Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 241518, 11 pages
doi:10.1155/2010/241518

Research Article

Multiple Solutions for Biharmonic Equations with Asymptotically Linear Nonlinearities

Ruichang Pei[1, 2]

[1] *Center for Nonlinear Studies, Northwest University, Xi'an 710069, China*
[2] *Department of Mathematics, Tianshui Normal University, Tianshui 741001, China*

Correspondence should be addressed to Ruichang Pei, prc211@163.com

Received 26 February 2010; Revised 2 April 2010; Accepted 22 April 2010

Academic Editor: Kanishka Perera

The existence of multiple solutions for a class of fourth elliptic equation with respect to the resonance and nonresonance conditions is established by using the minimax method and Morse theory.

1. Introduction

Consider the following Navier boundary value problem:

$$\begin{aligned} \Delta^2 u(x) &= f(x,u), \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N > 4)$, and $f(x,t)$ satisfies the following:

(H_1') $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$, $f(x,t)t \geq 0$ for all $x \in \Omega$, $t \in \mathbb{R}$;

(H_2') $\lim_{|t|\to 0}(f(x,t)/t) = f_0$, $\lim_{|t|\to\infty}(f(x,t)/t) = l$ uniformly for $x \in \Omega$, where f_0 and l are constants;

(H_3') $\lim_{|t|\to\infty}[f(x,t)t - 2F(x,t)] = -\infty$, where $F(x,t) = \int_0^t f(x,s)ds$.

In view of the condition (H_2'), problem (1.1) is called asymptotically linear at both zero and infinity. Clearly, $u = 0$ is a trivial solution of problem (1.1). It follows from (H_1') and (H_2') that the functional

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F(x,u)dx \tag{1.2}$$

is of C^2 on the space $H_0^1(\Omega) \cap H^2(\Omega)$ with the norm

$$\|u\| := \left(\int_\Omega |\Delta u|^2 dx\right)^{1/2}. \tag{1.3}$$

Under the condition (H_2'), the critical points of I are solutions of problem (1.1). Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the eigenvalues of $(\Delta^2, H^2(\Omega) \cap H_0^1(\Omega))$ and $\phi_1(x) > 0$ be the eigenfunction corresponding to λ_1. Let E_{λ_k} denote the eigenspace associated to λ_k. Throughout this paper, we denoted by $|\cdot|_p$ the $L^p(\Omega)$ norm.

If l in the above condition (H_2') is an eigenvalue of $(\Delta^2, H^2(\Omega)\cap H_0^1(\Omega))$, then problem (1.1) is called resonance at infinity. Otherwise, we call it non-resonance. A main tool of seeking the critical points of functional I is the mountain pass theorem (see [1–3]). To apply this theorem to the functional I in (1.2), usually we need the following condition [1], that is, for some $\theta > 2$ and $M > 0$,

(AR)

$$0 < \theta F(x,s) \le f(x,s)s \quad \text{for a.e. } x \in \Omega,\ |s| > M. \tag{1.4}$$

It is well known that the condition (AR) plays an important role in verifying that the functional I has a "mountain pass" geometry and a related $(PS)_c$ sequence is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$ when one uses the mountain pass theorem.

If $f(x,t)$ admits subcritical growth and satisfies (AR) condition by the standard argument of applying mountain pass theorem, we known that problem (1.1) has nontrivial solutions. Similarly, lase $f(x,t)$ is of critical growth (see, e.g., [4–7] and their references).

It follows from the condition (AR) that $\lim_{|t|\to\infty}(F(x,t)/t^2) = +\infty$ after a simple computation. That is, $f(x,t)$ must be superlinear with respect to t at infinity. Noticing our condition (H_2'), the nonlinear term $f(x,t)$ is asymptotically linear, not superlinear, with respect to t at infinity, which means that the usual condition (AR) cannot be assumed in our case. If the mountain pass theorem is used to seek the critical points of I, it is difficult to verify that the functional I has a "mountain pass" structure and the $(PS)_c$ sequence is bounded.

