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Research Article

Existence and Multiplicity of Positive Solutions to a Class of Quasilinear Elliptic Equations in $\mathbb{R}^N$

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We consider the following class of quasilinear elliptic equations $-h^p \Delta_p u + V_\varepsilon(x)|u|^{p-2}u = |u|^{q-2}u$, $u(x) > 0$ for all $x \in \mathbb{R}^N$, where $h > 0$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $2 \le p < N$, $p < q < p^* = Np/(N-p)$. We allow the potential V_ε to be unbounded below and prove the existence and multiplicity for positive solutions.

1. Introduction

In this paper we are concerned with the existence and multiplicity of positive solutions for the following class of quasilinear elliptic equations:

$$\begin{gathered} -h^p \Delta_p u + V_\varepsilon(x)|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}\left(\mathbb{R}^N\right) \quad \text{with } 2 \le p < N, \\ u(x) > 0, \quad \forall x \in \mathbb{R}^N, \end{gathered} \tag{$P_{h,\varepsilon}$}$$

where $h > 0$, $p < q < p^* = Np/(N-p)$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Moreover, we consider the perturbed potential V_ε satisfying

$$V_\varepsilon(x) = V(x) - \varepsilon(h)W(x), \quad \forall x \in \mathbb{R}^N, \tag{1.1}$$

where $\varepsilon : [0,+\infty) \to [0,+\infty), W : \mathbb{R}^N \to [0,+\infty)$ is a measurable function such that, for some $\alpha_1 > 0$ and $\alpha_2 \geq 0$, the inequality

$$\int_{\mathbb{R}^N} W(x)|u|^p \leq \alpha_1 \|\nabla u\|_p^p + \alpha_2 \|u\|_p^p \tag{1.2}$$

holds for any $u \in W^{1,p}(\mathbb{R}^N)$ and the "unperturbed" potential V is a continuous function satisfying

$$0 < V_0 = \inf_{\mathbb{R}^N} V < \liminf_{|x|\to\infty} V(x). \tag{1.3}$$

The last hypothesis was introduced by Rabinowitz in [1].

For the case $p = 2$, equations of the kind

$$-h^2 \Delta u + V(x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N \tag{P_*}$$

in different models, for example, are related with the existence of standing waves of the nonlinear Schrödinger equation

$$ih\frac{\partial \psi}{\partial t} = -h^2 \Delta \psi + (V(x) - \lambda)\psi - |\psi|^{q-2}\psi, \quad \forall x \in \mathbb{R}^N, \tag{NLS}$$

where $\lambda \in \mathbb{R}$ and $2 < q < 2N/(N-2)$. A standing wave of (NLS) is a solution of the form $\psi(x,t) = \exp(-i\lambda h^{-1}t)u(x)$. In this case, u is a solution of (P_*).

Existence and concentration of positive solutions for (P_*) have been extensively studied in the recent years; see, for example, Ambrosetti et al. [2, 3], Cingolani and Lazzo [4, 5], Floer and Weinstein [6], Oh [7–9], Rabinowitz [1], Serrin and Tang [10], Wang [11], and their references. In [12], Lazzo considers the potential in (P_*) perturbed by adding a negative potential. Under the assumptions (1.1)–(1.3) she obtained the existence and multiplicity results for positive solutions of the equation

$$-h^2 \Delta u + V_\varepsilon(x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $h > 0$, $2 < q < 2N/(N-2)$.

In this paper, we will adapt some variational arguments explored by Lazzo [12] and extend the results of [12] to the quasilinear case. In order to state our results we need the following standard notation: if Y is a closed subset of a topological space Z, $\operatorname{cat}_Z Y$ is the Ljusternik-Schnirelman category of Y in Z, namely, the least number of closed and contractible sets in Z which cover Y. If $Y = Z$, we set $\operatorname{cat}_Z(Z) = \operatorname{cat}(Y)$. Let

$$\begin{gathered} \varepsilon_0 = \limsup_{h\to 0} \frac{\varepsilon(h)}{h^p}, \\ M = \left\{x \in \mathbb{R}^N : V(x) = V_0\right\}. \end{gathered} \tag{1.5}$$

For $\delta > 0$, let $M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\}$.

