned}|Av(t)| &\le \int_0^{+\infty} G(t,s)\left|f\left(s, v(s)+\frac{a+b\delta}{\Delta}+bs, v'(s)+b\right)\right|ds \\ &\le \int_0^{+\infty}(1+v_0(s))\left[p(s)\left(\frac{|v(s)|}{1+v_0(s)}+b\right)+q(s)\left|v'(s)+b\right|+r(s)\right]ds \\ &\le (d_1+b)(P_2+Q_2)+R_2 < +\infty, \quad t\in\mathbb{R}^+,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\left|(Av)'(t)\right| &\le \int_0^{+\infty}\left|f\left(s, v(s)+\frac{a+b\delta}{\Delta}+bs, v'(s)+b\right)\right|ds \\ &\le \int_0^{+\infty}\left[p(s)\left(\frac{|v(s)|}{1+v_0(s)}+b\right)+q(s)\left|v'(s)+b\right|+r(s)\right]ds \\ &\le (d_1+b)(P_1+Q_1)+R_1 < +\infty.\end{aligned} \tag{2.24}$$

On the other hand, for any $t_1,\ t_2 \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$, by Remark 2.3, we have

$$\begin{aligned} &|G(t_1,s) - G(t_2,s)| \left| f\left(s, v(s) + \frac{a+b\delta}{\Delta} + bs, v'(s) + b\right)\right| \\ &\quad \le 2(1+v_0(s)) \left[p(s)\left(\frac{|v(s)|}{1+v_0(s)} + b\right) + q(s)\left|v'(s) + b\right| + r(s)\right] \\ &\quad \le 2(1+v_0(s))\left[(p(s)+q(s))(\|v\| + b) + r(s)\right]. \end{aligned} \tag{2.25}$$

Hence, by (H_1), the Lebesgue dominated convergence theorem, and the continuity of $G(t,s)$, for any $t_1, t_2 \in \mathbb{R}^+$, we have

$$\begin{aligned} |(Av)(t_1) - (Av)(t_2)| &\le \int_0^{+\infty} |G(t_1,s) - G(t_2,s)| \left| f\left(s, v(s) + \frac{a+b\delta}{\Delta} + bs, v'(s) + b\right)\right| ds \\ &\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2, \\ \left|(Av)'(t_1) - (Av)'(t_2)\right| &= \int_{t_1}^{t_2} \left| f\left(s, v(s) + \frac{a+b\delta}{\Delta} + bs, v'(s) + b\right)\right| ds \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \end{aligned} \tag{2.26}$$

So, $Av \in C^1(\mathbb{R}^+, R)$ for any $v \in E$.

We can show that $Av \in E$. In fact, by (2.23) and (2.24), we obtain

$$\begin{aligned} &\lim_{t\to+\infty} \frac{|Av(t)|}{1+v_0(t)} = 0, \quad \text{then } \lim_{t\to+\infty} \frac{Av(t)}{1+v_0(t)} = 0, \\ &\lim_{t\to+\infty} (Av)'(t) = \lim_{t\to+\infty} \int_t^{+\infty} f\left(s, v(s) + \frac{a+b\delta}{\Delta} + bs, v'(s) + b\right) ds = 0. \end{aligned} \tag{2.27}$$

Hence, $A : E \to E$ is well defined.

(2) We show that A is continuous.

Suppose $\{v_m\} \subseteq E,\ \overline{v} \in E$, and $\lim_{m\to\infty} v_m = \overline{v}$. Then, $v_m(t) \to \overline{v}(t),\ v'_m(t) \to \overline{v}'(t)$ as $m \to +\infty,\ t \in \mathbb{R}^+$, and there exists $r_0 > 0$ such that $\|v_m\|_\infty \le r_0,\ (m = 1,2,\ldots), \|\overline{v}\|_\infty \le r_0$. The continuity of f implies that

$$\left| f\left(t, v_m(t) + bt + \frac{a+b\delta}{\Delta}, v'_m(t) + b\right) - f\left(t, \overline{v}(t) + bt + \frac{a+b\delta}{\Delta}, \overline{v}'(t) + b\right)\right| \longrightarrow 0 \tag{2.28}$$

as $m \to \infty,\ t \in \mathbb{R}^+$. Moreover, since

$$\begin{aligned} &\left| f\left(t, v_m(t) + bt + \frac{a+b\delta}{\Delta}, v'_m(t) + b\right) - f\left(t, \overline{v}(t) + bt + \frac{a+b\delta}{\Delta}, \overline{v}'(t) + b\right)\right| \\ &\quad \le 2\left[(p(t)+q(t))(r_0 + b) + r(t)\right], \quad t \in \mathbb{R}^+, \end{aligned} \tag{2.29}$$