In [8], Zhou studied the following elliptic problem:

$$-\Delta u = f(x,u), \quad u \in H_0^1(\Omega), \tag{1.5}$$

where the conditions on $f(x,t)$ are similar to (H_1') and (H_2'). He provided a valid method to verify the (PS) sequence of the variational functional, for the above problem is bounded in $H_0^1(\Omega)$ (see also [9, 10]).

To the author's knowledge, there seems few results on problem (1.1) when $f(x,t)$ is asymptotically linear at infinity. However, the method in [8] cannot be applied directly to the biharmonic problems. For example, for the Laplacian problem, $u \in H_0^1(\Omega)$ implies $|u|$, $u_+, u_- \in H_0^1(\Omega)$, where $u_+ = \max(u,0)$, $u_- = \max(-u,0)$. We can use u_+ or u_- as a test function, which is helpful in proving a solution nonnegative. While for the biharmonic problems, this trick fails completely since $u \in H_0^2(\Omega)$ does not imply u_+, $u_- \in H_0^2(\Omega)$ (see [11, Remark 2.1.10]). As far as this point is concerned, we will make use of the methods in [12] to discuss in the following Lemma 2.3. In this paper we consider multiple solutions of problem (1.1) in the cases of resonance and non-resonance by using the mountain pass theorem and Morse theory. At first, we use the truncated skill and mountain pass theorem to obtain a positive solution and a negative solution of problem (1.1) under our more general condition (H_1') and (H_2') with respect to the conditions (H_1) and (H_3) in [8]. In the course of proving existence of positive solution and negative solution, the monotonicity condition (H_2) of [8] on the nonlinear term f is not necessary, this point is very important because we can directly prove existence of positive solution and negative solution by using Rabinowitz's mountain pass theorem. That is, the proof of our compact condition is more simple than that in [8]. Furthermore, we can obtain a nontrivial solution when the nonlinear term f is resonance or non-resonance at the infinity by using Morse theory.

2. Main Results and Auxiliary Lemmas

Let us now state the main results.

Theorem 2.1. *Assume that conditions (H_1') and (H_2') hold, $f_0 < \lambda_1$, and $l \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 2$; then problem* (1.1) *has at least three nontrivial solutions.*

Theorem 2.2. *Assume that conditions (H_1')–(H_3') hold, $f_0 < \lambda_1$, and $l = \lambda_k$ for some $k \geq 2$; then problem* (1.1) *has at least three nontrivial solutions.*

Consider the following problem:

$$\begin{gathered} \Delta^2 u = f_+(x,u), \quad x \in \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{gathered} \tag{2.1}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{2.2}$$

Define a functional $I_+ : H^2(\Omega) \cap H_0^1(\Omega) \to \mathbb{R}$ by

$$I_+(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F_+(x,u)dx, \tag{2.3}$$

where $F_+(x,t) = \int_0^t f_+(x,s)ds$, and then $I_+ \in C^2(H^2(\Omega) \cap H_0^1(\Omega), \mathbb{R})$.

Lemma 2.3. *I_+ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a sequence such that $|I'_+(u_n)| \leq c, < I'_+(u_n), \phi > \to 0$ as $n \to \infty$. Note that

$$\langle I'_+(u_n), \phi \rangle = \int_\Omega \Delta u_n \Delta \phi dx - \int_\Omega f_+(x, u_n) \phi dx = o(\|\phi\|) \tag{2.4}$$

for all $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that $|u_n|_2$ is bounded, taking $\phi = u_n$ in (2.4). By (H'_2), there exists $c > 0$ such that $|f_+(x, u_n(x))| \leq c|u_n(x)|$, a.e. $x \in \Omega$. So u_n is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. If $|u_n|_2 \to +\infty$, as $n \to \infty$, set $v_n = u_n/|u_n|_2$, and then $|v_n|_2 = 1$. Taking $\phi = v_n$ in (2.4), it follows that $\|v_n\|$ is bounded. Without loss of generality, we assume that $v_n \rightharpoonup v$ in $H^2(\Omega) \cap H_0^1(\Omega)$, and then $v_n \to v$ in $L^2(\Omega)$. Hence, $v_n \to v$ a.e. in Ω. Dividing both sides of (2.4) by $|u_n|_2$, we get

$$\int_\Omega \Delta v_n \Delta \phi dx - \int_\Omega \frac{f_+(x, u_n)}{|u_n|_2} \phi dx = o\left(\frac{\|\phi\|}{|u_n|_2} \right), \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.5}$$