Now we can describe our main results.

Theorem 1.1. *Suppose that the assumptions* (1.1)–(1.3) *hold. There exists* $\varepsilon^* > 0$ *such that if* $\varepsilon_0 < \varepsilon^*$, *then* $(P_{h,\varepsilon})$ *has a positive solution for* h *sufficiently small.*

Theorem 1.2. *Suppose that the assumptions* (1.1)–(1.3) *hold. For any* $\delta > 0$ *there exists* $\varepsilon^*(\delta) > 0$ *such that if* $\varepsilon_0 < \varepsilon^*(\delta)$, *then* $(P_{h,\varepsilon})$ *has at least* $\mathrm{cat}_{M_\delta}(M)$ *positive solutions for* h *sufficiently small.*

2. Existence of Solutions

In this section, we will give an existence result for $(P_{h,\varepsilon})$. We need some notations, definitions, and auxiliary results. Let us recall the definition of $W^{1,p}(\mathbb{R}^N)$,

$$W^{1,p}\left(\mathbb{R}^N\right) = \left\{u \in L^p\left(\mathbb{R}^N\right) : \partial_i u \in L^p\left(\mathbb{R}^N\right), i = 1,2,\dots,N\right\},$$
$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p, \tag{2.1}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{R}^N)$. The space $W^{1,p}(\mathbb{R}^N)$ is the completion of the space $D(\mathbb{R}^N)$ of C^∞-functions with compact support with respect to the norm $\|\cdot\|_{1,p}$ and

$$X = \left\{u \in W^{1,p}\left(\mathbb{R}^N\right) : \int V(x)|u|^p < +\infty\right\}, \tag{2.2}$$

X^* is the dual space of X and the integration set $\mathbb{R}^N$ will be understood.

In X we define the functionals

$$J_{h,\varepsilon}(u) = \int h^p|\nabla u|^p + V_\varepsilon(x)|u|^p,$$
$$J_{h,0}(u) = \int h^p|\nabla u|^p + V(x)|u|^p. \tag{2.3}$$

From (1.1)–(1.3) and if $0 < h^p \le V_0\alpha_1\alpha_2^{-1}$ (no restrictions on h if $\alpha_2 = 0$), then for any $u \in X$, we have

$$\left(1 - \alpha_1\frac{\varepsilon(h)}{h^p}\right)J_{h,0}(u) \le J_{h,\varepsilon}(u) \le J_{h,0}(u). \tag{2.4}$$

Indeed,

$$\int W(x)|u|^p \le \alpha_1\int|\nabla u|^p + \frac{\alpha_2}{V_0}\int V(x)|u|^p \le \frac{\alpha_1}{h^p}J_{h,0}(u). \tag{2.5}$$

As a consequence,

$$J_{h,0}(u) = J_{h,\varepsilon}(u) + \varepsilon(h)\int W(x)|u|^p \le J_{h,\varepsilon}(u) + \alpha_1\frac{\varepsilon(h)}{h^p}J_{h,0}(u) \tag{2.6}$$

whence (2.4) follows. From (2.4), if $\limsup_{h\to 0} \varepsilon(h)h^{-p} < \alpha_1^{-1}$ there exist $\alpha_0,\ h_0^* > 0$ such that

$$J_{h,\varepsilon}(u) \geq \min\{h^p, V_0\}\alpha_0\|u\|_{1,p}^p \tag{2.7}$$

for any $u \in X$, for any $0 < h < h_0^*$. As a result the set X, endowed with the norm $\|u\|_h^p = J_{h,\varepsilon}(u)$, is a Banach space and it is continuously embedded in $W^{1,p}(\mathbb{R}^N)$.

