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Research Article

Superlinear Singular Problems on the Half Line

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The paper studies the singular differential equation $(p(t)u')' = p(t)f(u)$, which has a singularity at $t = 0$. Here the existence of strictly increasing solutions satisfying $\sup\{|u(t)| : t \in [0,\infty)\} \geq L > 0$ is proved under the assumption that f has two zeros 0 and L and a superlinear behaviour near $-\infty$. The problem generalizes some models arising in hydrodynamics or in the nonlinear field theory.

1. Introduction

Let us consider the problem

$$(p(t)u')' = p(t)f(u), \tag{1.1}$$

$$u'(0) = 0, \qquad u(\infty) = L, \tag{1.2}$$

where L is a positive real parameter.

Definition 1.1. Let $c > 0$. A function $u \in C^1([0,c]) \cap C^2((0,c])$ satisfying (1.1) on $(0,c]$ is called *a solution of* (1.1) *on* $[0,c]$.

Definition 1.2. Let u be a solution of (1.1) on $[0,c]$ for each $c > 0$. Then u is called *a solution of* (1.1) *on* $[0,\infty)$. If u moreover fulfils conditions (1.2), it is called *a solution of problem* (1.1), (1.2).

Definition 1.3. A strictly increasing solution of problem (1.1), (1.2) is called *a homoclinic solution*.

In this paper we are interested in the existence of strictly increasing solutions and, in particular, of homoclinic solutions. In what follows we assume

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad f(0) = f(L) = 0, \tag{1.3}$$

$$f(x) < 0 \quad \text{for } x \in (0, L), \tag{1.4}$$

$$\text{there exists } \overline{B} < 0 \text{ such that } f(x) > 0 \text{ for } x \in \left[\overline{B}, 0\right), \tag{1.5}$$

$$F\left(\overline{B}\right) = F(L), \quad \text{where } F(x) = -\int_0^x f(z)\mathrm{d}z, \tag{1.6}$$

$$p \in C([0,\infty)) \cap C^1((0,\infty)), \quad p(0) = 0, \tag{1.7}$$

$$p'(t) > 0,\ t \in (0,\infty), \quad \lim_{t\to\infty}\frac{p'(t)}{p(t)} = 0. \tag{1.8}$$

Under assumptions (1.3)–(1.8) problem (1.1), (1.2) generalizes some models arising in hydrodynamics or in the nonlinear field theory (see [1–5]). If a homoclinic solution exists, many important properties of corresponding models can be obtained. Note that if we extend the function $p(t)$ in (1.1) from the half-line onto $\mathbb{R}$ (as an even function), then any solution of (1.1), (1.2) has the same limit L as $t \to -\infty$ and $t \to \infty$. This is a motivation for Definition 1.3. Equation (1.1) is singular at $t = 0$ because $p(0) = 0$. In [6, 7] we have proved that assumptions (1.3)–(1.8) are sufficient for the existence of strictly increasing solutions and homoclinic solutions provided

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty \text{ or there exists } L_0 < \overline{B} \text{ such that } f(L_0) = 0. \tag{1.9}$$

Here we assume that (1.9) is not valid. Then

$$f(x) > 0 \quad \text{for } x < 0, \tag{1.10}$$

and the papers [6, 8] provide existence theorems for problem (1.1), (1.2) if f has a sublinear or linear behaviour near $-\infty$. The case that f has a superlinear behaviour near $-\infty$ is studied in this paper. To this aim we consider the initial conditions

$$u(0) = B, \qquad u'(0) = 0, \tag{1.11}$$

where $B < 0$, and introduce the following definition.

Definition 1.4. Let $c > 0$ and let u be a solution of (1.1) on $[0, c]$ satisfying (1.11). Then u is called *a solution of problem* (1.1), (1.11) *on* $[0, c]$. If u moreover fulfils

$$u'(t) > 0 \quad \text{for } t \in (0, c], \qquad u(c) = L, \tag{1.12}$$

then u is called *an escape solution of problem* (1.1), (1.11).

We have proved in [6, 8] that for sublinear or linear f the existence of a homoclinic solution follows from the existence of an escape solution of problem (1.1), (1.11). Therefore our first task here is to prove that at least one escape solution of (1.1), (1.11) exists, provided (1.3)–(1.8), (1.10), and

$$f(x) = 0 \quad \text{for } x > L \tag{1.13}$$

hold, and f has a superlinear behaviour near $-\infty$. This is done in Section 2. Using the results of Section 2 "Theorem 2.10", and of [6, Theroms 13, 14 and 20] we get the existence of a homoclinic solution in Section 3.

Note that by Definitions 1.3 and 1.4 just the values of a solution which are less than L are important for a decision whether the solution is homoclinic or escape one. Therefore condition (1.13) can be assumed without any loss of generality.

Close problems about the existence of positive solutions have been studied in [9–11].

2. Escape Solutions

In this section we assume that (1.3)–(1.8), (1.10), and (1.13) hold. We will need some lemmas.

Lemma 2.1 (see [6, Lemma 3]). *For each $B < 0$, problem* (1.1), (1.11) *has a unique solution u on $[0, \infty)$ such that*

$$u(t) \geq B \quad \textit{for } t \in [0, \infty). \tag{2.1}$$

In what follows by a solution of (1.1), (1.11) we mean a solution on $[0, \infty)$.

Remark 2.2 (see [6, Remark 4]). Choose $a \geq 0$ and $A \leq L$, and consider the initial conditions

$$u(a) = A, \qquad u'(a) = 0. \tag{2.2}$$

Problem (1.1), (2.2) has a unique solution u on $[a, \infty)$. In particular, for $A = 0$ and $A = L$, we get $u \equiv 0$ and $u \equiv L$, respectively. Clearly, for $a > 0$, $u \equiv 0$ and $u \equiv L$ are solutions of (1.1) on the whole interval $[0, \infty)$.