Then for a.e. $x \in \Omega$, we deduce that $f_+(x, u_n)/|u_n|_2 \to l v_+$ as $n \to \infty$, where $v_+ = \max\{v, 0\}$. In fact, when $v(x) > 0$, by (H'_2) we have

$$\begin{gathered} u_n(x) = v_n(x)|u_n|_2 \longrightarrow +\infty, \\ \frac{f_+(x, u_n)}{|u_n|_2} = \frac{f_+(x, u_n)}{u_n} v_n \longrightarrow l v. \end{gathered} \tag{2.6}$$

When $v(x) = 0$, we have

$$\frac{f_+(x, u_n)}{|u_n|_2} \leq c|v_n| \longrightarrow 0. \tag{2.7}$$

When $v(x) < 0$, we have

$$\begin{gathered} u_n(x) = v_n(x)|u_n|_2 \longrightarrow -\infty, \\ \frac{f_+(x, u_n)}{|u_n|_2} = 0. \end{gathered} \tag{2.8}$$

Since $f_+(x, u_n)/|u_n|_2 \leq c|v_n|$, by (2.5) and the Lebesgue dominated convergence theorem, we arrive at

$$\int_\Omega \Delta v \Delta \phi dx - \int_\Omega l v_+ \phi dx = 0, \quad \text{for any } \phi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.9}$$

Choosing $\phi = \phi_1$, we deduce that

$$l \int_\Omega v_+ \phi_1 dx = \lambda_1 \int_\Omega v \phi_1 dx. \tag{2.10}$$

Notice that

$$\int_\Omega v_+ \phi_1 dx - \int_\Omega v \phi_1 dx = \int_{\Omega_-} -v \phi_1 dx \geq 0, \tag{2.11}$$

where $\Omega_- = \{x \in \Omega : v(x) < 0\}$.

Now we show that there is a contradiction in both cases of $|\Omega_-| = 0$ and $|\Omega_-| > 0$.

Case 1. Suppose $|\Omega_-| = 0$, then $v(x) \geq 0$ a.e. in Ω. By $v(x) \not\equiv 0$ we have $\int_\Omega v\phi_1 dx > 0$. Thus (2.11) implies that

$$l \int_\Omega v \phi_1 dx = l \int_\Omega v_+ \phi_1 dx = \lambda_1 \int_\Omega v \phi_1 dx \tag{2.12}$$

which contradicts to $l > \lambda_1$.

Case 2. Suppose $|\Omega_-| > 0$, then $\int_{\Omega_-} -v\phi_1 dx > 0$, and $\int_\Omega v_+\phi_1 dx > \int_\Omega v\phi_1 dx$. It follows from (2.11) that

$$l \int_\Omega v_+ \phi_1 dx = \lambda_1 \int_\Omega v \phi_1 dx < \lambda_1 \int_\Omega v_+ \phi_1 dx \tag{2.13}$$

which contradicts to $l > \lambda_1$ if $\int_\Omega v_+\phi_1 dx > 0$ and contradicts to $0 \not< 0$ if $\int_\Omega v_+\phi_1 dx = 0$.