we have from the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\|Av_m - Av_0\|_\infty \\
&\quad = \max\left\{ \sup_{t\in\mathbb{R}^+} \frac{|(Av_m)(t) - (A\overline{v})(t)|}{1+v_0(t)},\ \sup_{t\in\mathbb{R}^+}\left|(Av_m)'(t) - (A\overline{v})'(t)\right| \right\} \\
&\quad \le \int_0^{+\infty} \left| f\left(s, v_m(s) + bs + \frac{a+b\delta}{\Delta}, v_m'(s) + b\right) - f\left(s, \overline{v}(s) + bs + \frac{a+b\delta}{\Delta}, \overline{v}'(s) + b\right)\right| ds \\
&\quad \longrightarrow 0 \quad (m \longrightarrow \infty).
\end{aligned} \tag{2.30}$$

Thus, $A : E \to E$ is continuous.

(3) We show that $A : E \to E$ is relatively compact.

(a) Let $B \subset E$ be a bounded subset. Then, there exists $M > 0$ such that $\|v\|_\infty \le M$ for all $v \in B$. By the similar proof of (2.20) and (2.22), if $v \in B$, one has

$$\|Av\|_\infty \le (M+b)(P_1+Q_1) + R_1, \tag{2.31}$$

which implies that $A(B)$ is uniformly bounded.

(b) For any $T > 0$, if $t_1, t_2 \in [0,T],\ v \in B$, we have

$$\begin{aligned}
&\left| \frac{(Av)(t_1)}{1+v_0(t_1)} - \frac{(Av)(t_2)}{1+v_0(t_2)} \right| \\
&\quad \le \int_0^{+\infty} \left| \frac{G(t_1,s)}{1+v_0(t_1)} - \frac{G(t_2,s)}{1+v_0(t_2)} \right| \left| f\left(s, v(s) + bs + \frac{a+b\delta}{\Delta}, v'(s) + b\right)\right| ds \\
&\quad \le 2\int_0^{+\infty} \left| f\left(s, v(s) + bs + \frac{a+b\delta}{\Delta}, v'(s) + b\right)\right| ds \\
&\quad \le 2[(M+b)(P_1+Q_1) + R_1], \\
&\left|(Av)'(t_1) - (Av)'(t_2)\right| \\
&\quad \le \int_{t_1}^{t_2} \left| f\left(s, v(s) + bs + \frac{a+b\delta}{\Delta}, v'(s) + b\right)\right| ds \\
&\quad \le (M+b)(P_1+Q_1) + R_1.
\end{aligned} \tag{2.32}$$

Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $t_1, t_2 \in [0,T], |t_1 - t_2| < \delta,\ v \in B$, then

$$\left|\frac{(Av)(t_1)}{1+v_0(t_1)} - \frac{(Av)(t_2)}{1+v_0(t_2)}\right| < \varepsilon,$$
$$\left|(Av)'(t_1) - (Av)'(t_2)\right| < \varepsilon. \tag{2.33}$$

Since T is arbitrary, then $\{(AB)(t)/(1+v_0(t))\}$ and $\{(AB)'(t)\}$ are locally equicontinuous on $\mathbb{R}^+$.

(c) For $v \in B$, from (2.27), we have

$$\lim_{t\to+\infty}\left|\frac{(Av)(t)}{1+v_0(t)} - \lim_{s\to+\infty}\frac{(Av)(s)}{1+v_0(s)}\right| = \lim_{t\to+\infty}\left|\frac{(Av)(t)}{1+v_0(t)}\right| = 0,$$
$$\lim_{t\to+\infty}\left|(Av)'(t) - \lim_{s\to+\infty}(Av)'(s)\right| = \lim_{t\to+\infty}\left|(Av)'(t)\right| = 0, \tag{2.34}$$

which means that $\{(AB)(t)/(1+v_0(t))\}$ and $\{(AB)'(t)\}$ are equiconvergent at $+\infty$. By Lemma 2.1, $A : E \to E$ is relatively compact.

Therefore, $A : E \to E$ is completely continuous. The proof is complete. □

Lemma 2.5 (see [28, 29]). *Let E be Banach space, Ω be a bounded open subset of $E, \theta \in \Omega$, and $A : \overline{\Omega} \to E$ be a completely continuous operator. Then either there exist $x \in \partial\Omega, \lambda > 1$ such that $F(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Lemma 2.6 (see [28, 29]). *Let Ω be a bounded open set in real Banach space E, let P be a cone of E, $\theta \in \Omega$, and let $A : \overline{\Omega} \cap P \to P$ be completely continuous. Suppose that*

$$\lambda Ax \neq x, \quad \forall x \in \partial\Omega \cap P,\ \lambda \in (0,1]. \tag{2.35}$$

Then,

$$i(A, \Omega \cap P, P) = 1. \tag{2.36}$$

3. Existence Result

In this section, we present the existence of an unbounded solution for BVP (1.1) by using the Leray-Schauder nonlinear alternative.

