Research Article

Global Existence and Convergence of Solutions to a Cross-Diffusion Cubic Predator-Prey System with Stage Structure for the Prey

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We study a cubic predator-prey system with stage structure for the prey. This system is a generalization of the two-species Lotka-Volterra predator-prey model. Firstly, we consider the asymptotical stability of equilibrium points to the system of ordinary differential equations type. Then, the global existence of solutions and the stability of equilibrium points to the system of weakly coupled reaction-diffusion type are discussed. Finally, the existence of nonnegative classical global solutions to the system of strongly coupled reaction-diffusion type is investigated when the space dimension is less than 6, and the global asymptotic stability of unique positive equilibrium point of the system is proved by constructing Lyapunov functions.

1. Introduction and Mathematical Model

The predator-prey model as, which follows, the ordinary differential equation system

$$\frac{du}{dt} = \left(b_1 + b_2 u - b_3 u^2\right)u - b_4 uv,$$

$$\frac{dv}{dt} = -cv + (\alpha u - \beta v)v \tag{1.1}$$

is said to be the general Lotka-Volterra predator-prey model in [1–3], and to be cubic predator-prey system in [4], where u, v are the population densities of prey and predator species at time t, respectively. $b_3, b_4, c, \alpha, \beta$ are positive constants, b_1 is nonnegative as the intrinsic growth rate of prey population, and the sign of b_2 is undetermined. c is the net

mortality rate of predator population, and the survival of predator species is dependent on the survival state of prey species, and $b_2u - b_3u^2, \beta v$ are the respective density restriction terms of prey and predator species. b_4u is the predation rate of the predator, and αu is the conversion rate of the predator. In [4], three questions about system (1.1) are discussed: the stability of nonnegative equilibrium points, and the existence, as well as numbers of limit cycle.

Referring to [5], we establish cubic predator-prey system with stage structure for the prey as follows:

$$\frac{dx_1}{dt} = \eta_1 x_2 - r_1 x_1 - \eta_2 x_1 + b_2 x_1^2 - b_3 x_1^3 - b_4 x_1 x_3,$$

$$\frac{dx_2}{dt} = \eta_2 x_1 - r_2 x_2, \tag{1.2}$$

$$\frac{dx_3}{dt} = -cx_3 + (\alpha x_1 - \beta x_3)x_3,$$

where x_1 and x_2 are the population densities of the immature and mature prey species, respectively, and x_3 denotes the density of the predator species. The predators live only on the immature prey species, as well as the survival of the predator species is dependent on the survival state of the immature prey species. $\eta_1, \eta_2, r_1, r_2, b_3, b_4, c, \alpha, \beta$ are positive constants, and the sign of b_2 is undetermined. η_1 and r_1 are the birth rate and the mortality rate of the immature prey species, respectively. r_2 and c are the net mortality rate of the mature prey population and the predator population, and η_2 is the conversion rate of the immature prey to the mature prey species. $b_2x_1 - b_3x_1^2$ and βx_3 are the respective density restriction terms of the immature prey species and predator species. b_4x_1 is the predation rate of the predator to the immature prey population, and αx_1 is the conversion rate of the predator.

Using the scaling

$$u_1 = \frac{\alpha}{r_2}x_1, \qquad u_2 = \frac{\alpha}{\eta_2}x_2, \qquad u_3 = \frac{\beta}{r_2}x_3, \qquad d\tau = r_2 dt, \tag{1.3}$$

and redenoting τ by t, system (1.2) reduces to

$$\frac{du_1}{dt} = a_0 u_2 - a_1 u_1 + a_2 u_1^2 - a_3 u_1^3 - ku_1 u_3,$$

$$\frac{du_2}{dt} = u_1 - u_2, \tag{1.4}$$

$$\frac{du_3}{dt} = (-b + u_1 - u_3)u_3,$$

where $a_0 = \eta_1\eta_2/r_2^2, a_1 = (r_1+\eta_2)/r_2, a_2 = b_2/\alpha, a_3 = b_3/r_2, k = b_4/\beta$, and $b = c/r_2$ are positive constants, and $a_2 = b_2/\alpha$ is undetermined to the sign.

To take into account the inhomogeneous distribution of the predators and prey in different spatial locations within a fixed bounded domain $\Omega \subset \mathbb{R}^N$ at any given time, and the natural tendency of each species to diffuse to areas of smaller population concentration,

we derive the following PDE system of reaction-diffusion type:

$$u_{1t} - d_1 \Delta u_1 = a_0 u_2 - a_1 u_1 + a_2 u_1^2 - a_3 u_1^3 - k u_1 u_3, \quad x \in \Omega, \ t > 0,$$

$$u_{2t} - d_2 \Delta u_2 = u_1 - u_2, \quad x \in \Omega, \ t > 0,$$

$$u_{3t} - d_3 \Delta u_3 = (-b + u_1 - u_3) u_3, \quad x \in \Omega, \ t > 0, \quad (1.5)$$

$$\partial_\eta u_i = 0, \quad i = 1, 2, 3, \ x \in \partial\Omega, \ t > 0,$$

$$u_i(x, 0) = u_{i0}(x) \geq 0, \quad i = 1, 2, 3, \ x \in \Omega,$$

where $\partial_\eta = \partial/\partial_\eta$, η is the unit outward normal vector of the boundary $\partial\Omega$ which we will assume to be smooth. The homogeneous Neumann boundary condition indicates that the above system is self-contained with zero population flux across the boundary. The positive constants d_1, d_2, and d_3 are said to be the diffusion coefficients, and the initial values $u_{i0}(x)$ $(i = 1, 2, 3)$ are nonnegative smooth functions.

Note that, in recent years, there has been considerable interest to investigate the global behavior of a system of interacting populations by taking into account the effect of self as well as cross-diffusion. According to the ideas in [6–13], especially to [8, 9], the cross-diffusion term will be only included in the third equation, that is, the following cross-diffusion system:

$$u_t = \Delta \left(d_1 u + \alpha_{11} u^2 \right) + a_0 v - a_1 u + a_2 u^2 - a_3 u^3 - k u w, \quad x \in \Omega, \ t > 0,$$

$$v_t = \Delta \left(d_2 v + \alpha_{22} v^2 \right) + u - v, \quad x \in \Omega, \ t > 0,$$

$$w_t = \Delta \left(d_3 w + \alpha_{31} u w + \alpha_{32} v w + \alpha_{33} w^2 \right) + (-b + u - w) w, \quad x \in \Omega, \ t > 0, \quad (1.6)$$

$$u_\eta(x, t) = v_\eta(x, t) = w_\eta(x, t) = 0, \quad x \in \partial\Omega, \ t > 0,$$

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad w(x, 0) = w_0(x) \geq 0, \quad x \in \Omega.$$

In the above, d_i, α_{ii} $(i = 1, 2, 3), \alpha_{31}$, and α_{32} are positive constants. d_1, d_2 and d_3 are the diffusion rates of the three species, respectively. α_{ii} $(i = 1, 2, 3)$ are referred to as self-diffusion pressures. α_{31} and α_{32} are cross-diffusion pressures. The term self-diffusion implies the movement of individuals from a higher to a lower concentration region. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. Generally, the value of the cross-diffusion coefficient may be positive, negative, or zero. The term positive cross-diffusion coefficient denotes the movement of the species in the direction of lower concentration of another species, and negative cross-diffusion coefficient denotes that one species tends to diffuse in the direction of higher concentration of another species [9].

The main purpose of this paper is to study the asymptotic behavior of the solutions of the reaction-diffusion system (1.5) and the global existence of the solution of the cross-diffusion system (1.6). But it is necessary to denonstrate that the conclusion for the existence of global solution of system (1.6) in this paper is the generalization of the work to Lotka-Volterra competition model with cross-diffusion [11] and that the convergence of solution investigated in this paper which is not discussed in [11].

The paper will be organized as follows. In Section 2, we analyze the asymptotical stability of equilibrium points for the ODE system (1.4) via linearization and the Lyapunov method. In Section 3, we prove the global existence of solutions and the stability of the equilibrium points to the diffusion system (1.5). In Section 4, we investigate the existence of nonnegative classical global solutions by assuming a_0, a_1, a_2, a_3, k, b to be positive constants only for the simplicity of calculation, and the global asymptotic stability of unique positive equilibrium point to the cross-diffusion system (1.6).

2. Equilibrium Solution of the ODE System

In this section we discuss the stability of unique positive equilibrium point for system (1.4). The following theorem shows that the solution of system (1.4) is bounded.

Theorem 2.1. *Let $(u_1(t), u_2(t), u_3(t))$ be the solution of system (1.4) with initial values $u_i(0) > 0$ $(i = 1, 2, 3)$, and let $[0, T)$ be the maximal existence interval of the solution. Then $0 < u_i(t) \leq M_i$ $(i = 1, 2, 3), t \in [0, T)$, where*

$$M_1 = \max\left\{ u_1(0) + (a_0 + a_1)u_2(0), C_0\left(1 + \frac{a_0}{a_1}\right) \right\},$$

$$M_2 = \max\left\{ \frac{u_1(0)}{a_0 + a_1}, \frac{C_0}{a_1} \right\}, \tag{2.1}$$

$$M_3 = \max\{u_3(0), M_1 - b\}.$$

The above C_0 is a positive constant depending only on a_0, a_1, a_2, a_3, and further $T = +\infty$.