□

Lemma 2.4. *Let ϕ_1 be the eigenfunction corresponding to λ_1 with $\|\phi_1\| = 1$. If $f_0 < \lambda_1 < l$, then*

(a) *there exist $\rho,\ \beta > 0$ such that $I_+(u) \geq \beta$ for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$ with $\|u\| = \rho$;*

(b) *$I_+(t\phi_1) = -\infty$ as $t \to +\infty$.*

Proof. By (H_1') and (H_2'), if $l \in (\lambda_1, +\infty)$, for any $\varepsilon > 0$, there exist $A = A(\varepsilon) \geq 0$ and $B = B(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F_+(x, s) \leq \frac{1}{2}(f_0 + \varepsilon)s^2 + As^{p+1}, \tag{2.14}$$

$$F_+(x, s) \geq \frac{1}{2}(l - \varepsilon)s^2 - B, \tag{2.15}$$

where $p \in (1, (N+4)/(N-4))$ if $N > 4$.

Choose $\varepsilon > 0$ such that $f_0 + \varepsilon < \lambda_1$. By (2.14), the Poincaré inequality, and the Sobolev inequality, we get

$$\begin{aligned} I_+(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F_+(x,u)dx \\ &\geq \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{2}\int_\Omega \Big[(f_0+\varepsilon)u^2 + A|u|^{p+1}\Big]dx \\ &\geq \frac{1}{2}\left(1 - \frac{f_0+\varepsilon}{\lambda_1}\right)\|u\|^2 - c\|u\|^{p+1}. \end{aligned} \tag{2.16}$$

So, part (a) holds if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, if $l \in (\lambda_1, +\infty)$, take $\varepsilon > 0$ such that $l - \varepsilon > \lambda_1$. By (2.15), we have

$$I_+(u) \leq \frac{1}{2}\|u\|^2 - \frac{l-\varepsilon}{2}|u|_2^2 + B|\Omega|. \tag{2.17}$$

Since $l - \varepsilon > \lambda_1$ and $\|\phi_1\| = 1$, it is easy to see that

$$I_+(t\phi_1) \leq \frac{1}{2}\left(1 - \frac{l-\varepsilon}{\lambda_1}\right)t^2 + B|\Omega| \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty, \tag{2.18}$$

and part (b) is proved. □

Lemma 2.5. *Let $H^2(\Omega) \cap H_0^1(\Omega) = V \oplus W$, where $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$. If f satisfies (H_1')–(H_3'), then*

(i) *the functional I is coercive on W, that is,*

$$I(u) \longrightarrow +\infty \quad \textit{as } \|u\| \longrightarrow +\infty, \ u \in W \tag{2.19}$$

and bounded from below on W;

(ii) *the functional I is anticoercive on V.*

Proof. For $u \in W$, by (H_2'), for any $\varepsilon > 0$, there exists $B_1 = B_1(\varepsilon)$ such that for all $(x,s) \in \Omega \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{2}(l+\varepsilon)s^2 + B_1. \tag{2.20}$$

So we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{2}(l+\varepsilon)|u|_2^2 - B_1|\Omega| \\ &\geq \frac{1}{2}\left(1 - \frac{l+\varepsilon}{\lambda_{k+1}}\right)\|u\|^2 - B_1|\Omega|. \end{aligned} \tag{2.21}$$

Choose $\varepsilon > 0$ such that $l + \varepsilon < \lambda_{k+1}$. This proves (i).

(ii) We firstly consider the case $l = \lambda_k$. Write $G(x,t) = F(x,t) - (1/2)\lambda_k t^2$, $g(x,t) = f(x,t) - \lambda_k t$. Then (H_2') and (H_3') imply that

$$\lim_{|t|\to\infty}\left[g(x,t)t - 2G(x,t)\right] = -\infty, \tag{2.22}$$

$$\lim_{|t|\to\infty}\frac{2G(x,t)}{t^2} = 0. \tag{2.23}$$

It follows from (2.22) that for every $M > 0$, there exists a constant $T > 0$ such that

$$g(x,t)t - 2G(x,t) \le -M, \quad \forall t \in \mathbb{R},\ |t| \ge T,\ \text{a.e.}\ \ x \in \Omega. \tag{2.24}$$

For $\tau > 0$, we have

$$\frac{d}{d\tau}\frac{G(x,\tau)}{\tau^2} = \frac{g(x,\tau)\tau - 2G(x,\tau)}{\tau^3}. \tag{2.25}$$