Lemma 2.3. *Let $B < 0$ and let u be a solution of problem* (1.1), (1.11) *which is not an escape solution. Let us denote*

$$\theta = \sup\{t > 0 : u < 0 \textit{ in } (0, t)\}, \qquad b = \sup\{t > 0 : u' > 0 \textit{ in } (0, t)\}. \tag{2.3}$$

Then $0 < \theta \leq b \leq \infty$ holds and pu' is increasing on $(0, \theta)$. If $\theta < \infty$, then $\theta < b$ and

$$\max\{p(t)u'(t) : t \in [0, b)\} = p(\theta)u'(\theta). \tag{2.4}$$

Proof. The inequality $u(0) < 0$ yields $\theta > 0$. By (1.1) and (1.10), we get $(pu')' = pf(u) > 0$ on $(0, \theta)$ and hence pu' is increasing on $(0, \theta)$. As $p(0)u'(0) = 0$, one has $pu' > 0$ on $(0, \theta)$ and consequently $u' > 0$ on $(0, \theta)$. Therefore $\theta \leq b$.

Let $\theta < \infty$. Then θ is the first zero of u and $u'(\theta) > 0$. Remark 2.2 yields that $u'(\theta) = 0$ is not possible. This implies that $\theta < b$. As u is strictly increasing on (θ, b) and u is not an escape solution, we have $0 < u < L$ on (θ, b). Thus $(pu')' = pf(u) < 0$ on (θ, b) and hence pu' is decreasing on (θ, b). This gives (2.4). □

Lemma 2.4. *Let $B < 0$ and let u be a solution of problem* (1.1), (1.11) *which is not an escape solution. Assume that b is given by Lemma 2.3. Then*

$$u(b) \in [0, L], \quad u'(b) = 0. \tag{2.5}$$

Proof. From (1.1), we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)), \quad t > 0, \tag{2.6}$$

and, by multiplication and integration over $[0, t]$,

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \mathrm{d}s = F(u(0)) - F(u(t)), \ t > 0. \tag{2.7}$$

(1) Assume that $b = \infty$. The definition of b yields $u' > 0$ on $(0, \infty)$. Since u is not an escape solution, it is bounded above and there exists

$$u(b) = \lim_{t \to \infty} u(t) \in (B, L]. \tag{2.8}$$

Therefore the following integral is bounded and, since it is increasing, it has a limit

$$0 \le \lim_{t \to \infty} \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \mathrm{d}s \le F(B) - F(u(b)) < \infty. \tag{2.9}$$

So, by (2.7), $\lim_{t \to \infty} u'^2(t)$ exists. By virtue of (2.8), we get

$$u'(b) = \lim_{t \to \infty} u'(t) = \lim_{t \to \infty} u'^2(t) = 0. \tag{2.10}$$

If $u(b) \notin \{0, L\}$, then by (1.4), (1.10) and (2.6) we get $\lim_{t \to \infty} u''(t) = u''(b) = f(u(b)) \neq 0$, which contradicts (2.10). Hence, $u(b) \in \{0, L\}$. In particular, if θ is defined as in Lemma 2.3, then

$$u(b) = 0 \quad \text{for } \theta = \infty, \qquad u(b) = L \quad \text{for } \theta < \infty. \tag{2.11}$$

(2) Assume that $b < \infty$. Then the continuity of u' gives $u'(b) = 0$ and θ of Lemma 2.3 fulfils $0 < \theta < b$. We deduce that $0 < u < L$ on (θ, b) as in the proof of Lemma 2.3. Remark 2.2 yields that if $u'(b) = 0$, then neither $u(b) = 0$ nor $u(b) = L$ can occur. Therefore $u(b) \in (0, L)$. □

Denote

$$P(t) = \int_0^t p(s)\mathrm{d}s, \quad t \in [0, \infty). \tag{2.12}$$

Lemma 2.5. *Let $B < 0$ and let u be a solution of problem* (1.1), (1.11). *Further assume maximal $b > 0$ such that $u'(t) > 0$ and $u(t) \in (B, L)$ for $t \in (0, b)$. Then*

$$\int_0^t 2F(u(s))p(s)p'(s)\mathrm{d}s = F(u(t))p^2(t) + \frac{1}{2}p^2(t)u'^2(t), \quad t \in (0, b). \tag{2.13}$$

For $a \in [0, 1)$, let us denote

$$Q(x) := 2F(x) - a(L - x)f(x), \quad x \in (-\infty, L]. \tag{2.14}$$

Then

$$\begin{aligned} \int_0^t Q(u(s))p(s)ds < {} & P(t)\Big(2F(u(t)) + u'^2(t)\Big) \\ & + \int_0^t \left(\frac{2p'(s)P(s)}{p^2(s)} - (a+1)\right)p(s)u'^2(s)\mathrm{d}s, \quad t \in (0, b). \end{aligned} \tag{2.15}$$

Proof. For equality (2.13) see Lemma 4.6 in [8]. Let us prove (2.15). Using the per partes integration, we get for $t \in (0, b)$

$$\begin{aligned} \int_0^t Q(u(s))p(s)\mathrm{d}s &= \int_0^t \big(2F(u(s)) - a(L - u(s))f(u(s))\big)p(s)\mathrm{d}s \\ &= 2F(u(t))P(t) + I_1 + I_2, \end{aligned} \tag{2.16}$$

where

$$I_1 = 2\int_0^t f(u(s))u'(s)P(s)\mathrm{d}s, \qquad I_2 = -a\int_0^t (L - u(s))f(u(s))p(s)\mathrm{d}s. \tag{2.17}$$