Proof. It is easy to see that (1.4) has a unique positive local solution $(u_1(t), u_2(t), u_3(t))$. Let $T \in (0, +\infty]$ be the maximal existence time of the solution, and combin u_1 and u_2 linearly, that is, $u_1 + (a_0 + a_1)u_2$, it follows from (1.4) that

$$\frac{d}{dt}[u_1 + (a_0 + a_1)u_2] \leq -a_1 u_2 + a_0 u_1 + a_2 u_1^2 - a_3 u_1^3. \tag{2.2}$$

Using Young inequality, we can check that there exists a positive constant C_0 depending only on a_0, a_1, a_2, and a_3 such that

$$a_0 u_1 + a_2 u_1^2 - a_3 u_1^3 \leq C_0 - \frac{a_1}{a_0 + a_1} u_1. \tag{2.3}$$

It follows that

$$\frac{d}{dt}[u_1 + (a_0 + a_1)u_2] \leq C_0 - \frac{a_1}{a_0 + a_1}[u_1 + (a_0 + a_1)u_2], \tag{2.4}$$

which implies that there exist M_1 and M_2 referring to (2.1) such that $0 < u_1 \le M_1, 0 < u_2 \le M_2$, and $t \in [0, T)$.

Finally, we note that $du_3/dt = (-b + u_1 - u_3)u_3 \le (-b + M_1 - u_3)u_3$. Let $M_3 = \max\{u_3(0), M_1 - b\}$, then

$$\frac{d(u_3 - M_3)}{dt} + (u_3 - M_3)u_3 \le 0 \quad u_3(0) - M_3 \le 0. \tag{2.5}$$

From the comparison inequality for the ODE, we have $u_3 - M_3 \le 0$, $t \in [0, T)$.

Thus the solutions for system (1.4) are bounded. Further, from the extension theorem of solutions, we have $T = +\infty$. □

By the simple calculation, the sufficient conditions for system (1.4) having a unique positive equilibrium point as follows:

(i) $a_2 - k = 2\sqrt{a_3(a_1 - a_0 - kb)} > 2a_3b$; (ii) $\max\{a_1 - a_0, (a_2 - 2a_3b)b\} \le bk \le (a_2 - a_3b)b$, where the left equal sign holds if and only if $a_1 - a_0 < (a_2 - 2a_3b)b$; (iii) $a_1 - a_0 \le b(a_2 - a_3b) \le kb$; (iv) $a_1 - a_0 < kb \le (a_2 - 2a_3b)b$; (v) $kb < a_1 - a_0 \le (a_2 - a_3b)b$ and $a_2 - k \ge \max\{2a_3b, 2\sqrt{a_3(a_1 - a_0 - kb)}\}$, where the second equal sign holds if and only if $a_3b^2 > a_1 - a_0 - kb$; (vi) $2\sqrt{a_3(a_1 - a_0 - kb)} < a_2 - k \le 2a_3b$ and $kb < a_1 - a_0 < (a_2 - a_3b)b$.

If one of the above conditions holds, then system (1.4) has the unique positive equilibrium point $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$, where

$$\overline{u}_1 = \overline{u}_2 = \frac{(a_2 - k) + \sqrt{(a_2 - k)^2 + 4a_3(kb + a_0 - a_1)}}{2a_3}, \qquad \overline{u}_3 = \overline{u}_1 - b. \tag{2.6}$$

Theorem 2.2. *System (1.4) has the unique positive equilibrium point $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ when one of the above conditions (i), (ii), (iii), (iv), (v), and (vi) holds. If $a_1 - a_0 < ((a_2^2 - 2ka_2)/4a_3) + kb$ holds, then $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ is locally asymptotically stable.*

Theorem 2.2 is easy to be obtained by using linearization; therefore, we omit its proof. The objective of this section is to prove the following result.

Theorem 2.3. *System (1.4) has the unique positive equilibrium point $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ when one of the above conditions (i), (ii), (iii), (iv), (v), and (vi) holds. If $a_1 - a_0 < \min\{((a_2^2 - 2ka_2)/4a_3) + kb, kb - a_2k/a_3\}$ holds, then $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ is globally asymptotically stable.*

Proof. We make use of the general Lyapunov function

$$V(u(t)) = \sum_{i=1}^{3} \alpha_i \left(u_i - \overline{u}_i - \overline{u}_i \ln \frac{u_i}{\overline{u}_i} \right), \tag{2.7}$$

where α_i $(i = 1, 2, 3)$ are positive constants. It holds that $V(u(t)) \ge 0$ for any $t > 0$. Calculating the derivative along each solution of system (1.4), we have

$$\frac{dV}{dt} = \sum_{i=1}^{3} \alpha_i \frac{u_i - \overline{u}_i}{u_i} \frac{du_i}{dt}$$

$$= \alpha_1 \frac{u_1 - \overline{u}_1}{u_1}$$

$$\times \left\{ \frac{a_0}{\overline{u}_1} [u_1(u_2 - \overline{u}_2) - u_2(u_1 - \overline{u}_1)] - u_1(u_1 - \overline{u}_1)(-a_2 + a_3 u_1 + a_3 \overline{u}_1) - k u_1(u_3 - \overline{u}_3) \right\}$$

$$+ \alpha_2 \frac{u_2 - \overline{u}_2}{u_2} \frac{1}{\overline{u}_2} [u_2(u_1 - \overline{u}_1) - u_1(u_2 - \overline{u}_2)] + \alpha_3 u_3 [(u_1 - \overline{u}_1) - (u_3 - \overline{u}_3)] \frac{u_3 - \overline{u}_3}{u_3}$$

$$= -\alpha_1(-a_2 + a_3 u_1 + a_3 \overline{u}_1)(u_1 - \overline{u}_1)^2 - \alpha_3(u_3 - \overline{u}_3)^2$$

$$+ (\alpha_3 - k\alpha_1)(u_1 - \overline{u}_1)(u_3 - \overline{u}_3) + a_0 \alpha_1(u_1 - \overline{u}_1)$$

$$\times \left[-\frac{u_2}{u_1 \overline{u}_1}(u_1 - \overline{u}_1) + \frac{1}{\overline{u}_1}(u_2 - \overline{u}_2) \right] + \alpha_2(u_2 - \overline{u}_2) \frac{(u_1 - \overline{u}_1)u_2 - (u_2 - \overline{u}_2)u_1}{u_2 \overline{u}_2}.$$

$$(2.8)$$

Let $\alpha_2 = a_0 \alpha_1$ and $\alpha_3 = k\alpha_1$. Then

$$\frac{dV}{dt} = -\alpha_1(-a_2 + a_3 u_1 + a_3 \overline{u}_1)(u_1 - \overline{u}_1)^2 - \alpha_3(u_3 - \overline{u}_3)^2$$

$$- a_0 \alpha_1 \frac{1}{\overline{u}_1} \left[\sqrt{\frac{u_2}{u_1}}(u_1 - \overline{u}_1) - \sqrt{\frac{u_1}{u_2}}(u_2 - \overline{u}_2) \right]^2.$$

$$(2.9)$$

We observe that

$$a_1 - a_0 < kb - \frac{a_2 k}{a_3} \tag{2.10}$$

is a sufficient condition of $-a_2 + a_3 u_1 + a_3 \overline{u}_1 > 0$. So, when condition (2.10) holds, we have

$$\frac{dV}{dt} \leq 0. \tag{2.11}$$

Set $D = \{u \in \text{Int}\,\mathbb{R}_+^3 : dV/dt = 0\} = \{(\overline{u}_1, \overline{u}_2, \overline{u}_3)\}$. According to the Lyapunov-LaSalle invariance principle [14], $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ is global asymptotic stability if inequality (2.10) and all conditions of Theorem 2.2 are satisfied. Theorem 2.3 is, thus, proved. $\square$

3. Stability of the PDE System without Cross-Diffusion

In this section, we first prove the global existence and uniform boundedness of solutions, then discuss the stability of unique positive equilibrium solution for the weakly coupled reaction-diffusion system (1.5).

Denote that $F(\mathbf{u}) = (f_1, f_2, f_3)^{\mathrm{T}}$, where $\mathbf{u} = (u_1, u_2, u_3)$, $f_1 = a_0 u_2 - a_1 u_1 + a_2 u_1^2 - a_3 u_1^3 - k u_1 u_3$, $f_2 = u_1 - u_2$, and $f_3 = (-b + u_1 - u_3)u_3$. It is easy to see that $f_1, f_2, f_3 \in C^1(\overline{R}_+^3)$ with $\overline{R}_+^3 = \{(u_1, u_2, u_3) \mid u_i \geq 0, i = 1, 2, 3\}$. The standard PDE theory [15] shows that (1.5) has the unique solution $(u_1, u_2, u_3) \in [C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))]^3$, where $T \leq +\infty$ is the maximal existence time. The following theorem shows that the solution of (1.5) is uniformly bounded, and thus $T = +\infty$.