Integrating (2.25) over $[t,s] \subset [T,+\infty)$, we deduce that

$$\frac{G(x,s)}{s^2} - \frac{G(x,t)}{t^2} \le \frac{M}{2}\left(\frac{1}{s^2} - \frac{1}{t^2}\right). \tag{2.26}$$

Let $s \to +\infty$ and use (2.23); we see that $G(x,t) \ge M/2$, for $t \in \mathbb{R}$, $t \ge T$, a.e. $x \in \Omega$. A similar argument shows that $G(x,t) \ge M/2$, for $t \in \mathbb{R}$, $t \le -T$, a.e. $x \in \Omega$. Hence

$$\lim_{|t|\to\infty} G(x,t) \longrightarrow +\infty, \quad \text{a.e.}\ \ x \in \Omega. \tag{2.27}$$

By (2.27), we get

$$\begin{aligned} I(v) &= \frac{1}{2}\int_\Omega |\Delta v|^2 dx - \int_\Omega F(x,v)dx \\ &= \frac{1}{2}\int_\Omega |\Delta v|^2 dx - \frac{1}{2}\lambda_k\int_\Omega v^2 dx - \int_\Omega G(x,v)dx \\ &\le -\delta\left\|v^-\right\|^2 - \int_\Omega G(x,v)dx \longrightarrow -\infty \end{aligned} \tag{2.28}$$

for $v \in V$ with $\|v\| \to +\infty$, where $v^- \in E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_{k-1}}$.

In the case of $\lambda_k < l < \lambda_{k+1}$, we do not need the assumption (H_3') and it is easy to see that the conclusion also holds. □

Lemma 2.6. *If $\lambda_k < l < \lambda_{k+1}$, then I satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a sequence such that $|I(u_n)| \le c$, $< I'(u_n), \phi > \to 0$. One has

$$\langle I'(u_n), \phi \rangle = \int_\Omega \Delta u_n \Delta \phi dx - \int_\Omega f(x, u_n) \phi dx = o(\|\phi\|) \tag{2.29}$$

for all $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$. If $|u_n|_2$ is bounded, we can take $\phi = u_n$. By (H_2'), there exists a constant $c > 0$ such that $|f(x, u_n(x))| \le c|u_n(x)|$, a.e. $x \in \Omega$. So u_n is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. If $|u_n|_2 \to +\infty$, as $n \to \infty$, set $v_n = u_n/|u_n|_2$, and then $|v_n|_2 = 1$. Taking $\phi = v_n$ in (2.29), it follows that $\|v_n\|$ is bounded. Without loss of generality, we assume $v_n \rightharpoonup v$ in $H^2(\Omega) \cap H_0^1(\Omega)$, and then $v_n \to v$ in $L^2(\Omega)$. Hence, $v_n \to v$ a.e. in Ω. Dividing both sides of (2.29) by $|u_n|_2$, we get

$$\int_\Omega \Delta v_n \Delta \phi dx - \int_\Omega \frac{f(x, u_n)}{|u_n|_2} \phi dx = o\left(\frac{\|\phi\|}{|u_n|_2}\right) \quad \text{for any } \phi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.30}$$

Then for a.e. $x \in \Omega$, we have $f(x, u_n)/|u_n|_2 \to lv$ as $n \to \infty$. In fact, if $v(x) \not= 0$, by (H_2'), we have

$$\begin{gathered} |u_n(x)| = |v_n(x)||u_n|_2 \longrightarrow +\infty, \\ \frac{f(x, u_n)}{|u_n|_2} = \frac{f(x, u_n)}{u_n} v_n \longrightarrow lv. \end{gathered} \tag{2.31}$$

If $v(x) = 0$, we have

$$\frac{|f(x, u_n)|}{|u_n|_2} \le c|v_n| \longrightarrow 0. \tag{2.32}$$

Since $|f(x, u_n)|/|u_n|_2 \le c|v_n|$, by (2.30) and the Lebesgue dominated convergence theorem, we arrive at

$$\int_\Omega \Delta v \Delta \phi dx - \int_\Omega lv \phi dx = 0, \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.33}$$