By multiplication and integration of (1.1) we obtain

$$2\int_0^t u''(s)u'(s)P(s)\mathrm{d}s + 2\int_0^t u'^2(s)\frac{p'(s)}{p(s)}P(s)\mathrm{d}s = I_1, \tag{2.18}$$

and by the per partes integration,

$$I_1 = P(t)u'^2(t) + \int_0^t \left(\frac{2p'(s)}{p(s)}P(s) - p(s)\right)u'^2(s)\mathrm{d}s. \tag{2.19}$$

To compute I_2, we use (1.1) and get

$$I_2 = -a\int_0^t (L-u(s))(p(s)u'(s))'\mathrm{d}s. \tag{2.20}$$

By the per partes integration we derive

$$I_2 = -ap(t)u'(t)(L-u(t)) - a\int_0^t p(s)u'^2(s)\mathrm{d}s < -a\int_0^t p(s)u'^2(s)\mathrm{d}s. \tag{2.21}$$

We have proved that (2.15) is valid. □

Lemma 2.6. *Let $C < \overline{B}$, $\{B_n\}_{n=1}^{\infty} \subset (-\infty, C)$ and let $\{u_n\}_{n=1}^{\infty}$ be solutions of problem* (1.1), (1.11) *with $B = B_n$, $n \in \mathbb{N}$. Let us denote*

$$b_n = \sup\{t > 0 : u_n \in (B, L),\ u_n' > 0 \text{ in } (0,t)\}, \quad n \in \mathbb{N}. \tag{2.22}$$

Then for each $n \in \mathbb{N}$ there exists a unique $\gamma_n \in (0, b_n)$ satisfying

$$u_n(\gamma_n) = C. \tag{2.23}$$

If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, then there exists an escape solution in $\{u_n\}_{n=1}^{\infty}$.

Proof. Choose $n \in \mathbb{N}$. The monotonicity and continuity of u_n in $(0, b_n)$ give a unique $\gamma_n \in (0, b_n)$. If $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded we argue as in the proof of Lemma 4.8 in [8]. □

Let $C < \overline{B}$ and let $\{B_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences from Lemma 2.6. Assume that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (1.1), (1.11). Lemma 2.6 implies that

$$\Gamma := \sup\{\gamma_n : n \in \mathbb{N}\} < \infty. \tag{2.24}$$

We can assume that that either there exists $b_0 > 0$ such that

$$b_n \le b_0, \quad n \in \mathbb{N}, \tag{2.25}$$

or

$$b_n > \Gamma + 1, \quad n \in \mathbb{N}. \tag{2.26}$$

Otherwise we take a subsequence. Some additional properties of $\{u_n\}_{n=1}^{\infty}$ are given in the next two lemmas.

Lemma 2.7. *Denote*

$$\theta_n := \sup\{t > 0 : u_n < 0 \text{ in } (0,t)\}, \quad n \in \mathbb{N}, \tag{2.27}$$

and assume that the sequence $\{\theta_n\}_{n=1}^{\infty}$ *is bounded above. Then there exists* $K > 0$ *such that*

$$p(t)u_n'(t) \le K \quad \text{for } t \in [0, b_n),\ n \in \mathbb{N}. \tag{2.28}$$

Proof. By Lemma 2.4 we have

$$u_n(b_n) \in [0, L], \quad u_n'(b_n) = 0, \quad n \in \mathbb{N}. \tag{2.29}$$

Step 1 (sequence $\{p(\gamma_n)u_n'(\gamma_n)\}_{n=1}^{\infty}$ is bounded). Assume on the contrary that $\{p(\gamma_n)u_n'(\gamma_n)\}_{n=1}^{\infty}$ is unbounded. We may write

$$\lim_{n\to\infty} p(\gamma_n)u_n'(\gamma_n) = \infty \tag{2.30}$$

(otherwise we take a subsequence). Equality (2.13) yields for $n \in \mathbb{N}$ and $t \in (\gamma_n, b_n)$,

$$\begin{aligned} 0 &< \int_{\gamma_n}^{t} 2F(u_n(s))p(s)p'(s)\mathrm{d}s \\ &= F(u_n(t))p^2(t) + \frac{1}{2}p^2(t)u_n'^2(t) - F(u_n(\gamma_n))p^2(\gamma_n) - \frac{1}{2}p^2(\gamma_n)u_n'^2(\gamma_n). \end{aligned} \tag{2.31}$$

Using (1.4), (1.6), (1.10), $C < \overline{B}$ and the fact that $u_n(t) \in (C, L)$ for $t \in (\gamma_n, b_n)$, we get

$$F(u_n(t)) < F(C) \quad \text{for } t \in (\gamma_n, b_n). \tag{2.32}$$

Consequently, inequality in (2.31) leads to

$$F(C)p^2(\gamma_n) + \frac{1}{2}p^2(\gamma_n)u_n'^2(\gamma_n) < F(C)p^2(t) + \frac{1}{2}p^2(t)u_n'^2(t) \tag{2.33}$$

for $t \in (\gamma_n, b_n)$. Therefore

$$p^2(\gamma_n)u_n'^2(\gamma_n) - 2F(C)p^2(t) < p^2(t)u_n'^2(t), \quad t \in (\gamma_n, b_n),\ n \in \mathbb{N}. \tag{2.34}$$

We will consider two cases.