Theorem 3.1. *Let* $(u_1, u_2, u_3) \in [C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))]^3$ *be the solution of system* (1.5) *with initial values* $u_{i0}(x) \geq 0$ $(i = 1, 2, 3)$, *and let* T *be the maximal existence time. Then* $0 \leq u_1(x,t), u_2(x,t) \leq M_1$, $0 \leq u_3(x,t) \leq M_2$, *and* $t \in [0, T)$, *where* M_1 *is a positive constant depending only on* Ω *and all coefficients of* (1.5) *and* $\|u_{i0}\|_{L^\infty(\Omega)}$ $(i = 1, 2)$, $M_2 = \max\{\|u_{30}\|_{L^\infty(\Omega)}, M_1 - b\}$. *Furthermore,* $T = +\infty$ *and* $u_i(x,t) > 0$ *on* $\overline{\Omega}$ *for any* $t > 0$ *if* $u_{i0} \geq (\not\equiv)0$ $(i = 1, 2, 3)$.

Proof. Let (u_1, u_2, u_3) be the solution of (1.5) with initial values $u_{i0}(x) \geq 0$ $(i = 1, 2, 3)$. From the maximum principle for parabolic equations [16], it is not hard to verify that $u_i(x,t) \geq 0$ for $(x,t) \in \Omega \times [0, T)$ $(i = 1, 2, 3)$, where T is the maximal existence time of the solution (u_1, u_2, u_3). Furthermore, we know by the strong maximum principle that $u_i(x,t) > 0$ on $\overline{\Omega}$ for all $t > 0$ if $u_{i0} \geq (\not\equiv)0$ $(i = 1, 2, 3)$. Next we prove that the solution (u_1, u_2, u_3) is bounded on $\Omega \times [0, T)$.

Integrating the first two equations of (1.5) over Ω and adding the results linearly, we have that, by Young inequality,

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega [u_1 + (a_0 + a_1)u_2] dx &\leq \int_\Omega (a_0 u_1 - a_1 u_2) dx + \int_\Omega \left(a_2 u_1^2 - a_3 u_1^3\right) dx \\
&\leq -a_1 \int_\Omega u_2 dx + \int_\Omega \left[\left(a_0 + \frac{a_2^2}{a_3}\right)u_1 - a_2 u_1^2\right] dx \qquad (3.1) \\
&\leq C - \frac{a_1}{a_0 + a_1} \int_\Omega [u_1 + (a_0 + a_1)u_2] dx
\end{aligned}
$$

for some positive constant C depending only on the coefficients of (1.5). Therefore, $\|u_{1,2}(t)\|_{L^1(\Omega)}$ is bounded in $[0, \infty)$. Using [17, Exercise 5 of Section 3.5], we obtain that $\|u_{1,2}(t)\|_{L^\infty(\Omega)}$ is also bounded in $[0, \infty)$. Now note that $\sup_{\overline{\Omega} \times [0, \infty)} u_{1,2}(x,t) \leq M_1$. The maximum principle gives $u_3(x,t) \leq \max\{\|u_{30}\|_{L^\infty(\Omega)}, M_1 - b\} := M_2$. The proof of Theorem 3.1 is completed. $\qquad \square$

In order to prove the global stability of unique positive equilibrium solution for system (1.5), we first recall the following lemma which can be found in [7, 17].

Lemma 3.2. *Let* a *and* b *be positive constants. Assume that* $\phi, \varphi \in C^1([a, \infty))$, $\varphi(t) > 0$, *and* ϕ *is bounded from below. If* $\phi'(t) \leq -b\varphi(t)$, *and* $\varphi'(t) \leq K$ *in* $[a, \infty)$ *for some constant* K, *then* $\lim_{t \to \infty} \varphi(t) = 0$.

Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$. Denote that $X = \{\mathbf{u} \in [C^1(\overline{\Omega})]^3 \partial_\eta \mathbf{u} = 0, x \in \partial\Omega\}$, $\{\phi_{ij}; j = 1, 2, \ldots, \dim E(\mu_i)\}$ is an orthonormal basis of $E(\mu_i)$, and $X_{ij} = \{\mathbf{c} \cdot \phi_{ij} \mid \mathbf{c} \in R^3\}$. Then

$$X = \bigoplus_{i=1}^{\infty} X_i, \quad X_i = \bigoplus_{j=1}^{\dim E(\mu_i)} X_{ij}. \tag{3.2}$$

Next we present the clear proof of the the global stability by two steps:

Step 1 (Local Stability). Let $\mathfrak{D} = \mathrm{diag}(d_1, d_2, d_3)$ and $\mathfrak{L} = \mathfrak{D} + F_{\mathbf{u}}(\overline{\mathbf{u}})\mathfrak{D}\Delta + \{a_{ij}\}$, where

$$F_{\mathbf{u}}(\overline{\mathbf{u}}) = \begin{pmatrix} -2a_3\overline{u}_1^2 + a_2\overline{u}_1 - a_0 & a_0 & -k\overline{u}_1 \\ 1 & -1 & 0 \\ \overline{u}_3 & 0 & -\overline{u}_3 \end{pmatrix} \triangleq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{3.3}$$

The linearization of (1.5) at $\overline{\mathbf{u}}$ is

$$\mathbf{u}_t = \mathfrak{L}\mathbf{u}. \tag{3.4}$$

For each $i \geq 1$, X_i is invariant under the operator $\mathfrak{L}$, and λ is an eigenvalue of $\mathfrak{L}$ on X_i if and only if it is an eigenvalue of the matrix $-\mu_i\mathfrak{D} + F_{\mathbf{u}}(\overline{\mathbf{u}})$.

The characteristic polynomial of $-\mu_i\mathfrak{D} + F_{\mathbf{u}}(\overline{\mathbf{u}})$ is given by

$$\varphi_i(\lambda) = \lambda^3 + A_i\lambda^2 + B_i\lambda + C_i, \tag{3.5}$$

where

$$A_i = \mu_i(d_1 + d_2 + d_3) - a_{11} - a_{33} + 1,$$

$$B_i = \mu_i^2(d_1d_2 + d_1d_3 + d_2d_3) + \mu_i[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})]$$

$$+ a_{11}a_{33} - a_{13}a_{31} - a_{33} - (a_{11} + a_0), \tag{3.6}$$

$$C_i = \mu_i^3 d_1d_2d_3 + \mu_i^2(d_1d_3 - a_{11}d_2d_3 - a_{33}d_1d_2)$$

$$- \mu_i[d_1a_{33} - d_2(a_{11}a_{33} - a_{13}a_{31}) + d_3(a_{11} + a_0)] + a_{33}(a_{11} + a_0) - a_{13}a_{31}.$$

Thus

$$H_i = A_iB_i - C_i = P_3\mu_i^3 + P_2\mu_i^2 + P_1\mu_i + P_0, \tag{3.7}$$

where P_0, P_1, P_2, and P_3 are given by

$$
\begin{aligned}
P_0 &= (a_{11} + a_{33})[a_{13}a_{31} + a_{33} - a_{11}a_{33} + (a_{11} + a_0)] - a_{33}(1 + a_0) - (a_{11} + a_0), \\
P_1 &= d_1[a_{11}a_{33} - a_{13}a_{31} - (a_{11} + a_0)] - d_2[(a_{11} + a_0) + a_{33}] + d_3(a_{11}a_{33} - a_{33} - a_{13}a_{31}) \\
&\quad - (a_{11} + a_{33} - 1)[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})], \\
P_2 &= (d_1 + d_2 + d_3)[d_1(1 - a_{33}) - d_2(a_{11} + a_{33}) + d_3(1 - a_{11})] \\
&\quad - a_{11}d_1(d_2 + d_3) + d_2(d_1 + d_3) - a_{33}d_3(d_1 + d_2), \\
P_3 &= (d_1 + d_2)(d_1d_2 + d_1d_3 + d_2d_3) + d_3^2(d_1 + d_2).
\end{aligned}
\tag{3.8}
$$

According to the Routh-Hurwitz criterion [18], for each $i \geq 1$, the three roots $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$ of $\varphi_i(\lambda) = 0$ all have negative real parts if and only if $A_i > 0$, $C_i > 0$, and $H_i > 0$. Noting that $a_{33} < 0$ and $a_{13}a_{31} < 0$, the three roots have negative real parts if $a_{11} + a_0 < 0$. A direct calculation shows that $a_{11} + a_0$ is negative if

$$
a_1 - a_0 < \frac{a_2^2 - 2a_2k}{4a_3} + kb.
\tag{3.9}
$$

Now we can conclude that there exists a positive constant δ such that

$$
\mathrm{Re}\{\lambda_{i,1}\}, \mathrm{Re}\{\lambda_{i,2}\}, \mathrm{Re}\{\lambda_{i,3}\} \leq -\delta, \quad i \geq 1.
\tag{3.10}
$$

In fact, let $\lambda = \mu_i \xi$, then

$$
\varphi_i(\lambda) = \mu_i^3 \xi^3 + A_i \mu_i^2 \xi^2 + B_i \mu_i \xi + C_i \triangleq \tilde{\varphi}_i(\xi).
\tag{3.11}
$$

Since $\mu_i \to \infty$ as $i \to \infty$, it follows that

$$
\lim_{i \to \infty} \frac{\tilde{\varphi}_i(\xi)}{\mu_i^3} = \xi^3 + (d_1 + d_2 + d_3)\xi^2 + (d_1d_2 + d_2d_3 + d_1d_3)\xi + d_1d_2d_3 \triangleq \tilde{\varphi}(\xi).
\tag{3.12}
$$