Theorem 3.1. *Suppose that conditions* (H_1)–(H_3) *hold. Then BVP* (1.1) *has at least one unbounded solution.*

Proof. Since $f(t,0,0)\not\equiv 0$, by (H_1), we have $r(t) \geq |f(t,0,0)|$, a.e. $t \in \mathbb{R}^+$, which implies that $R_1 > 0$. Set

$$R = \frac{b(P_1+Q_1)+R_1}{1-P_1-Q_1}, \quad \Omega_R = \{v \in E : \|v\|_\infty < R\}. \tag{3.1}$$

From Lemmas 2.2 and 2.4, BVP (1.1) has a solution $v = v(t)$ if and only if v is a fixed point of A in E. So, we only need to seek a fixed point of A in E.

Suppose $v \in \partial\Omega_R$, $\lambda > 1$ such that $Av = \lambda v$. Then

$$\begin{aligned}
\lambda R = \lambda\|v\|_\infty = \|Av\|_\infty &= \max\left\{\sup_{t\in\mathbb{R}^+}\frac{|(Av)(t)|}{1+v_0(t)},\ \sup_{t\in\mathbb{R}^+}\left|(Av)'(t)\right|\right\} \\
&\leq \int_0^{+\infty}\left|f\left(s, v(s)+bs+\frac{a+b\delta}{\Delta}, v'(s)+b\right)\right|ds \\
&\leq (P_1+Q_1)\|v\|_\infty + (P_1+Q_1)b + R_1 \\
&= (P_1+Q_1)R + (P_1+Q_1)b + R_1.
\end{aligned} \tag{3.2}$$

Therefore,

$$\lambda \leq (P_1+Q_1) + \frac{(P_1+Q_1)b+R_1}{R} = 1, \tag{3.3}$$

which contradicts $\lambda > 1$. By Lemma 2.5, A has a fixed point $v^* \in \overline{\Omega}_R$. Letting $u^*(t) = v^*(t) + bt + ((a+b\delta)/\Delta)$, $t \in \mathbb{R}^+$, boundary conditions imply that u^* is an unbounded solution of BVP (1.1). □

4. Existence and Uniqueness of Positive Solution

In this section, we restrict the nonlinearity $f \geq 0$ and discuss the existence and uniqueness of positive solution for BVP (1.1).

Define the cone $P \subset E$ as follows:

$$P = \left\{u \in E : u(t) \geq \gamma(t)\sup_{s\in\mathbb{R}^+}\left|\frac{u(s)}{1+v_0(s)}\right|,\ t \in \mathbb{R}^+,\ \frac{u(0)}{1+v_0(0)} \geq \frac{\beta}{\delta+\Delta+a/b}\sup_{s\in\mathbb{R}^+}\left|u'(s)\right|\right\}. \tag{4.1}$$

Lemma 4.1. *Suppose that* (H_1) *and* (H_2) *hold. Then,* $A : P \to P$ *is completely continuous.*

Proof. Lemma 2.4 shows that $A : P \to E$ is completely continuous, so we only need to prove $A(P) \subset P$. Since $f \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$, $(Av)(t) \geq 0$, $t \in \mathbb{R}^+$, and from Remark 2.3, we have

$$
\begin{aligned}
(Av)(t) &= \int_0^{+\infty} G(t,s) f\left(s, v(s) + \frac{a+b\delta}{\Delta} + bs, v'(s) + b\right) ds \\
&\geq \gamma(t) \int_0^{+\infty} \frac{G(\tau,s)}{1+v_0(\tau)} f\left(s, v(s) + \frac{a+b\delta}{\Delta} + bs, v'(s) + b\right) ds \\
&= \gamma(t) \frac{\int_0^{+\infty} G(\tau,s) f(s, v(s) + (a+b\delta)/\Delta + bs, v'(s) + b) ds}{1+v_0(\tau)} \\
&= \gamma(t) \frac{Av(\tau)}{1+v_0(\tau)}, \quad \forall t, \tau \in \mathbb{R}^+.
\end{aligned} \tag{4.2}
$$