Case 1. If (2.25) holds, then (2.34) gives for $n \in \mathbb{N}$

$$p^2(\gamma_n)u_n'^2(\gamma_n) - 2F(C)p^2(b_0) < p^2(b_0)u_n'^2(t), \quad t \in (\gamma_n, b_n). \tag{2.35}$$

By (2.30), for each sufficiently large $n \in \mathbb{N}$, we get

$$p^2(\gamma_n)u_n'^2(\gamma_n) > (2F(C)+1)p^2(b_0). \tag{2.36}$$

Putting it to (2.35), we have $1 \le u_n'(b_n)$, contrary to (2.29).

Case 2. If (2.26) holds, then (2.34) gives for $n \in \mathbb{N}$

$$p^2(\gamma_n)u_n'^2(\gamma_n) - 2F(C)p^2(\Gamma+1) < p^2(\Gamma+1)u_n'^2(t), \quad t \in (\gamma_n, \Gamma+1]. \tag{2.37}$$

Due to (2.30), we have

$$p^2(\gamma_n)u_n'^2(\gamma_n) > \left(2F(C) + (L-C)^2\right)p^2(\Gamma+1) \tag{2.38}$$

for each sufficiently large $n \in \mathbb{N}$. Putting it to (2.37), we get $L - C < u_n'(t)$ for $t \in (\gamma_n, \Gamma+1]$. Integrating it over $[\gamma_n, \Gamma+1]$, we obtain $L < u_n(\Gamma+1)$. Equation (1.1) and condition (1.13) yield $u_n'(t) > 0$ for $t \ge \Gamma+1$, and so $L < u_n(b_n)$, contrary to (2.29).

We have proved that there exists $K_0 > 0$ such that

$$p(\gamma_n)u_n'(\gamma_n) \le K_0, \quad n \in \mathbb{N}. \tag{2.39}$$

Step 2 (estimate for pu_n'). Choose $n \in \mathbb{N}$. By (2.32) we get

$$\int_{\gamma_n}^t 2F(u_n(s))p(s)p'(s)\mathrm{d}s < 2F(C)\int_{\gamma_n}^t p(s)p'(s)\mathrm{d}s < F(C)p^2(t), \quad t \in (\gamma_n, b_n). \tag{2.40}$$

This together with (2.31) and (2.39) imply

$$\frac{1}{2}p^2(t)u_n'^2(t) < 2F(C)p^2(t) + \frac{1}{2}K_0^2, \quad t \in (\gamma_n, b_n). \tag{2.41}$$

According to (2.27) and Lemma 2.3 we see that $\theta_n \in (\gamma_n, b_n)$ is the first zero of u_n. Since the sequence $\{\theta_n\}_{n=1}^{\infty}$ is bounded above, there exists $\Gamma_0 < \infty$ such that $\theta_n \le \Gamma_0$, $n \in \mathbb{N}$. Then (1.8) and (2.41) give

$$\frac{1}{2}p^2(\theta_n)u_n'^2(\theta_n) < 2F(C)p^2(\Gamma_0) + \frac{1}{2}K_0^2. \tag{2.42}$$

Put

$$\frac{1}{2}K^2 = 2F(C)p^2(\Gamma_0) + \frac{1}{2}K_0^2. \tag{2.43}$$

Then, by virtue of (2.4), inequality (2.28) is valid. □

Lemma 2.8. *Consider* $C < \overline{B}$ *and* Γ *satisfying* (2.23) *and* (2.24). *Let* θ_n, $n \in \mathbb{N}$ *be given by* (2.27). *Assume that*

$$\theta_n > \Gamma + 2, \quad n \in \mathbb{N}. \tag{2.44}$$

Then there exists $K \in (0, \infty)$ *such that*

$$p(t)u_n'(t) \le K \quad \text{for } t \in [0, \Gamma + 1], \quad n \in \mathbb{N}. \tag{2.45}$$

Proof. Assume on the contrary that

$$\sup\{p(t)u_n'(t) : t \in [0, \Gamma + 1],\ n \in \mathbb{N}\} = \infty. \tag{2.46}$$

By Lemma 2.3, pu_n' is increasing on $(0, \theta_n)$, $n \in \mathbb{N}$. Therefore

$$\sup\{p(\Gamma + 1)u_n'(\Gamma + 1) : n \in \mathbb{N}\} = \infty, \tag{2.47}$$

and therefore there exists $n_0 \in \mathbb{N}$ such that

$$p(\Gamma + 1)u_{n_0}'(\Gamma + 1) > |C|p(\Gamma + 2). \tag{2.48}$$

Moreover (2.23), (2.24), (2.27), (2.44), and the monotonicity of u_{n_0} and pu_{n_0}' yield

$$u_{n_0}(t) \in (C, 0), \quad p(t)u_{n_0}'(t) > |C|p(\Gamma + 2) \quad \text{for } t \in [\Gamma + 1, \Gamma + 2]. \tag{2.49}$$

Integrating the last inequality over $(\Gamma + 1, \Gamma + 2)$, we obtain $u_{n_0}(\Gamma + 2) - u_{n_0}(\Gamma + 1) > |C|$, so $|u_{n_0}(\Gamma + 1)| > |C|$, a contradiction. □

Lemma 2.9. *Let real sequences* $\{B_n\}_{n=1}^{\infty}$, $\{\kappa_n\}_{n=1}^{\infty}$, $\{\sigma_n\}_{n=1}^{\infty}$ *be given and assume that*

$$\lim_{n \to \infty} B_n = -\infty, \qquad \{\kappa_n\}_{n=1}^{\infty} \subset \left[\frac{1}{2}, 1\right], \quad \{\sigma_n\}_{n=1}^{\infty} \subset \left[\frac{1}{2}, 1\right]. \tag{2.50}$$