It is easy to see that d_1, d_2, d_3 are the three roots of $\tilde{\varphi}(\xi) = 0$. Thus, there exists a positive constant $\bar{\delta}$ such that

$$
\mathrm{Re}\{-d_1\}, \mathrm{Re}\{-d_2\}, \mathrm{Re}\{-d_3\} \leq -\bar{\delta}.
\tag{3.13}
$$

By continuity, we see that there exists i_0 such that the three roots $\xi_{i,1}$, $\xi_{i,2}$, $\xi_{i,3}$ of $\tilde{\varphi}_i(\xi) = 0$ satisfy

$$
\mathrm{Re}\{\xi_{i,1}\}, \mathrm{Re}\{\xi_{i,2}\}, \mathrm{Re}\{\xi_{i,3}\} \leq -\frac{\bar{\delta}}{2}, \quad i \geq i_0.
\tag{3.14}
$$

So

$$\text{Re}\{\lambda_{i,1}\}, \text{Re}\{\lambda_{i,2}\}, \text{Re}\{\lambda_{i,3}\} \leq -\frac{\mu_i \overline{\delta}}{2} \leq -\frac{\widehat{\delta}}{2}, \quad i \geq i_0. \tag{3.15}$$

Let

$$-\widetilde{\delta} = \max_{1 \leq i \leq i_0}\{\text{Re}\{\lambda_{i,1}\}, \text{Re}\{\lambda_{i,2}\}, \text{Re}\{\lambda_{i,3}\}\}, \tag{3.16}$$

then $\widetilde{\delta} > 0$, and (3.10) holds for $\delta = \min\{\widetilde{\delta}, \widehat{\delta}/2\}$.

Consequently, the spectrum of $\mathfrak{L}$, consisting only of eigenvalues, lies in $\{\text{Re }\lambda \leq -\delta\}$ if (3.9) holds, and the local stability of $\overline{\mathbf{u}}$ follows [19, Theorem 5.1.1].

Step 2 (*Global Stability*). In the following, C denotes a generic positive constant which does not depend on $x \in \overline{\Omega}$ and $t \geq 0$. Let $\mathbf{u}$ be the unique positive solution. Then it follows from Theorem 3.1 that $\mathbf{u}(\cdot, t)$ is bounded uniformly on $\overline{\Omega}$, that is, $\|u_i(\cdot, t)\|_\infty \leq C$ for all $t \geq 0$. By [20, Theorem A_2],

$$\|u_i(\cdot, t)\|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \quad \forall t \geq 1. \tag{3.17}$$

Define the Lyapunov function

$$\begin{aligned}
E(t) &= \int_\Omega \left(u_1 - \overline{u}_1 - \overline{u}_1 \ln \frac{u_1}{\overline{u}_1}\right) dx + a_0 \int_\Omega \left(u_2 - \overline{u}_2 - u_2 \ln \frac{u_2}{\overline{u}_2}\right) dx \\
&\quad + k \int_\Omega \left(u_3 - \overline{u}_3 - \overline{u}_3 \ln \frac{u_3}{\overline{u}_3}\right) dx.
\end{aligned} \tag{3.18}$$

Then $E(t) \geq 0$ for all $t > 0$. Using (1.5) and integrating by parts, we have

$$\begin{aligned}
E'(t) &= \int_\Omega \left(\frac{u_1 - \overline{u}_1}{u_1} u_{1t} + a_0 \frac{u_2 - \overline{u}_2}{u_2} u_{2t} + k \frac{u_3 - \overline{u}_3}{u_3} u_{3t}\right) dx \\
&= -\int_\Omega \left(\frac{d_1 \overline{u}_1}{u_1^2}|\nabla u_1|^2 + a_0 \frac{d_2 \overline{u}_2}{u_2^2}|\nabla u_2|^2 + k \frac{d_3 \overline{u}_3}{u_3^2}|\nabla u_3|^2\right) dx \\
&\quad + \int_\Omega \left[\frac{u_1 - \overline{u}_1}{u_1} f_1(\mathbf{u}) + a_0 \frac{u_2 - \overline{u}_2}{u_2} f_2(\mathbf{u}) + k \frac{u_3 - \overline{u}_3}{u_3} f_3(\mathbf{u})\right] dx \\
&\leq \int_\Omega \Bigg\{ -(-a_2 + a_3 u_1 + a_3 \overline{u}_1)(u_1 - \overline{u}_1)^2 - k(u_3 - \overline{u}_3)^2 \\
&\quad + a_0(u_1 - \overline{u}_1)\left[-\frac{u_2}{u_1 \overline{u}_1}(u_1 - \overline{u}_1) + \frac{1}{\overline{u}_1}(u_2 - \overline{u}_2)\right] \\
&\quad + a_0(u_2 - \overline{u}_2)\frac{(u_1 - \overline{u}_1)u_2 - (u_2 - \overline{u}_2)u_1}{u_2 \overline{u}_2} \Bigg\} dx
\end{aligned}$$

$$= \int_\Omega \left[-(-a_2 + a_3 u_1 + a_3 \bar{u}_1)(u_1 - \bar{u}_1)^2 - k(u_3 - \bar{u}_3)^2 \right] dx$$

$$- \frac{a_0}{\bar{u}_1} \int_\Omega \left[\sqrt{\frac{u_2}{u_1}} (u_1 - \bar{u}_1) - \sqrt{\frac{u_1}{u_2}} (u_2 - \bar{u}_2) \right]^2 dx$$

$$\leq - \int_\Omega (-a_2 + a_3 u_1 + a_3 \bar{u}_1)(u_1 - \bar{u}_1)^2 dx - k \int_\Omega (u_3 - \bar{u}_3)^2 dx.$$

$$(3.19)$$

Taking $l_1 = -a_2 + a_3 \bar{u}_1$ and $l_3 = k$, we have that

$$E'(t) \leq -l_1 \int_\Omega (u_1 - \bar{u}_1)^2 dx - l_3 \int_\Omega (u_3 - \bar{u}_3)^2 dx, \qquad (3.20)$$

where $l_1 > 0$ holds for

$$(a_1 - a_0) < \frac{(a_3 b - a_2)k}{a_3}. \qquad (3.21)$$

From Theorem 3.1 the solution u_i of (1.5) is bounded, and so are the derivatives of $(u_1 - \bar{u}_1)^2$ and $(u_3 - \bar{u}_3)^2$ by equations in (1.5). Applying Lemma 3.2, we obtain

$$\lim_{t \to \infty} \int_\Omega (u_1 - \bar{u}_1)^2 dx = 0, \qquad \lim_{t \to \infty} \int_\Omega (u_3 - \bar{u}_3)^2 dx = 0. \qquad (3.22)$$

As $u_i^2 \leq C$, it follows that

$$E'(t) \leq - \int_\Omega \left(\frac{d_1 \bar{u}_1}{u_1^2} |\nabla u_1|^2 + a_0 \frac{d_2 \bar{u}_2}{u_2^2} |\nabla u_2|^2 + k \frac{d_3 \bar{u}_3}{u_3^2} |\nabla u_3|^2 \right) dx$$

$$\leq -C \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2 \right) dx \qquad (3.23)$$

$$\triangleq -\varphi(t).$$

Using inequality (3.17) and system (1.5), the derivative of $\varphi(t)$ is bounded in $[1, \infty)$. From Lemma 3.2, we conclude that $\varphi(t) \to 0$ as $t \to \infty$. Therefore

$$\lim_{t \to \infty} \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2 \right) dx = 0. \qquad (3.24)$$

Using the Poincaré inequality yields

$$\lim_{t \to \infty} \int_\Omega (u_1 - \tilde{u}_1)^2 dx = \lim_{t \to \infty} \int_\Omega (u_2 - \tilde{u}_2)^2 dx = \lim_{t \to \infty} \int_\Omega (u_3 - \tilde{u}_3)^2 dx = 0, \qquad (3.25)$$

where $\tilde{u}_i(t) = (1/|\Omega|)\int_\Omega u_i dx, i = 1, 2, 3$. Thus, it follows from (3.22) and (3.25) that

$$|\Omega|(\tilde{u}_1(t) - \overline{u}_1)^2 = \int_\Omega (\overline{u}_1 - \overline{u}_1)^2 dx \leq 2\int_\Omega (\tilde{u}_1 - u_1)^2 dx + 2\int_\Omega (u_1 - \overline{u}_1)^2 dx \longrightarrow 0 \qquad (3.26)$$

as $t \to \infty$. So we have $\tilde{u}_1(t) \to \overline{u}_1$ as $t \to \infty$. Similarly, $\tilde{u}_3(t) \to \overline{u}_3$ as $t \to \infty$. Therefore, there exists a sequence $\{t_m\}$ with $t_m \to \infty$ such that $\tilde{u}_1'(t_m) \to 0$. As $\tilde{u}_2(t_m)$ is bounded, there exists a subsequence of $\{t_m\}$, still denoted by the same notation, and nonnegative constant $\hat{u}_2$ such that

$$\tilde{u}_2(t_m) \longrightarrow \hat{u}_2. \qquad (3.27)$$

At $t = t_m$, from the first equation of (1.5), we have

$$
\begin{aligned}
|\Omega|\tilde{u}_1'(t_m) = \int_\Omega u_{1t} dx\bigg|_{t_m} &= \int_\Omega (d\Delta u_1 + f_1(u)) dx\bigg|_{t_m} \\
&= \int_\Omega \frac{a_0}{\overline{u}_1}[u_1(u_2 - \overline{u}_2) - u_2(u_1 - \overline{u}_1)] dx\bigg|_{t_m} \\
&\quad + \int_\Omega -u_1(u_1 - \overline{u}_1)(-a_2 + a_3 u_1 + \overline{u}_1) dx\bigg|_{t_m} \\
&\quad + \int_\Omega -k u_1(u_3 - \overline{u}_3) dx\bigg|_{t_m} \longrightarrow 0.
\end{aligned}
\qquad (3.28)
$$