Weak solution to $(P_{h,\varepsilon})$ can be found by looking for critical points of $J_{h,\varepsilon}(u)$ on the manifold $\Sigma = \{u \in X : \int |u|^q = 1\}$. Indeed, $J_{h,\varepsilon}$ is well defined and smooth on Σ; moreover, for any critical point u of $J_{h,\varepsilon}$ on Σ, $(J_{h,\varepsilon}(u))^{1/(q-p)}u$ is a weak solution for $(P_{h,\varepsilon})$. Therefore, in order to prove existence of solutions to $(P_{h,\varepsilon})$ it suffices to solve the following minimization problem:

$$c_h = \inf_{u\in\Sigma} J_{h,\varepsilon}(u). \tag{P}$$

Problem (P) is affected by a lack of compactness, due to the noncompact Sobolev embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. One way is to guarantee that c_h is attained and to prove that $J_{h,\varepsilon}$ satisfies the Palais-Smale condition below $c_h + \alpha$, for some positive α. This is indeed the case: as we prove below, the Palais-Smale condition holds below some level, related to $\liminf_{|x|\to\infty} V(x)$. In order to state this result more precisely, we need some notations. First, let us recall some facts about ground state solution of the equation

$$-h^p\Delta_p u + \lambda|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{Q}$$

where $h, \lambda > 0$. By [13, Propositions 2.1 and 2.2], there is a positive radially symmetric ground state solution $\widetilde{w}(h,\lambda)$ of (Q). By adopting arguments similar to those in Li and Yan [14, Theorem 3.1], we obtain that $\widetilde{w}(h,\lambda) \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$ and that $\widetilde{w}(h,\lambda)$ decays exponentially at infinity (also see Alves and Carrião [15, Lemma 2.1]). The infimum

$$m(h;\lambda) = \inf\left\{\frac{h^p\|\nabla u\|_p^p + \lambda\|u\|_p^p}{\|u\|_q^p} : u \in W^{1,p}\left(\mathbb{R}^N\right),\ u \not\equiv 0\right\} \tag{2.8}$$

is achieved by $w(h;\lambda) = \widetilde{w}(h,\lambda)/\|\widetilde{w}(h,\lambda)\|_q$. It is easy to see that

$$m(h;\lambda) = h^\theta m(1;\lambda) \qquad \text{with } \theta = \frac{N(q-p)}{q}. \tag{2.9}$$

By (1.3), we can choose $V_\infty \in \mathbb{R}$ such that

$$V_0 < V_\infty \leq \liminf_{|x|\to\infty} V(x). \tag{2.10}$$

Let us denote

$$m_0 = m(1;V_0), \qquad m_\infty = m(1;V_\infty), \tag{2.11}$$

being the map $\lambda \to m(1;\lambda)$ strictly increasing, (2.10) implies

$$m_0 < m_\infty. \tag{2.12}$$

We are ready to state our compactness result.

Proposition 2.1. *Suppose that assumptions* (1.1)–(1.3) *hold and*

$$\varepsilon_0 < \frac{1}{\alpha_1}\left(1 - \frac{m_0}{m_\infty}\right). \tag{2.13}$$

Then there exists $k_1^* \in (0, m_\infty - m_0)$ *and* $h_1^* > 0$ *such that* $J_{h,\varepsilon}$ *satisfies the Palais-Smale condition in the sublevel* $\{u \in \Sigma : J_{h,\varepsilon}(u) < (m_0 + k_1^*)h^\theta\}$, *for any* $0 < h < h_1^*$.