It is easy to see that $v \not\equiv 0$. In fact, if $v \equiv 0$, then $|v|_2 = 0$ contradicts to $\lim_{n\to\infty} |v_n|_2 = |v|_2 = 1$. Hence, l is an eigenvalue of $(\Delta^2, H^2(\Omega) \cap H_0^1(\Omega))$. This contradicts our assumption. □

Lemma 2.7. *Suppose that $l = \lambda_k$ and f satisfies (H_3'). Then the functional I satisfies the (C) condition which is stated in [13].*

Proof. Suppose $u_n \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies

$$I(u_n) \longrightarrow c \in \mathbb{R}, \qquad (1 + \|u_n\|) \|I'(u_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.34}$$

In view of (H_2'), it suffices to prove that u_n is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. Similar to the proof of Lemma 2.6, we have

$$\int_\Omega \Delta v \Delta \phi dx - \int_\Omega l v \phi dx = 0, \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.35}$$

Therefore $v \not\equiv 0$ is an eigenfunction of λ_k, then $|u_n(x)| \to \infty$ for a.e. $x \in \Omega$. It follows from (H_3') that

$$\lim_{n \to +\infty} \left[f(x, u_n(x)) u_n(x) - 2F(x, u_n(x)) \right] = -\infty \tag{2.36}$$

holds uniformly in $x \in \Omega$, which implies that

$$\int_\Omega \left(f(x, u_n) u_n - 2F(x, u_n) \right) dx \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty. \tag{2.37}$$

On the other hand, (2.34) implies that

$$2I(u_n) - \left\langle I'(u_n), u_n \right\rangle \longrightarrow 2c \quad \text{as } n \longrightarrow \infty. \tag{2.38}$$

Thus

$$\int_\Omega \left(f(x, u_n) u_n - 2F(x, u_n) \right) dx \longrightarrow 2c \quad \text{as } n \longrightarrow \infty, \tag{2.39}$$

which contradicts to (2.37). Hence u_n is bounded. □

It is well known that critical groups and Morse theory are the main tools in solving elliptic partial differential equation. Let us recall some results which will be used later. We refer the readers to the book [14] for more information on Morse theory.

Let H be a Hilbert space, let $I \in C^1(H, \mathbb{R})$ be a functional satisfying the (PS) condition or (C) condition, let $H_q(X, Y)$ be the qth singular relative homology group with integer coefficients. Let u_0 be an isolated critical point of I with $I(u_0) = c$, $c \in \mathbb{R}$, and let U be a neighborhood of u_0. The group

$$C_q(I, u_0) := H_q(I^c \cap U, I^c \cap U \setminus \{u_0\}), \quad q \in Z \tag{2.40}$$

is said to be the qth critical group of I at u_0, where $I^c = \{u \in H : I(u) \le c\}$.

Let $K := \{u \in H : I'(u) = 0\}$ be the set of critical points of I and $a < \inf I(K)$; the critical groups of I at infinity are formally defined by (see [15])

$$C_q(I, \infty) := H_q(H, I^a), \quad q \in Z. \tag{2.41}$$

The following result comes from [14, 15] and will be used to prove the results in this paper.

Proposition 2.8 (see [15]). *Assume that* $H = H_\infty^+ \oplus H_\infty^-$, I *is bounded from below on* H_∞^+ *and* $I(u) \to -\infty$ *as* $\|u\| \to \infty$ *with* $u \in H_\infty^-$. *Then*

$$C_k(I, \infty) \ncong 0, \quad \text{if } k = \dim H_\infty^- < \infty. \tag{2.42}$$