In view of (3.22) and (3.27), it follows from (3.28) that $\hat{u}_2 = \overline{u}_2$, thus

$$\lim_{m\to\infty} \tilde{u}_2(t_m) = \overline{u}_2. \qquad (3.29)$$

According to (3.17), there exists a subsequence of $\{t_m\}$, denoted still by $\{t_m\}$, and nonnegative functions $w_i \in C^2(\overline{\Omega})$, such that

$$\lim_{m\to\infty} \|u_i(\cdot, t_m) - w_i(\cdot)\|_{C^2(\overline{\Omega})} = 0, \quad i = 1, 2, 3. \qquad (3.30)$$

In view of (3.29) and noting that in fact $\tilde{u}_1(t) \to \overline{u}_1$ and $\tilde{u}_3(t) \to \overline{u}_3$, we know that $w_i \equiv \overline{u}_i, i = 1, 2, 3$. Therefore,

$$\lim_{m\to\infty} \|u_i(\cdot, t_m) - \overline{u}_i\|_{C^2(\overline{\Omega})} = 0, \quad i = 1, 2, 3. \qquad (3.31)$$

The global asymptotic stability of $\overline{u}$ follows from (3.31) and the local stability of $\overline{u}$.

Theorem 3.3. *System* (1.5) *has the unique positive equilibrium point* $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ *when one of the conditions (i), (ii), (iii), (iv), (v), and (vi) in Section 2 holds. If* (3.9) *and* (3.21) *hold, then* $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ *is globally asymptotically stable.*

4. Global Existence of Classical Solutions and Convergence

In this section, we discuss the existence of nonnegative classical global solutions and the global asymptotic stability of unique positive equilibrium point of system (1.6).

Some notations throughout this section are as follows: $Q_T = \Omega \times [0, T]$, $u \in W_p^k(\Omega)$ means that $D^\alpha u \in L^p(\Omega)$ for any $|\alpha| \le k$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\|u\|_{W_p^k(\Omega)} = (\int_\Omega \sum_{|\alpha| \le k} |D^\alpha u|^p dx)^{1/p}$, $u \in W_p^{2,1}(Q_T)$ means that u, u_{x_i}, $u_{x_i x_j}(i, j = 1, 2, \ldots, n)$ and u_t are in $L^p(Q_T)$, $\|u\|_{W_p^{2,1}(Q_T)} = (\int_{Q_T} (|u|^p + |Du|^p + |D^2u|^p + |u_t|^p) dx\, dt)^{1/p}$, and $\|u\|_{V_2(Q_T)} = \sup_{0 \le t \le T} \|u(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}$ with $V_2(Q_T) = L^\infty((0, T), L^2(\Omega)) \cap W_2^{1,0}(Q_T)$.

To obtain $C^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$ normal estimates of the solution for (1.6), we present a series of lemmas in the following.

Lemma 4.1. Let (u, v, w) be the solution of (1.6). Then there exists a positive constant $M_0(\ge 1)$ such that

$$0 \le u, v \le M_0, \quad 0 \le w, \quad \forall t \ge 0. \tag{4.1}$$

Proof. By applying the comparison principle [20] to system (1.6), we have $u \ge 0$, $v \ge 0$, and $w \ge 0$ in $\overline{Q_T}$. To prove that $u, v \le M_0$ in the following, we consider the auxiliary problem

$$u_t - (d_1 + 2\alpha_{11}u)\Delta u + 2\alpha_{11}\sum_{i=1}^n u_{x_i}u_{x_i} = f_1, \quad x \in \Omega, \ t > 0,$$

$$v_t - (d_2 + 2\alpha_{22}v)\Delta v + 2\alpha_{22}\sum_{i=1}^n v_{x_i}v_{x_i} = f_2, \quad x \in \Omega, \ t > 0, \tag{4.2}$$

$$\partial_\eta u(x, t) = \partial_\eta v(x, t) = 0, \quad x \in \partial\Omega, \ t > 0,$$

$$u(x, 0) = u_0(x) \ge (\not\equiv)0, \quad v(x, 0) = v_0(x) \ge (\not\equiv)0, \quad x \in \Omega.$$

Notice that the functions f_1 and f_2 are sufficiently smooth in $\mathbb{R}^2$, and are quasimonotone in $\mathbb{R}_+^2$. Let $(0, 0)$ and (M, N) be a pair of upper-lower solutions for (4.2), where M and N are positive constants. Direct calculation with inequalities

$$a_0 N - a_1 M + a_2 M^2 - a_3 M^3 \le 0,$$

$$M - N \le 0, \tag{4.3}$$

$$u_0 \le M, \qquad v_0 \le N$$

yields $M = \max\{(a_2 + \sqrt{a_2^2 + 4a_3|a_0 - a_1|})/2a_3, \|u_0\|_{L^\infty(\Omega)}\}$ and $N = \max\{M, \|v_0\|_{L^\infty(\Omega)}\}$. It follows that there exists $M_0 = K \max\{M, N\} \ge 1$ for any $t \ge 0$, where K is a big enough positive constant such that (4.1) holds. $\qquad\square$

Lemma 4.2. *Let* $X = (d_1 + \alpha_{11}u)u$, *and* $u \in L^\infty(Q_T)$ *for the solution to following equation:*

$$
\begin{aligned}
u_t &= \Delta[(d_1 + \alpha_{11}u)u] + f_1, \quad (x,t) \in \Omega \times (0,T), \\
\partial_\eta u &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
u(x,0) &= u_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\tag{4.4}
$$

where d_1, α_{11} *are positive constants and* $0 \leq w \in L^2(Q_T)$. *Then there exists a positive constant* $C(T)$, *depending on* $\|u_0\|_{W_2^1(\Omega)}$ *and* $\|u_0\|_{L^\infty(\Omega)}$, *such that*

$$
\|X\|_{W_2^{2,1}(Q_T)} \leq C(T).
\tag{4.5}
$$

Furthermore,

$$
\nabla X \in V_2(Q_T), \qquad \nabla u \in L^{2(n+2)/n}(Q_T).
\tag{4.6}
$$

Proof. It is easy to check, from $X = (d_1 + \alpha_{11}u)u$, that

$$
X_t = (d_1 + 2\alpha_{11}u)\Delta X + C_1 - C_2 w,
\tag{4.7}
$$

where $C_1 = (d_1 + 2\alpha_{11}u)(a_0 v - a_1 u + a_2 u^2 - a_3 u^3)$ and $C_2 = k(d_1 + 2\alpha_{11}u)u$. C_1 and C_2 are bounded in Q_T from (4.1). Multiplying (4.7) by $-\Delta X$, and integrating by parts over Q_t, yields

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla X(x,t)|^2 dx - \frac{1}{2}\int_\Omega |\nabla X(x,0)|^2 dx + d_1 \int_{Q_t} |\Delta X|^2 dx\, dt \\
&\leq \int_{Q_T} |C_1 + C_2 w||\Delta X| dx\, dt.
\end{aligned}
\tag{4.8}
$$

Using Hölder inequality and Young inequality to estimate the right side of (4.8), we have

$$
\begin{aligned}
\|C_1 + C_2 w\|_{L^2(Q_T)}\|\Delta X\|_{L^2(Q_T)} &\leq m_1\left(1 + \|w\|_{L^2(Q_T)}\right)\|\Delta X\|_{L^2(Q_T)} \\
&\leq \frac{m_1^2(1 + M_3)^2}{2d_1} + \frac{d_1}{2}\|\Delta X\|_{L^2(Q_T)}^2
\end{aligned}
\tag{4.9}
$$

with some $m_1 > 0$. Substituting (4.9) into (4.8) yields

$$
\sup_{0 \leq t \leq T}\int_\Omega |\nabla X(x,t)|^2 dx + d_1\int_{Q_t} |\Delta X|^2 dx\, dt \leq m_2,
\tag{4.10}
$$

where m_2 depends on $\|u_0\|_{W_2^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$. Since $X \in L^2(Q_T)$, the elliptic regularity estimate [10, Lemma 2.3] yields

$$\int_{Q_T} |X_{x_i x_j}|^2 dx\, dt \le m_3, \quad i,j = 1,\dots,n. \tag{4.11}$$

From (4.7), we have $X_t \in L^2(Q_T)$. Hence, $\|X\|_{W_2^{2,1}(Q_T)} \le C(T)$. Moreover, the Sobolev embedding theorem shows that (4.6) holds. $\qquad\square$

Lemma 4.3 (Lemma 4.3 can be presented by combining Lemmas 2.3 and 2.4 in [11]). *Let $p > 1, \tilde{p} = 2 + 4p/n(q+1)$, and let w satisfy*

$$\sup_{0\le t\le T} \|w\|_{L^{2p/(p+1)}(\Omega)} + \|\nabla w\|_{L^2(Q_T)} < \infty, \tag{4.12}$$

and there exist positive constants $\beta \in (0,1)$ and C_T such that $\int_\Omega |w(\cdot,t)|^\beta dx \le C_T$ ($\forall t \le T$). Then there exists a positive constant M' independent of w but possibly depending on n, Ω, p, β, and C_T such that

$$\|w\|_{L^{\tilde{q}}(Q_T)} \le M' \left\{ 1 + \left(\sup_{0\le t\le T} \|w(t)\|_{L^{2p/(p+1)}(\Omega)} \right)^{4p/n(p+1)\tilde{p}} \|\nabla w\|_{L^2(Q_T)}^{2/\tilde{p}} \right\}. \tag{4.13}$$

Finally, one proposes some standard embedding results which are important to obtain the $C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ normal estimates of the solution for (1.6).