Proof. Let $\beta \in (m_0, (1-\alpha_1\varepsilon_0)m_\infty)$ and fix $\eta_0 > 0$ such that

$$\beta + \alpha_1\eta_0 m_\infty < (1-\alpha_1\varepsilon_0)m_\infty, \tag{2.14}$$

obviously, for h small we have

$$\frac{\varepsilon(h)}{h^p} \le \varepsilon_0 + \eta_0. \tag{2.15}$$

Next, let $\gamma < \beta$ and let $\{u_n\} \subset \Sigma$ be a Palais-Smale sequence for $J_{h,\varepsilon}$ on Σ at the level $\gamma_h \equiv \gamma h^\theta$, namely,

$$J_{h,\varepsilon}(u_n) = \gamma_h + o(1), \tag{2.16}$$

$$-h^p\Delta_p u_n + V_\varepsilon(x)|u_n|^{p-2}u_n - \lambda_n|u_n|^{q-2}u_n = o(1) \quad \text{in } X^*, \tag{2.17}$$

as $n \to \infty$, it is easily seen that $\lambda_n = \gamma_h + o(1)$. By standard calculations, we can see that $\{u_n\}$ is bounded in X. Therefore there exists $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in X. Moreover, adapting arguments found in [16–18], it follows that u is a weak solution of the following equation:

$$-h^p\Delta_p u + V_\varepsilon(x)|u|^{p-2}u = \gamma_h|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{E}$$

In order to prove that $\{u_n\}$ converges to u strongly in X we apply Lions Concentration-Compactness Lemma (see [19, 20]) to the sequence of measures $\rho_n = h^p|\nabla u_n|^p + V_\varepsilon(x)|u_n|^p$. By [20, Lemma I.1], and the fact that $u_n \in \Sigma$, we can exclude that vanishing occurs. If dichotomy occurs, there exists $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 = \gamma_h$ such that for any $\xi > 0$ there are $y_n \in \mathbb{R}^N$, $R > 0$, $R_n \to \infty$ such that

$$\int_{|x-y_n|<R} \rho_n \ge \delta_1 - \xi, \qquad \int_{|x-y_n|>2R_n} \rho_n \ge \delta_2 - \xi. \tag{2.18}$$

As a consequence,

$$\int_{R<|x-y_n|<2R_n} \rho_n \leq 2\xi. \tag{2.19}$$

Let $\zeta : [0,+\infty) \to [0,1]$ be a smooth, nonincreasing function, such that $\zeta(t) = 1$ if $0 \leq t \leq 1$, $\zeta(t) = 0$ if $t \geq 2$. If we define

$$u_n^1(x) = u_n(x)\zeta\left(\frac{x-y_n}{R}\right), \qquad u_n^2(x) = u_n(x) - u_n(x)\zeta\left(\frac{x-y_n}{R_n}\right), \tag{2.20}$$

then (2.18) yields

$$\int h^p|\nabla u_n^i|^p + V_\varepsilon(x)|u_n^i|^p \geq \delta_i - \xi, \quad i = 1,2. \tag{2.21}$$

From the definition of u_n^i, $i = 1,2$, and (2.19) we get

$$\begin{gathered}
\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n^i = \int \left|\nabla u_n^i\right|^p + O(\xi), \\
\int V_\varepsilon(x)|u_n|^{p-2}u_n u_n^i = \int V_\varepsilon(x)\left|u_n^i\right|^p + O(\xi), \\
\int |u_n|^{q-2}u_n u_n^i = \int \left|u_n^i\right|^q + O(\xi),
\end{gathered} \tag{2.22}$$

whence, by taking (2.17) into account,

$$J_{h,\varepsilon}\left(u_n^i\right) = \int h^p|\nabla u_n^i|^p + V_\varepsilon(x)|u_n^i|^p = \gamma_h \int |u_n^i|^q + o(1) + O(\xi). \tag{2.23}$$