3. Proof of the Main Results

Proof of Theorem 2.1. By Lemmas 2.32.4 and the mountain pass theorem, the functional I_+ has a critical point u_1 satisfying $I_+(u_1) \ge \beta$. Since $I_+(0) = 0$, $u_1 \neq 0$, and by the maximum principle, we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1.1) and satisfies

$$C_1(I_+, u_1) \neq 0, \quad u_1 > 0. \tag{3.1}$$

Using the results in [14], we obtain

$$C_q(I, u_1) = C_q\left(I_{C_0^1(\Omega)}, u_1\right) = C_q\left(I_+|_{C_0^1(\Omega)}, u_1\right) = C_q(I_+, u_1) = \delta_{q1} Z. \tag{3.2}$$

Similarly, we can obtain another negative critical point u_2 of I satisfying

$$C_q(I, u_2) = \delta_{q,1} Z. \tag{3.3}$$

Since $f_0 < \lambda_1$, the zero function is a local minimizer of I, and then

$$C_q(I, 0) = \delta_{q,0} Z. \tag{3.4}$$

On the other hand, by Lemmas 2.52.6 and Proposition 2.8, we have

$$C_k(I, \infty) \ncong 0. \tag{3.5}$$

Hence I has a critical point u_3 satisfying

$$C_k(I, u_3) \ncong 0. \tag{3.6}$$

Since $k \ge 2$, it follows from (3.2)–(3.6) that u_1, u_2, and u_3 are three different nontrivial solutions of problem (1.1). □

Proof of Theorem 2.2. By Lemmas 2.52.7 and the Proposition 2.8, we can prove the conclusion (3.5). The other proof is similar to that of Theorem 2.1. □

Acknowledgments

The author would like to thank the referees for valuable comments and suggestions for improving this paper. This work was supported by the National NSF (Grant no. 10671156) of China.

References

[1] A. Ambrosetti and P. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.
[2] H. Brézis and L. Nirenberg, "Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents," *Communications on Pure and Applied Mathematics*, vol. 36, no. 4, pp. 437–477, 1983.
[3] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMs Regional Conference Series in Mathematics, no. 65, American Mathematical Society, Providence, RI, USA, 1986.
[4] F. Bernis, J. García-Azorero, and I. Peral, "Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order," *Advances in Differential Equations*, vol. 1, no. 2, pp. 219–240, 1996.
[5] Y. B. Deng and G. S. Wang, "On inhomogeneous biharmonic equations involving critical exponents," *Proceedings of the Royal Society of Edinburgh*, vol. 129, no. 5, pp. 925–946, 1999.
[6] F. Gazzola, H.-C. Grunau, and M. Squassina, "Existence and nonexistence results for critical growth biharmonic elliptic equations," *Calculus of Variations and Partial Differential Equations*, vol. 18, no. 2, pp. 117–143, 2003.
[7] E. S. Noussair, C. A. Swanson, and J. Yang, "Critical semilinear biharmonic equations in $\mathbb{R}^N$," *Proceedings of the Royal Society of Edinburgh*, vol. 121, no. 1-2, pp. 139–148, 1992.
[8] H.-S. Zhou, "Existence of asymptotically linear Dirichlet problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, pp. 909–918, 2001.
[9] C. A. Stuart and H. S. Zhou, "Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$," *Communications in Partial Differential Equations*, vol. 24, no. 9-10, pp. 1731–1758, 1999.
[10] G. B. Li and H.-S. Zhou, "Multiple solutions to p-Laplacian problems with asymptotic nonlinearity as u^{p-1} at infinity," *Journal of the London Mathematical Society*, vol. 65, no. 1, pp. 123–138, 2002.
[11] W. P. Ziemer, *Weakly Differentiable Functions*, vol. 120 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1989.
[12] Y. Liu and Z. P. Wang, "Biharmonic equations with asymptotically linear nonlinearities," *Acta Mathematica Scientia*, vol. 27, no. 3, pp. 549–560, 2007.
[13] J. B. Su and L. G. Zhao, "An elliptic resonance problem with multiple solutions," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 604–616, 2006.
[14] K.-C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, Mass, USA, 1993.
[15] T. Bartsch and S. J. Li, "Critical point theory for asymptotically quadratic functionals and applications to problems with resonance," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 3, pp. 419–441, 1997.