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^N$ be a fixed bounded domain and $\partial\Omega \subset C^2$. Then for all $u \in W_q^{2,1}(Q_T)$ with $q \ge 1$, one has*

    (1) $\|\nabla u\|_{L^p(Q_T)} \le C\|u\|_{W_q^{2,1}(Q_T)}$, *for all $1 \le p \le (n+2)q/(n+2-q)$, $q < n+2$,*

    (2) $\|\nabla u\|_{L^p(Q_T)} \le C\|u\|_{W_q^{2,1}(Q_T)}$, *for all $1 \le p \le \infty$, $q = n+2$,*

    (3) $\|\nabla u\|_{C^{\alpha,\alpha/2}(Q_T)} \le C\|u\|_{W_q^{2,1}(Q_T)}$ *for all $1 - (n+2)/q \le \alpha \le 1$, $q > n+2$,*

where C is a positive constant dependent on q, n, Ω, and T.

The main result about the global existence of nonnegative classical solution for the cross-diffusion system (1.6) is given as follows.

Theorem 4.5. *Assume that $u_0 \ge 0$, $v_0 \ge 0$, and $w_0 \ge 0$ satisfy homogeneous Neumann boundary conditions and belong to $C^{2+\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$. Then system (1.6) has a unique nonnegative solution $u, v, w \in C^{2+\lambda,1+\lambda/2}(\overline{\Omega} \times [0,\infty))$ when the space dimension is $n \le 5$.*

Proof.

Step 1. L^1-, L^2-Estimates and L^q-Estimates of w. Firstly, integrating the third equation of (1.6) over Ω, we have

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega w \, dx &\leq \frac{1}{2} \int_\Omega u^2 dx - \frac{1}{2} \int_\Omega w^2 dx \\
&\leq \frac{|\Omega|}{2} M_0^2 - \frac{1}{2|\Omega|} \left(\int_\Omega w \, dx \right)^2.
\end{aligned}
\tag{4.14}
$$

Thus

$$
\int_\Omega w \, dx \leq \max\left\{ M_0|\Omega|, \int_\Omega w_0 dx \right\} := M_1, \quad \forall t \geq 0.
\tag{4.15}
$$

Furthermore

$$
\|w\|_{L^1(Q_T)} \leq \int_0^T M_1 dt := M_2.
\tag{4.16}
$$

Integrating (4.14) in $[0, T]$ and moving terms yield

$$
\|w\|_{L^2(Q_T)} \leq \left(M_0^2 |\Omega| T + 2\|w_0\|_{L_1(\Omega)} \right)^{1/2} := M_3.
\tag{4.17}
$$

Secondly, multiplying the third equation of (1.6) by qw^{q-1} $(q > 1)$ and integrating over Ω, we have

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega w^q dx &\leq -\frac{4(q-1)d_3}{q} \int_\Omega \left| \nabla \left(w^{q/2} \right) \right|^2 dx - \frac{8q(q-1)\alpha_{33}}{(q+1)^2} \int_\Omega \left| \nabla \left(w^{(q+1)/2} \right) \right|^2 dx \\
&\quad - (q-1)\alpha_{31} \int_\Omega \nabla u \cdot \nabla(w^q) dx - (q-1)\alpha_{32} \int_\Omega \nabla v \cdot \nabla(w^q) dx \\
&\quad + q \int_\Omega w^q (-b + u - w) dx.
\end{aligned}
\tag{4.18}
$$

Integrating the above expression in $[0, t]$ $(t \leq T)$ yields

$$\int_\Omega w^q(x, t)dx + \frac{4(q-1)d_3}{q}\int_{Q_t}\left|\nabla\left(w^{q/2}\right)\right|^2 dx\,dt + \frac{8q(q-1)\alpha_{33}}{(q+1)^2}\int_{Q_t}\left|\nabla\left(w^{(q+1)/2}\right)\right|^2 dx\,dt$$

$$\leq \int_\Omega w_0^q(x)dx - (q-1)\alpha_{31}\int_{Q_t}\nabla u\cdot\nabla(w^q)dx\,dt - (q-1)\alpha_{32}\int_{Q_t}\nabla v\cdot\nabla(w^q)dx\,dt$$

$$+ q\int_{Q_t}w^q(-b+u-w)dx\,dt.$$

$$(4.19)$$

Since $\nabla u \in L^{2(n+2)/n}(Q_T)$ from Lemma 4.2, and using Hölder inequality and Young inequality, we have

$$-(q-1)\alpha_{31}\int_{Q_t}\nabla u\cdot\nabla(w^q)dx\,dt - (q-1)\alpha_{32}\int_{Q_t}\nabla v\cdot\nabla(w^q)dx\,dt$$

$$\leq \frac{2q(q-1)\alpha_{31}}{q+1}\left|\int_{Q_t}w^{(q-1)/2}\nabla\left(w^{(q+1)/2}\right)\cdot\nabla u\,dx\,dt\right|$$

$$+ \frac{2q(q-1)\alpha_{32}}{q+1}\left|\int_{Q_t}w^{(q-1)/2}\nabla\left(w^{(q+1)/2}\right)\cdot\nabla v\,dx\,dt\right|$$

$$\leq \frac{2q(q-1)}{q+1}\left(\alpha_{31}\|\nabla u\|_{L^{2(n+2)/n}(Q_T)} + \alpha_{32}\|\nabla v\|_{L^{2(n+2)/n}(Q_T)}\right)$$

$$\times \|w^{(q-1)/2}\|_{L^{n+2}(Q_T)}\|\nabla(w^{(q+1)/2})\|_{L^2(Q_T)}$$

$$\leq C_3\left\|\nabla\left(w^{(q+1)/2}\right)\right\|_{L^2(Q_T)}\|w^{(q-1)/2}\|_{L^{n+2}(Q_T)}$$

$$\leq C_3\varepsilon\left\|\nabla(w^{(q+1)/2})\right\|_{L^2(Q_T)}^2 + \frac{C_3}{4\varepsilon}\|w^{(q-1)/2}\|_{L^{n+2}(Q_T)}^2.$$

$$(4.20)$$

From (4.1) and $\alpha^p\beta \leq (p/(p+1))\alpha^{p+1} + (1/(p+1))\beta^{p+1}$ $(\alpha, \beta > 0)$, it holds that

$$q\int_{Q_t}w^q(-b+u-w)dx\,dt \leq q(-b+M_0)\int_{Q_t}w^q dx\,dt - q\int_{Q_t}w^{q+1}dx$$

$$\leq q(-b+M_0)|Q_T|^{1/(q+1)}\|w\|_{L^{q+1}(Q_t)}^q - q\|w\|_{L^{q+1}(Q_t)}^{q+1}$$

$$\leq (-b+M_0)^2|Q_T| \triangleq C_4.$$

$$(4.21)$$

Taking $\overline{w} = w^{(q+1)/2}$ and selecting a proper ε such that $C_3\varepsilon \leq 4q(q-1)\alpha_{33}/(q+1)^2$, then applying (4.20) and (4.21) to (4.19) yields

$$\int_\Omega \overline{w}^{2q/(q+1)}(x,t)dx + \frac{4(q-1)d_3}{q}\int_{Q_t}\left|\nabla\left(\overline{w}^{q/(q+1)}\right)\right|^2 dx\,dt + \frac{4q(q-1)\alpha_{33}}{(q+1)^2}\int_{Q_t}|\nabla\overline{w}|^2 dx\,dt$$

$$\leq \|w_0\|_{L^q(\Omega)}^q + \frac{C_3}{4\varepsilon}\|\overline{w}\|_{L^{(q-1)(n+2)/(q+1)}(Q_T)}^{2(q-1)/(q+1)} + C_4$$

$$\leq C_5\left(1 + \|\overline{w}\|_{L^{(q-1)(n+2)/(q+1)}(Q_T)}^{2(q-1)/(q+1)}\right).$$

$$(4.22)$$

Denote that $E \equiv \sup_{0<t<T}\int_\Omega \overline{w}^{2q/(q+1)}(x,t)dx + \int_{Q_T}|\nabla\overline{w}|^2 dx\,dt$. Then it follows from (4.22) that

$$E \leq C_6\left(1 + \|\overline{w}\|_{L^{(q-1)(n+2)/(q+1)}(Q_T)}^{2(q-1)/(q+1)}\right).$$

$$(4.23)$$

It is easy to see that $2q/(q+1) < 2 < \tilde{q}$ and $(q-1)(n+2)/(q+1) < \tilde{q} = 2 + 4q/n(q+1)$ for any $q < n(n+4)/(n^2-4)$; hence