Let $k \ge 2$ *and*

$$1 < r < \frac{k + 2}{k - 2} \tag{2.51}$$

(for $k = 2$ we assume $r \in (1, \infty)$) be such that

$$0 < \lim_{x \to -\infty} \frac{|x|^r}{f(x)} < \infty. \tag{2.52}$$

Assume that Q is given by (2.14) *with $a \in (0, 2/(r+1))$. Then*

$$\lim_{n \to \infty} Q(\kappa_n B_n) \left(\frac{|B_n|}{f(\sigma_n B_n)} \right)^{k/2} = \infty. \tag{2.53}$$

Proof. By (2.50), $\lim_{n\to\infty} \kappa_n B_n = \lim_{n\to\infty} \sigma_n B_n = -\infty$. Condition (2.52) yields that there exists $\lambda \in (0, \infty)$ such that

$$\lim_{x \to -\infty} \frac{|x|^r}{f(x)} = \lambda, \qquad \lim_{x \to -\infty} \frac{F(x)}{|x|^{r+1}} = \frac{1}{(r+1)\lambda}. \tag{2.54}$$

Therefore

$$\lim_{x \to -\infty} \frac{2F(x)}{(L-x)f(x)} = 2 \lim_{x \to -\infty} \frac{|x|}{L-x} \frac{F(x)}{|x|^{r+1}} \frac{|x|^r}{f(x)} = \frac{2}{r+1}. \tag{2.55}$$

Hence

$$\begin{aligned} \lim_{x \to -\infty} Q(x) &= \lim_{x \to -\infty} (L-x) f(x) \left(\frac{2F(x)}{(L-x)f(x)} - a \right) \\ &= a_0 \lim_{x \to -\infty} (L-x) f(x), \end{aligned} \tag{2.56}$$

where $a_0 := 2/(r+1) - a > 0$. Consequently,

$$\begin{aligned} &\lim_{n \to \infty} Q(\kappa_n B_n) \left(\frac{|B_n|}{f(\sigma_n B_n)} \right)^{k/2} \\ &\quad = a_0 \lim_{n \to \infty} (L - \kappa_n B_n) f(\kappa_n B_n) \left(\frac{|B_n|}{f(\sigma_n B_n)} \right)^{k/2} \\ &\quad = a_0 \lim_{n \to \infty} (L - \kappa_n B_n) \frac{f(\kappa_n B_n)}{|\kappa_n B_n|^r} \left(\frac{|\sigma_n B_n|^r}{f(\sigma_n B_n)} \right)^{k/2} \kappa_n^r \frac{1}{\sigma_n^{rk/2}} |B_n|^{r-(k/2)(r-1)} \\ &\quad \geq a_0 \left(\frac{1}{2} \right)^{r+1} \lambda^{k/2-1} \lim_{n \to \infty} |B_n|^{r_0}, \end{aligned} \tag{2.57}$$

where $r_0 = r + 1 - k(r-1)/2 > 0$, because r is less than the critical value $(k+2)/(k-2)$. We have proved (2.53). □

Now we are ready to prove the following main result of this paper.

Theorem 2.10. *Assume that*

$$\lim_{t\to 0+}\frac{p'(t)}{t^{k-2}} \in (0,\infty) \tag{2.58}$$

for some $k \geq 2$*. Further, let* r *and* f *be such that* (2.51) *and* (2.52) *are valid. Then there exists* $B < \overline{B}$ *such that the corresponding solution of problem* (1.1), (1.11) *is an escape solution.*

Proof. Assumption (2.51) implies $(k-2)/k < 2/(r+1) < 1$, and hence we can choose $a \in ((k-2)/k, 2/(r+1))$ and define Q by (2.14). According to (1.4), (1.10), and (2.56), there exists $C < \overline{B}$ such that $Q(x) > 0$ for $x \in (-\infty, C) \cup (0, L]$. Consequently, we can find $\widetilde{Q} \in [0,\infty)$ such that

$$Q(x) \geq -\widetilde{Q} \quad \text{for } x \in (-\infty, L]. \tag{2.59}$$

Let $\{B_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences defined in Lemma 2.6. Moreover, let

$$\lim_{n\to\infty} B_n = -\infty. \tag{2.60}$$

Assume that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (1.1), (1.11). By Lemma 2.4 we have

$$u_n(b_n) \in [0, L], \quad u_n'(b_n) = 0, \quad n \in \mathbb{N}. \tag{2.61}$$

Condition (2.60) gives $n_0 \in \mathbb{N}$ such that

$$B_n < 2C \quad \text{for } n \in \mathbb{N}, \quad n \geq n_0. \tag{2.62}$$

Choose an arbitrary $n \geq n_0$. We will construct a contradiction.

Step 1 (inequality for u_n'). Since u_n is increasing on $(0, b_n)$, (2.62) gives a unique $\overline{\gamma}_n \in (0, \gamma_n)$ satisfying

$$u_n(\overline{\gamma}_n) = \frac{1}{2}B_n. \tag{2.63}$$

By (2.59) we have

$$\int_0^t Q(u_n(s))p(s)\mathrm{d}s > \int_0^{\overline{\gamma}_n} Q(u_n(s))p(s)\mathrm{d}s - \widetilde{Q}\int_{\gamma_n}^t p(s)\mathrm{d}s, \quad t \in [\gamma_n, b_n), \tag{2.64}$$

because $Q(u_n(t)) > 0$ for $t \in (\overline{\gamma}_n, \gamma_n)$. Further, there exists $\kappa_n \in [1/2, 1]$ satisfying

$$Q(\kappa_n B_n) = \min\left\{ Q(x) : x \in \left[B_n, \frac{B_n}{2}\right]\right\}. \tag{2.65}$$