Then,

$$
\begin{aligned}
(Av)(t) &\geq \gamma(t) \sup_{\tau \in \mathbb{R}^+} \frac{Av(\tau)}{1+v_0(\tau)}, \quad t \in \mathbb{R}^+, \\
\frac{Av(0)}{1+v_0(0)} &= \frac{\int_0^{+\infty} G(0,s) f(s, v(s) + (a+b\delta)/\Delta + bs, v'(s) + b) ds}{1+v_0(0)} \\
&\geq \frac{(\beta/\Delta) \int_0^{+\infty} f(s, v(s) + (a+b\delta)/\Delta + bs, v'(s) + b) ds}{1 + (a/b + \delta)/\Delta} \\
&\geq \frac{\beta}{\delta + \Delta + a/b} \sup_{t \in \mathbb{R}^+} \left|(Av)'(t)\right|.
\end{aligned} \tag{4.3}
$$

Therefore, $A(P) \subset P$. □

Theorem 4.2. *Suppose that conditions* (H_2) *and* (H_3) *hold and the following condition holds:*

(H_1') *suppose that* $f(t,0,0)$, $tf(t,0,0) \in L^1[0,+\infty)$, $f(t,0,0) \not\equiv 0$ *and there exist nonnegative functions* $p(t), q(t) \in L^1[0,+\infty)$ *with* $tp(t), tq(t) \in L^1[0,+\infty)$ *such that*

$$
\begin{aligned}
&\left| f(t, (1+v_0(t))u_1, v_1) - f(t, (1+v_0(t))u_2, v_2 \right| \\
&\quad \leq p(t)|u_1 - u_2| + q(t)|v_1 - v_2|, \quad \text{a.e. } (t, u_i, v_i) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, i = 1, 2.
\end{aligned} \tag{4.4}
$$

Then, BVP (1.1) *has a unique unbounded positive solution.*

Proof. We first show that (H_1') implies (H_1). By (4.4), we have

$$
\left| f(t, (1+v_0(t))u, v) \right| \leq p(t)|u| + q(t)|v| + \left| f(t,0,0) \right|, \quad \text{a.e. } (t,u,v) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}. \tag{4.5}
$$

By Lemma 4.1, $A : P \to P$ is completely continuous. Let $\overline{R} = \int_0^{\infty} f(t,0,0) dt$. Then, $\overline{R} > 0$. Set

$$
R > \frac{b(P_1 + Q_1) + \overline{R}}{1 - P_1 - Q_1}, \quad \Omega = \{ v \in E : \|v\|_\infty < R \}. \tag{4.6}
$$

For any $v \in P \cap \partial\Omega$, by (4.5), we have

$$\begin{aligned}
\frac{|(Av)(t)|}{1+v_0(t)} &= \left| \int_0^{+\infty} \frac{G(t,s)}{1+v_0(t)} f\left(s, v(s)+bs+\frac{a+b\delta}{\Delta}, v'(s)+b\right) ds \right| \\
&\le (R+b)(P_1+Q_1)+\overline{R} < R, \quad t \in \mathbb{R}^+, \\
\left|(Av)'(t)\right| &= \left| \int_t^{+\infty} f\left(s, v(s)+bs+\frac{a+b\delta}{\Delta}, v'(s)+b\right) ds \right| \\
&\le \int_0^{+\infty} \left| f\left(s, v(s)+bs+\frac{a+b\delta}{\Delta}, v'(s)+b\right) \right| ds \\
&\le (R+b)(P_1+Q_1)+\overline{R} < R, \quad t \in \mathbb{R}^+.
\end{aligned} \tag{4.7}$$

Therefore, $\|Av\|_\infty < \|v\|_\infty$, for all $v \in P \cap \partial\Omega$, that is, $\lambda Av \neq v$ for any $\lambda \in (0,1]$, $v \in P \cap \partial\Omega$. Then, Lemma 2.6 yields $i(A, P\cap\Omega, P) = 1$, which implies that A has a fixed point $v^* \in P \cap \Omega$. Let $u^*(t) = v^*(t) + bt + ((a+b\delta)/\Delta)$, $t \in \mathbb{R}^+$. Then, u^* is an unbounded positive solution of BVP (1.1).