$$E \leq C_7\left(1 + \|\overline{w}\|_{L^{\tilde{q}}(Q_T)}^{2(q-1)/(q+1)}\right).$$

$$(4.24)$$

Take $\beta = 2/(q+1) \in (0,1)$. Then it follows from $L^1(\Omega)$-estimates of w, namely (4.15), that

$$\|\overline{w}\|_{L^\beta(\Omega)} = \left(\int_\Omega |\overline{w}(\cdot,t)|^\beta dx\right)^{1/\beta} = \|w\|_{L^1(\Omega)}^{1/\beta} \leq M_1^{1/\beta}, \quad \forall t \leq T.$$

$$(4.25)$$

It follows from Lemma 4.3 and (4.24) that

$$E \leq C_7\left[1 + \left(M' + M'\sup_{0<t<T}\|\overline{w}(\cdot,t)\|_{L^{2q/(q+1)}(\Omega)}^{4q/n(q+1)\tilde{q}}\|\nabla\overline{w}\|_{L^2(Q_T)}^{2/\tilde{q}}\right)^{2(q-1)/(q+1)}\right]$$

$$\leq C_8\left[1 + \left(\sup_{0<t<T}\|\overline{w}(\cdot,t)\|_{L^{2q/(q+1)}(\Omega)}^{2q/(q+1)}\right)^{4(q-1)/n(q+1)\tilde{q}}\left(\|\nabla\overline{w}\|_{L^2(Q_T)}^2\right)^{2(q-1)/(q+1)\tilde{q}}\right]$$

$$\leq C_8\left(1 + E^{((2+n)/n\tilde{q})\cdot 2((q-1)/(q+1-))}\right).$$

$$(4.26)$$

Since $((2+n)/n\tilde{q}) \cdot (2(q-1)/(q+1)) \in (0,1)$, E is bounded by contrary proof. It follows that $\|\overline{w}\|_{L^{\tilde{q}}(Q_T)}$ is bounded, that is, $w \in L^{(q+1)\tilde{q}/2}(Q_T)$. It is easy to check that $(q+1)\tilde{q}/2 \in (1, 2(n+1)/(n-2))$ for all $q < n(n+4)/(n^2-4)$ still denote $(q+1)\tilde{q}/2$ by q, then

$$w \in L^q(Q_T), \quad \forall q \in \left(1, \frac{2(n+1)}{n-2}\right).$$

$$(4.27)$$

Finally, we observe that $q = 2$ satisfies $(n^2 - 4)q < n^2 + 4n$ with $n = 2, 3, 4, 5$. So take $q = 2$ for (4.17) and (4.19). Then there exists a positive constant M_4 such that

$$\|w\|_{V_2(Q_T)} \leq M_4. \tag{4.28}$$

Step 2. L^∞-Estimates of w. We rewrite the third equation of (1.6) as a linear parabolic equation

$$\frac{\partial w}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial w}{\partial x_j}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x,t)w) - (b - u + w)w, \tag{4.29}$$

where $a_{ij}(x,t) = (d_3 + \alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w)\delta_{ij}$, $a_i(x,t) = \alpha_{31}(\partial u/\partial x_i) + \alpha_{32}(\partial v/\partial x_i)$, δ_{ij} are Kronecker symbols.

To apply the maximum principle [15, Theorem 7.1, page 181] to (4.15) to obtain $w \in L^\infty(Q_T)$, we need to verify that the following conditions hold: (1) $\|w\|_{V_2(Q_T)}$ is bounded; (2) $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \nu \sum_{i=1}^{n} \xi_i^2$; (3) $\|\sum_{i=1}^{n} a_i^2(x,t), b - u + w\|_{L^{q,r}(Q_T)} \leq \mu_1$, where ν and μ_1 are positive constants, and q and r satisfy

$$\frac{1}{r} + \frac{n}{2q} = 1 - \chi, \quad 0 < \chi < 1,$$
$$q \in \left[\frac{n}{2(1-\chi)}, +\infty\right) \quad r \in \left[\frac{1}{1-\chi}, +\infty\right), \quad n \geq 2. \tag{4.30}$$

Next we verify conditions (1)–(3) in turn. From (4.28), condition (1) is true for $n \leq 5$. One can choose $\nu = d_3$ such that condition (2) holds. To verify condition (3), the first equation of (1.6) is written in the divergence form

$$u_t = \nabla \cdot [(d_1 + 2\alpha_{11}u)\nabla u] + a_0 v - a_1 u + a_2 u^2 - a_3 u^3 - kuw, \tag{4.31}$$

where $d_1 + 2\alpha_{11}u$ is bounded in $\overline{Q}_T$ by Lemma 4.1, and $a_0 v - a_1 u + a_2 u^2 - a_3 u^3 - kuw \in L^q(Q_T)$ for $q \in ((n+2)/2, 2(n+1)/(n-2))$ from (4.27). Application of the Hölder continuity result [15, Theorem 10.1, page 204] to (4.19) yields

$$u \in C^{\beta,\beta/2}\left(\overline{Q}_T\right), \quad \beta \in (0,1). \tag{4.32}$$

Returning to (4.7), since $C_1 + C_2 w \in L^q(Q_T)$ for any $q \in ((n+2)/2, 2(n+1)/n-2)$ by (4.1) and (4.27), and $d_1 + \alpha_{11}u \in C^{\beta,\beta/2}(\overline{Q}_T)$ by (4.32), then by applying the parabolic regularity theorem [15, Theorem 9.1, pages 341-342] to (4.7) we have

$$X \in W_q^{2,1}(Q_T), \quad \forall q \in \left(\frac{n+2}{2}, \frac{2(n+1)}{n-2}\right). \tag{4.33}$$

Hence $\nabla X \in L^{(n+2)q/(n+2-q)}(Q_T)$ from Lemma 4.4, which shows that $\nabla u \in L^{(n+2)q/(n+2-q)}(Q_T)$. Similarly, $\nabla v \in L^{(n+2)q/(n+2-q)}(Q_T)$ by the second equation of (1.6). Now we can show that

$|\nabla u|^2, |\nabla v|^2 \in L^{(n+2)q/2(n+2-q)}(Q_T)$, which imply that $\sum_{i=1}^{n} a_i^2(x,t) \in L^{(n+2)q/2(n+2-q)}(Q_T)$. In addition, $b - u + w$ obviously belongs to $L^q(Q_T)$. It follows that one can select $q = r = (n+2)p/2(n+2-p)$. Now the above three conditions are satisfied, and $w \in L^\infty(Q_T)$ from [15, Theorem 7.1, page 181]. Recalling **Lemma 4.1**, thus there exists a positive constant M_5 for any $T > 0$ such that

$$\|u\|_{L^\infty(Q_T)}, \qquad \|v\|_{L^\infty(Q_T)}, \qquad \|w\|_{L^\infty(Q_T)} \leq M_5. \tag{4.34}$$

Step 3. The Proof of the Classical Solution (u,v,w) of (1.6) in Q_T for Any $T > 0$. Because $(d_1 u + \alpha_{11} u^2 \alpha)_t = (d_1 + 2\alpha_{11} u)\Delta(d_1 u + \alpha_{11} u^2) + (d_1 + 2\alpha_{11} u)f_1$, we have from (4.34) that $X = d_1 u + \alpha_{11} u^2 \in W_q^{2,1}(Q_T)$ for any $q > 1$. So $\nabla X \in C^{\beta^*, \beta^*/2}(\overline{Q}_T)$ for all $\beta^* \in (0,1)$. It follows from [15, Lemma 3.3, page 80] that $X \in C^{1+\beta^*, (1+\beta^*)/2}(\overline{Q}_T)$. And direct calculation $X = (d_1 + \alpha_{11} u)u$ yields $u = (-d_1 + \sqrt{d_1^2 + 4\alpha_{11} X})/2\alpha_{11}$. So we have

$$u \in C^{1+\beta^*, (1+\beta^*)/2}(\overline{Q}_T), \quad \forall \beta^* \in (0,1). \tag{4.35}$$

The third equation of (1.6) can be written as

$$w_t = \nabla \cdot [(d_3 + \alpha_{31} u + \alpha_{32} v + 2\alpha_{33} w)\nabla w + (\alpha_{31} \nabla u + \alpha_{32} \nabla v)w] + (-b + u - w)w. \tag{4.36}$$

Summarizing the above conclusions that are proved, we know that $(-b + u - w)w \in L^\infty(Q_T)$ and $u, v, w, \nabla u, \nabla v$ are all bounded in Q_T. It follows from [15, Theorem 10.1, page 204] that there exists $\sigma_1 \in (0,1)$ such that

$$w \in C^{\sigma_1, \sigma_1/2}(\overline{Q}_T). \tag{4.37}$$

The proof of **Lemma 4.2** is similar. Then we have $\nabla v \in L^2(Q_T)$, that is, $v \in V_2(Q_T)$. Applying the [13, Theorem 10.1, page 204] to the second equation (1.6), there exists $\sigma_2 \in (0,1)$ such that

$$v \in C^{\sigma_2, \sigma_2/2}(\overline{Q}_T). \tag{4.38}$$

Furthermore, applying Schauder estimate [15, page 320-321] yields $v \in C^{2+\sigma^*, 1+\sigma^*/2}(\overline{Q}_T)$ for $\sigma^* = \min\{\sigma_2, \lambda\}$. Selecting $\alpha = \min\{\sigma_1, \sigma^*\}$ and using Sobolev embedding theorem, we have $C^{2+\sigma^*, 1+\sigma^*/2}(\overline{Q}_T) \hookrightarrow C^{\alpha, \alpha/2}(\overline{Q}_T)$. Still applying Schauder estimate, we have

$$v \in C^{2+\sigma, 1+\sigma/2}(\overline{Q}_T), \quad \sigma = \min\{\alpha, \lambda\}. \tag{4.39}$$