Therefore, according to (2.12),

$$\int_0^t Q(u_n(s))p(s)\mathrm{d}s > Q(\kappa_n B_n)P(\overline{\gamma}_n) - \widetilde{Q}P(t), \quad t \in [\gamma_n, b_n). \tag{2.66}$$

Let us put

$$I_n(t) = \int_0^t \left(\frac{2p'(s)P(s)}{p^2(s)} - (a+1)\right)p(s)u_n'^2(s)\mathrm{d}s, \quad t \in (0, b_n). \tag{2.67}$$

Then inequalities (2.15) and (2.66) imply

$$Q(\kappa_n B_n)P(\overline{\gamma}_n) - \widetilde{Q}P(t) < P(t)\left(2F(u_n(t)) + u_n'^2(t)\right) + I_n(t), \quad t \in [\gamma_n, b_n). \tag{2.68}$$

Step 2 (estimate of $P(\overline{\gamma}_n)$ from below). Since u_n is a solution of (1.1) on $[0, \infty)$, we have

$$u_n'(t) = \frac{1}{p(t)}\int_0^t p(s)f(u_n(s))\mathrm{d}s, \quad t \in (0, \overline{\gamma}_n]. \tag{2.69}$$

Therefore

$$f(\sigma_n B_n)\frac{P(t)}{p(t)} \geq u_n'(t) \geq f(\rho_n B_n)\frac{P(t)}{p(t)}, \quad t \in (0, \overline{\gamma}_n], \tag{2.70}$$

where $\rho_n, \sigma_n \in [1/2, 1]$, are such that

$$\begin{aligned} f(\rho_n B_n) &= \min\left\{ f(x) : x \in \left[B_n, \frac{B_n}{2}\right]\right\}, \\ f(\sigma_n B_n) &= \max\left\{ f(x) : x \in \left[B_n, \frac{B_n}{2}\right]\right\}. \end{aligned} \tag{2.71}$$

Integrating (2.70) over $(0, \overline{\gamma}_n)$, we get

$$f(\sigma_n B_n)\int_0^{\overline{\gamma}_n} \frac{P(s)}{p(s)}\mathrm{d}s \geq \frac{1}{2}|B_n| \geq f(\rho_n B_n)\int_0^{\overline{\gamma}_n} \frac{P(s)}{p(s)}\mathrm{d}s. \tag{2.72}$$

Hence

$$2^{r-1}\frac{|\rho_n B_n|^r}{f(\rho_n B_n)}\frac{1}{|B_n|^{r-1}} \ge \int_0^{\overline{\gamma}_n}\frac{P(s)}{p(s)}\mathrm{d}s. \tag{2.73}$$

By (2.52), (2.60) and $r > 1$, we deduce that

$$\lim_{n\to\infty}\int_0^{\overline{\gamma}_n}\frac{P(s)}{p(s)}\mathrm{d}s = 0. \tag{2.74}$$

Since $P(t)/p(t) > 0$ for $t > 0$, we get

$$\lim_{n\to\infty}\overline{\gamma}_n = 0. \tag{2.75}$$

Due to (2.58), there exists $\mu \in (0,\infty)$ such that $\lim_{t\to 0+}p'(t)/((k-1)t^{k-2}) = \mu$. Then $\lim_{t\to 0+}p(t)/t^{k-1} = \mu$ and $\lim_{t\to 0+}kP(t)/t^k = \mu$. Hence for each $\epsilon \in (0,\mu)$ there exists $\delta > 0$ such that, for $t \in (0,\delta]$,

$$\begin{gathered}(\mu-\epsilon)(k-1)t^{k-2} < p'(t) < (\mu+\epsilon)(k-1)t^{k-2},\\ (\mu-\epsilon)t^{k-1} < p(t) < (\mu+\epsilon)t^{k-1},\\ (\mu-\epsilon)\frac{t^k}{k} < P(t) < (\mu+\epsilon)\frac{t^k}{k}.\end{gathered} \tag{2.76}$$

Consequently

$$\frac{P(t)}{p(t)} < \frac{t(\mu+\epsilon)}{k(\mu-\epsilon)}, \quad \frac{p'(t)P(t)}{p^2(t)} < \frac{k-1}{k}\left(\frac{\mu+\epsilon}{\mu-\epsilon}\right)^2, \quad t \in (0,\delta]. \tag{2.77}$$

Having in mind (2.75), we can choose n_0 in (2.62) such that for all $n \ge n_0$ the inequality $\overline{\gamma}_n \le \delta$ holds. Hence (2.72) and the first inequality in (2.77) yield

$$\frac{|B_n|}{2f(\sigma_n B_n)} < \int_0^{\overline{\gamma}_n}\frac{s(\mu+\epsilon)}{k(\mu-\epsilon)}\mathrm{d}s = \frac{\overline{\gamma}_n^2(\mu+\epsilon)}{2k(\mu-\epsilon)}. \tag{2.78}$$

Put $\mu_0^2 = k(\mu-\epsilon)/(\mu+\epsilon)$. Then $\mu_0^2|B_n|/f(\sigma_n B_n) \le \overline{\gamma}_n^2$, and

$$\overline{\gamma}_n \ge \mu_0\left(\frac{|B_n|}{f(\sigma_n B_n)}\right)^{1/2}. \tag{2.79}$$

On the other hand, by (2.76),

$$P(\overline{\gamma}_n) = \int_0^{\overline{\gamma}_n}p(t)\mathrm{d}t > \int_0^{\overline{\gamma}_n}(\mu-\epsilon)t^{k-1}\mathrm{d}t = \frac{\mu-\epsilon}{k}\overline{\gamma}_n^k. \tag{2.80}$$