Next, we show the uniqueness of positive solution for BVP (1.1). We will show that A is a contraction. In fact, by (4.4), we have

$$\begin{aligned}
&\|Av_1 - Av_2\|_\infty \\
&\quad = \max\left\{ \sup_{t\in\mathbb{R}^+} \frac{|(Av_1)(t)-(Av_2)(t)|}{1+v_0(t)},\ \sup_{t\in\mathbb{R}^+} \left|(Av_1)'(t)-(Av_2)'(t)\right| \right\} \\
&\quad \le \int_0^{+\infty} \left| f\left(s, v_1(s)+bs+\frac{a+b\delta}{\Delta}, v_1'(s)+b\right) - f\left(s, v_2(s)+bs+\frac{a+b\delta}{\Delta}, v_2'(s)+b\right) \right| ds \\
&\quad \le \int_0^{+\infty} \left[p(s)\frac{|v_1(s)-v_2(s)|}{1+v_0(s)} + q(s)\left|v_1'(s)-v_2'(s)\right| \right] ds \\
&\quad \le (P_1+Q_1)\|v_1-v_2\|_\infty.
\end{aligned} \tag{4.8}$$

So, A is indeed a contraction. The Banach contraction mapping principle yields the uniqueness of positive solution to BVP (1.1). □

5. Examples

Example 5.1. Consider the following BVP:

$$\begin{aligned}
&u''(t) + 2e^{-4t}\frac{u^2(t)}{1+u^2(t)} + 2e^{-3t}\frac{(u'(t))^3}{1+(u'(t))^4} - \frac{\arctan t}{1+t^3} = 0, \quad t \in \mathbb{R}^+, \\
&89u(0) - 3u'(0) - \sum_{i=1}^{7} iu\left(\frac{i+3}{4}\right) = 2, \qquad \lim_{t\to+\infty} u'(t) = 1,
\end{aligned} \tag{5.1}$$

We have

$$\Delta = 89 - \sum_{i=1}^{7} i = 61, \qquad \delta = 3 + \sum_{i=1}^{7} i\frac{i+3}{4} = 59, \qquad v_0(t) = t + \frac{2+59}{61} = t+1,$$
$$f(t,x,y) = 2e^{-4t}\frac{x^2}{1+x^2} + 2e^{-3t}\frac{y^3}{1+y^4} - \frac{\arctan t}{1+t^3} \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \tag{5.2}$$
$$\left|f(t,(2+t)x,y)\right| \leq (2+t)e^{-4t}|x| + e^{-3t}\left|y\right| + \frac{\arctan t}{1+t^3}.$$

Let

$$p(t) = (2+t)e^{-4t}, \qquad q(t) = e^{-3t}, \qquad r(t) = \frac{\arctan t}{1+t^3}. \tag{5.3}$$

Then, $p(t), q(t), r(t) \in L^1[0,\infty), tp(t), tq(t), tr(t) \in L^1[0,\infty)$, and it is easy to prove that $(\mathbf{H_1})$ is satisfied. By direct calculations, we can obtain that $P_1 = 9/16,\ Q_1 = 1/3,\ P_1 + Q_1 < 1$. By Theorem 3.1, BVP (5.1) has an unbounded solution.

Example 5.2. Consider the following BVP:

$$u''(t) + \frac{1}{(2+t)^2}e^{-4t}\frac{u^2(t)}{1+u^2(t)} + \frac{1}{4}e^{-3t}\frac{|u'(t)|^3}{1+(u'(t))^4} + \frac{\arctan t}{1+t^3} = 0, \quad t \in \mathbb{R}^+,$$
$$89u(0) - 3u'(0) - \sum_{i=1}^{7} iu\left(\frac{i+3}{4}\right) = 2, \qquad \lim_{t\to+\infty} u'(t) = 1. \tag{5.4}$$

In this case, we have

$$\Delta = 61, \qquad \delta = 59, \qquad v_0(t) = t+1,$$
$$f(t,x,y) = \frac{1}{(2+t)^2}e^{-4t}\frac{x^2}{1+x^2} + \frac{1}{4}e^{-3t}\frac{\left|y\right|^3}{1+y^4} + \frac{\arctan t}{1+t^3} \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+), \tag{5.5}$$
$$\left|f(t,(2+t)x_1,y_1) - f(t,(2+t)x_2,y_2)\right| \leq 2e^{-4t}|x_1 - x_2| + \frac{3}{4}e^{-3t}\left|y_1 - y_2\right|.$$

Let

$$p(t) = 2e^{-4t}, \qquad q(t) = \frac{3}{4}e^{-3t}. \tag{5.6}$$

Then, $P_1 = 1/2,\ Q_1 = 1/4,\ P_1 + Q_1 < 1$. By Theorem 4.2, BVP (5.4) has a unique unbounded positive solution.

Acknowledgments

The authors are grateful to the referees for valuable suggestions and comments. The first and second authors were supported financially by the National Natural Science Foundation of China (11071141, 10771117) and the Natural Science Foundation of Shandong Province of China (Y2007A23, Y2008A24). The third author was supported financially by the Australia Research Council through an ARC Discovery Project Grant.

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