Let $\overline{w} = (d_3 + \alpha_{31} u + \alpha_{32} v + \alpha_{33} w)w$. Then $\overline{w}$ satisfies

$$\overline{w} = (d_3 + \alpha_{31} u + \alpha_{32} v + 2\alpha_{33} w)\Delta\overline{w} + f(x,t), \tag{4.40}$$

where $f(x,t) = (d_3 + \alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w)(-b + u - w)w + (\alpha_{31}u_t + \alpha_{32}v_t)w$. By (4.35)–(4.38), we have $d_3 + \alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w, f(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}_T)$. So applying Schauder estimate to (4.40) yields $\overline{w} \in C^{2+\sigma,1+\sigma/2}(\overline{Q}_T)$. Since $w = (-(d_3 + \alpha_{31}u + \alpha_{32}v) + \sqrt{(d_3 + \alpha_{31}u + \alpha_{32}v)^2 + 4\alpha_{33}\overline{w}})/2\alpha_{33}$, we have

$$w \in C^{2+\sigma,1+\sigma/2}\left(\overline{Q}_T\right), \quad \sigma = \min\{\alpha, \lambda\}. \tag{4.41}$$

The first equation of (1.6) can be written as

$$u_t = (d_1 + 2\alpha_{11}u)\Delta u + g(x,t), \tag{4.42}$$

where $g(x,t) = 2\alpha_{11}|\nabla u|^2 + a_0 v - a_1 u + a_2 u^2 - a_3 u^3 - kuw$. By (4.35), (4.39), and (4.41), we have $d_1 + 2\alpha_{11}u, g(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}_T)$. So applying Schauder estimate to (4.42) yields

$$u \in C^{2+\sigma,1+\sigma/2}\left(\overline{Q}_T\right), \quad \sigma = \min\{\alpha, \lambda\}. \tag{4.43}$$

In particular, if $\lambda < \alpha$, then $\sigma = \lambda$; in other words, Theorem 4.5 is proved. For the case $\alpha < \lambda$, from Sobolev embedding theorem, we have $C^{2+\sigma,1+\sigma/2}(\overline{Q}_T) \hookrightarrow C^{\lambda,\lambda/2}(\overline{Q}_T)$. Repeating the above bootstrap and Shauder estimate arguments, this completes the proof of Theorem 4.5. About space dimension $n = 1$, see [21]. □

Theorem 4.6. *System (1.6) has the unique positive equilibrium point $(\overline{u}, \overline{v}, \overline{w})$ when one of the conditions (i), (ii), (iii), (iv), (v), and (vi) in Section 2 holds. Let the space dimension be $n \leq 5$, and let the initial values u_0, v_0, w_0 be nonnegative smooth functions and satisfy the homogenous Neumann boundary conditions. If the following condition (4.44) holds, then the solution (u, v, w) of (1.6) converges to $(\overline{u}, \overline{v}, \overline{w})$ in $L^2(\Omega)$:*

$$4\alpha\overline{u}d_1 d_2 d_3 > \beta\overline{w}\left[\alpha\overline{u}\alpha_{32}^2(d_1 + 2\alpha_{11}M_0) + (\alpha_{31}M_0)^2(d_2 + 2\alpha_{22}M_0)\right], \tag{4.44}$$

where $\alpha = 2/a_0^2 + 1/2a_3\overline{u}$, and $\beta = (k + a_3\overline{u})\alpha$.

Proof. Define the Lyapunov function

$$H(t) = \alpha \int_\Omega \left(u - \overline{u} - \overline{u}\ln\frac{u}{\overline{u}}\right)dx + \frac{1}{2}\int_\Omega (v - \overline{v})^2 dx + \beta\int_\Omega\left(w - \overline{w} - \overline{w}\ln\frac{w}{\overline{w}}\right)dx, \tag{4.45}$$

where α and β have been given in Theorem 4.6. Obviously, $H(t)$ is nonnegative, and $H(t) = 0$ if and only if $u = \overline{u}$, $v = \overline{v}$, and $w = \overline{w}$. When (u, v, w) is a positive solution of system

(1.6), $H(t)$ is well posed for all $t \geq 0$ from Theorem 4.5. According to system (1.6), the time derivative of $H(t)$ satisfies

$$
\begin{aligned}
\frac{dH(t)}{dt} = &-\int_\Omega \left[\frac{\alpha \overline{u}}{u^2}(d_1 + 2\alpha_{11}u)|\nabla u|^2 + (d_2 + 2\alpha_{22}v)|\nabla v|^2 \right. \\
&\left. + \frac{\beta \overline{w}}{w^2}(d_3 + \alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w)|\nabla w|^2 + \beta \frac{\alpha_{31}\overline{w}}{w}\nabla u \nabla w + \beta \frac{\alpha_{32}\overline{w}}{w}\nabla v \nabla w \right] dx \\
&-\int_\Omega \left\{ \alpha \left[\left(\frac{a_0}{u} - a_2 \right) + a_3(u + \overline{u}) \right](u - \overline{u})^2 + (v - \overline{v})^2 + \beta(w - \overline{w})^2 \right. \\
&\left. - \left(1 + \frac{a_0\alpha}{u} \right)(u - \overline{u})(v - \overline{v}) + (k\alpha - \beta)(u - \overline{u})(w - \overline{w}) \right\} dx \\
\leq &-\int_\Omega \left[\frac{\alpha \overline{u}}{u^2}(d_1 + 2\alpha_{11}u)|\nabla u|^2 + (d_2 + 2\alpha_{22}v)|\nabla v|^2 \right. \\
&\left. + \frac{\beta \overline{w}}{w^2}(d_3 + \alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w)|\nabla w|^2 + \beta \frac{\alpha_{31}\overline{w}}{w}\nabla u \nabla w + \beta \frac{\alpha_{32}\overline{w}}{w}\nabla v \nabla w \right] dx \\
&-\int_\Omega \left\{ \alpha \left[\frac{a_0}{u} - \frac{a_2(M + a_0/a_2 + 1/a_2 + a_0^2/4a_2a_3\overline{u})}{u} \right](u - \overline{u})^2 \right. \\
&\left. - \frac{a_0\alpha}{u}(u - \overline{u})(v - \overline{v}) - \frac{1}{u}(v - \overline{v})^2 \right\} dx \\
&-\int_\Omega \left[\alpha \frac{a_3\overline{u}}{2}(u - \overline{u})^2 - (u - \overline{u})(v - \overline{v}) + (v - \overline{v})^2 \right] dx \\
&-\int_\Omega \left[\alpha \frac{a_3\overline{u}}{2}(u - \overline{u})^2 + (k\alpha - \beta)(u - \overline{u})(w - \overline{w}) + \beta(w - \overline{w})^2 \right] dx.
\end{aligned}
$$

$$(4.46)$$

It is easy to check that the final three integrands on the right side of the above expression are positive definite because of the electing of α, β, and the sufficient and necessary conditions of the first integrand being positive definite are the following:

$$
\begin{aligned}
4\alpha\beta\overline{u}\,\overline{w}(d_1 + 2\alpha_{11}u)(d_2 + 2\alpha_{22}v)(d_3 + \alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w) \\
> \alpha\overline{u}(\beta\alpha_{32}\overline{w})^2(d_1 + 2\alpha_{11}u) + (\beta\alpha_{31}\overline{w}u)^2(d_2 + 2\alpha_{22}v).
\end{aligned}
$$

$$(4.47)$$

Noticing that (4.44) is the sufficient conditions of (4.47), so there exists a positive constant $\delta > 0$ such that

$$
\begin{aligned}
\frac{dH(t)}{dt} &\leq -\delta \int_\Omega \left[(u - \overline{u})^2 + (v - \overline{v})^2 + (w - \overline{w})^2 \right] dx, \\
\frac{dH(t)}{dt} &< 0 \quad (u, v, w) \neq (\overline{u}, \overline{v}, \overline{w}).
\end{aligned}
$$

$$(4.48)$$

Similar to the tedious calculations of $dH(t)/dt$, using integration by parts, Hölder inequality, and (4.34), one can verify that $(d/dt)\int_{\Omega}[(u-\overline{u})^2 + (v-\overline{v})^2 + (w-\overline{w})^2]dx$ is bounded from above. Thus we have from (4.48) and Lemma 3.2 in Section 3 that

$$|u(\cdot,t) - \overline{u}|_2 \longrightarrow 0, \quad |v(\cdot,t) - \overline{v}|_2 \longrightarrow 0, \quad |w(\cdot,t) - \overline{w}|_2 \longrightarrow 0 \quad (t \longrightarrow \infty). \tag{4.49}$$

In addition, $H(t)$ is decreasing for $t \geq 0$, so we can conclude that the solution $(\overline{u}, \overline{v}, \overline{w})$ is globally asymptotically stable. The proof of Theorem 4.6 is completed. $\square$

Acknowledgments

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