By (2.79), this yields

$$P(\overline{\gamma}_n) > \frac{\mu-\epsilon}{k}\mu_0^k\left(\frac{|B_n|}{f(\sigma_n B_n)}\right)^{k/2} =: \widetilde{\mu}_0\left(\frac{|B_n|}{f(\sigma_n B_n)}\right)^{k/2}. \tag{2.81}$$

Step 3 (estimate of $\{I_n\}_{n=n_0}^{\infty}$). The inequality $a > (k-2)/k$ gives $a+1 > 2(k-1)/k$. Hence there exists $\epsilon \in (0,\mu)$ such that

$$a+1 = \frac{2(k-1)(\mu+\epsilon)^2}{k(\mu-\epsilon)^2}. \tag{2.82}$$

Having in mind (2.76), we choose δ to this ϵ and then, by the second inequality in (2.77), we obtain

$$\frac{2p'(t)P(t)}{p^2(t)} < a+1, \quad \text{for } t \in (0,\delta]. \tag{2.83}$$

Therefore

$$I_n(t) < 0 \quad \text{for } t \in (0,\delta],\ n \in \mathbb{N},\ n \geq n_0. \tag{2.84}$$

By Lemmas 2.7 and 2.8 there exists $K > 0$ such that

$$p(t)u_n'(t) \leq K, \quad \text{for } t \in J_n,\ n \in \mathbb{N}. \tag{2.85}$$

Here $J_n = [0,b_n)$, $n \in \mathbb{N}$, if (2.25) holds and $J_n = [0,\Gamma+1]$, $n \in \mathbb{N}$, if (2.26) holds. In addition there exists $\widetilde{b} > \Gamma+1$ such that $J_n \subset [0,\widetilde{b}]$, $n \in \mathbb{N}$. (Note that if $\{\theta_n\}_{n=1}^{\infty}$ in Lemma 2.8 is not bounded but does not fulfil (2.44), we work with a proper subsequence fulfilling (2.44).) By virtue of (2.84) and (2.85) we get

$$I_n(t) \leq K^2\int_\delta^t\left(\frac{2p'(s)P(s)}{p^2(s)} - (a+1)\right)\frac{1}{p(s)}\mathrm{d}s, \quad \text{for } t \in J_n,\ t \geq \delta,\ n \in \mathbb{N},\ n \geq n_0. \tag{2.86}$$

Inequalities (2.84) and (2.86) yield

$$I_n(t) < K^2\widetilde{b}\frac{1}{p(\delta)}\left(\frac{2p'\left(\widetilde{b}\right)P\left(\widetilde{b}\right)}{p^2(\delta)} + a + 1\right) =: \widetilde{K}, \quad \text{for } t \in J_n,\ n \in \mathbb{N},\ n \geq n_0. \tag{2.87}$$

Step 4 (final contradictions). Putting (2.81) and (2.87) to (2.68) and using (1.6), (1.10) and $C < \overline{B}$, we obtain

$$\begin{aligned} &Q(\kappa_n B_n)\left(\frac{|B_n|}{f(\sigma_n B_n)}\right)^{k/2}\widetilde{\mu}_0 - P(t)\left(\widetilde{Q} + 2F(C)\right) - \widetilde{K} \\ &\quad < P(t)u_n'^2(t), \quad \text{for } t \in [\gamma_n, b_n),\ n \in \mathbb{N},\ n \geq n_0. \end{aligned} \tag{2.88}$$

First, let us assume that (2.26) holds and $J_n = [0, \Gamma + 1]$, $n \in \mathbb{N}$. So, conditions (2.85), and (2.88) yield

$$\begin{aligned} &Q(\kappa_n B_n)\left(\frac{|B_n|}{f(\sigma_n B_n)}\right)^{k/2}\widetilde{\mu}_0 - P(\Gamma+1)\left(\widetilde{Q} + 2F(C)\right) - \widetilde{K} \\ &\quad < P(\Gamma+1)u_n'^2(\Gamma+1) < K^2\frac{P(\Gamma+1)}{p^2(\Gamma+1)} < \infty, \quad \text{for } n \in \mathbb{N},\ n \geq n_0. \end{aligned} \tag{2.89}$$

Letting $n \to \infty$ we get a contradiction to (2.53).

Finally, let us assume that (2.25) holds and $J_n = [0, b_n)$, $n \in \mathbb{N}$. Then (2.61), (2.88), and $J_n \subset [0, \widetilde{b}]$ yield

$$\begin{aligned} &Q(\kappa_n B_n)\left(\frac{|B_n|}{f(\sigma_n B_n)}\right)^{k/2}\widetilde{\mu}_0 - P\left(\widetilde{b}\right)\left(\widetilde{Q} + 2F(C)\right) - \widetilde{K} \\ &\quad < P\left(\widetilde{b}\right)u_n'^2(b_n) = 0, \quad \text{for } n \in \mathbb{N},\ n \geq n_0, \end{aligned} \tag{2.90}$$

contrary to (2.53). □

Remark 2.11. We assume that $k \geq 2$ in Theorem 2.10. In particular for $k = 2$ and $p(t) = ct$, $c > 0$, the function f can behave in neighbourhood of $-\infty$ as a function $|x|^r$ for arbitrary $r > 1$.

Now, let (2.58) hold for $k < 2$. Then $\lim_{t\to 0+} t^{k-1}/p(t) \in (0, \infty)$ and therefore

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} = \int_0^1 \frac{s^{k-1}}{p(s)}\frac{1}{s^{k-1}}\mathrm{d}s < \infty, \tag{2.91}$$

which is the first condition in (1.9). We have proved in [6, 7] that, in this case, assumptions (1.3)–(1.8) are sufficient for the existence of an escape solution.

Example 2.12. Let $\alpha \in (-1, 1)$ and $p(t) = t^3 + \alpha \sin t^3$, $t \in [0, \infty)$. Then $p'(t) = 3t^2(1 + \alpha\cos t^3)$ and

$$\lim_{t\to 0+}\frac{p'(t)}{t^2} = \lim_{t\to 0+} 3\left(1 + \alpha\cos t^3\right) = 3(1+\alpha). \tag{2.92}$$

Hence, for $k = 4$ condition (2.58) is satisfied. The critical value $(k+2)/(k-2)$ is equal to 3. By Theorem 2.10, if f fulfils (2.52) with $r \in (1, 3)$, problem (1.1), (1.11) has an escape solution.

Example 2.13. Let $p(t) = t\ln(t+1)$, $t \in [0,\infty)$. Then $p'(t) = \ln(t+1) + t/(t+1)$ and

$$\lim_{t\to 0+} \frac{p'(t)}{t} = \lim_{t\to 0+}\left(\frac{\ln(t+1)}{t} + \frac{1}{t+1}\right) = 2. \tag{2.93}$$

Hence, for $k = 3$ condition (2.58) is satisfied. The critical value $(k+2)/(k-2)$ is equal to 5. By Theorem 2.10, if f fulfils (2.52) with $r \in (1,5)$, problem (1.1), (1.11) has an escape solution.

3. Homoclinic Solutions

Having an escape solution we can deduce the existence of a homoclinic solution by the same arguments as in [6]. For completeness we bring here the main ideas. Remember that our basic assumptions (1.3)–(1.8), (1.10) and (1.13) are fulfilled in this section.

By Lemma 11 in [6], a solution u of problem (1.1), (1.11) is homoclinic if and only if

$$\sup\{u(t) : t \in [0,\infty)\} = L. \tag{3.1}$$

By Theorem 16 in [6], a solution u of problem (1.1), (1.11) is an escape solution if and only if

$$\sup\{u(t) : t \in [0,\infty)\} > L. \tag{3.2}$$

The third type of solutions of problem (1.1), (1.11) is characterized in the next definition.

Definition 3.1. A solution u of problem (1.1), (1.11) is called damped, if

$$\sup\{u(t) : t \in [0,\infty)\} < L. \tag{3.3}$$

The following properties of damped and escape solutions are important for the existence of homoclinic solutions.

Theorem 3.2 (see [6, Theorem 13] (on damped solutions)). *Let $\overline{B}$ be of* (1.5) *and* (1.6). *Assume that u is a solution of problem* (1.1), (1.11) *with $B \in [\overline{B}, 0)$. Then u is damped.*

Theorem 3.3 (see [6, Theorem 14]). *Let $\mathcal{M}_d$ be the set of all $B < 0$ such that corresponding solutions of problem* (1.1), (1.11) *are damped. Then $\mathcal{M}_d$ is open in* $(-\infty, 0)$.

Theorem 3.4 (see [6, Theorem 20]). *Let $\mathcal{M}_e$ be the set of all $B < 0$ such that corresponding solutions of problem* (1.1), (1.11) *are escape ones. Then $\mathcal{M}_e$ is open in* $(-\infty, 0)$.

Having these theorems we get the main result of this section.

Theorem 3.5 (On a homoclinic solution). *Assume that the assumptions of Theorem 2.10 are satisfied. Then problem* (1.1), (1.2) *has a homoclinic solution.*

Proof. By Theorems 3.2 and 3.3, the set $\mathcal{M}_d$ is nonempty and open in $(-\infty, 0)$. By Theorem 3.4, the set $\mathcal{M}_e$ is open in $(-\infty, 0)$. Using Theorem 2.10, we get that $\mathcal{M}_e$ is nonempty. Therefore the set $\mathcal{M}_h = (-\infty, 0) \setminus (\mathcal{M}_d \cup \mathcal{M}_e)$ is nonempty and if $B \in \mathcal{M}_h$, then the corresponding solution of problem (1.1), (1.11) is neither damped nor an escape solution. Therefore $\sup\{u(t) : t \in [0, \infty)\} = L$, and by Lemma 11 in [6], such solution u is homoclinic. □

The proof of Theorem 3.5 implies that if problem (1.1), (1.11) has an escape solution, then it has also a homoclinic solution. Hence the following corollary is true.

Corollary 3.6. *Assume that the assumptions of Theorem 2.10 are satisfied. Let problem* (1.1), (1.11) *have no homoclinic solution. Then it has no escape solution.*

If we assume (2.51) and (2.52), then the growth of f at $-\infty$ is less than the critical value $(k+2)/(k-2)$. This is necessary for the existence of homoclinic solutions of some types of (1.1). See the next example.

Example 3.7. Let $k, r \in \mathbb{N}$, $k > 2$, $r > 1$. Consider (1.1), where $p(t) = t^{k-1}$ and $f(x) = (1-x)^r - (1-x)$ for $x \le 1$ and $f(x) = 0$ for $x > 1$. Then p and f satisfy conditions (1.3)–(1.8), (1.10), (1.13), (2.52) and (2.58) with $L = 1$. By Theorem 3.5, if

$$r < \frac{k+2}{k-2}, \tag{3.4}$$

then problem (1.1), (1.11) has a homoclinic solution. But if

$$r \ge \frac{k+2}{k-2}, \tag{3.5}$$

then we have proved in [12] that problem (1.1), (1.11) has no homoclinic solution and consequently no escape solution.

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