Now, if the sequence $\{y_n\}$ is unbounded in $\mathbb{R}^N$, for large n we have $V(x) \geq V_\infty - \xi$ for any $x \in B_R(y_n)$. Thus from (2.4), (2.15), the definition of $m(h;V_\infty)$, and (2.23) we have

$$\begin{aligned}
J_{h,\varepsilon}\left(u_n^1\right) &\geq \left(1 - \alpha_1\frac{\varepsilon(h)}{h^p}\right)\int h^p|\nabla u_n^1|^p + V(x)|u_n^1|^p \\
&\geq O(\xi) + (1 - \alpha_1(\varepsilon_0 + \eta_0))\int h^p|\nabla u_n^1|^p + V_\infty|u_n^1|^p \\
&\geq O(\xi) + (1 - \alpha_1(\varepsilon_0 + \eta_0))m(h;V_\infty)\left\|u_n^1\right\|_q^p \\
&= O(\xi) + o(1) + (1 - \alpha_1(\varepsilon_0 + \eta_0))m(h;V_\infty)\left(\frac{J_{h,\varepsilon}(u_n^1)}{\gamma_h}\right)^{p/q},
\end{aligned} \tag{2.24}$$

whence

$$J_{h,\varepsilon}\left(u_n^1\right) \geq O(\xi) + o(1) + \left(1 - \alpha_1\left(\varepsilon_0 + \eta_0\right)\right)^{q/(q-p)} m(h; V_\infty)^{q/(q-p)} \gamma_h^{p/(p-q)}. \tag{2.25}$$

From (2.16) and (2.25) we can deduce

$$\begin{aligned} \gamma_h + o(1) &\geq J_{h,\varepsilon}\left(u_n^1\right) + O(\xi) \\ &\geq O(\xi) + o(1) + \left(1 - \alpha_1\left(\varepsilon_0 + \eta_0\right)\right)^{q/(q-p)} m(h; V_\infty)^{q/(q-p)} \gamma_h^{p/(p-q)}, \end{aligned} \tag{2.26}$$

letting $\xi \to 0$, $n \to \infty$ and dividing by h^θ yields

$$\gamma \geq \left(1 - \alpha_1\left(\varepsilon_0 + \eta_0\right)\right) m_\infty \tag{2.27}$$

and, from (2.14), $\gamma > \beta$, a contradiction. If the sequence $\{y_n\}$ is bounded in $\mathbb{R}^N$, for large n we have $V(x) \geq V_\infty - \xi$ for any x such that $|x - y_n| > R_n$, and we get again a contradiction by taking u_n^2 into account. Dicotomy is therefore ruled out in any case. As a result, the sequence $\{\rho_n\}$ is tight; there exists $\{y_n\} \subset \mathbb{R}^N$ such that for any $\xi > 0$

$$\int_{|x-y_n|<R} h^p |\nabla u_n|^p + V_\varepsilon(x)|u_n|^p \geq \gamma_h - \xi \tag{2.28}$$

for a suitable $R > 0$. If the sequence $\{y_n\}$ is unbounded in $\mathbb{R}^N$, we could define u_n^1 as in (2.20) and, noticing that

$$\int h^p |\nabla u_n^1|^p + V_\varepsilon(x)|u_n^1|^p \geq \gamma_h - \xi, \tag{2.29}$$

we could get a contradiction exactly as before. So $\{y_n\}$ is bounded in $\mathbb{R}^N$, and for some $\overline{R}$ we have

$$\int_{|x|>\overline{R}} h^p |\nabla u_n|^p + V_\varepsilon(x)|u_n|^p < \xi + o(1). \tag{2.30}$$

By the compactness of the embedding $W^{1,p} \hookrightarrow L^q$ on bounded domains implies that $\{u_n\} \to u$ strongly in L^q and u is a weak solution of (E), we get

$$\begin{aligned} \int h^p |\nabla u_n|^p + V_\varepsilon(x)|u_n|^p &= \gamma_h \int |u_n|^q + o(1) = \gamma_h \int |u|^q + o(1) \\ &= \int h^p |\nabla u|^p + V_\varepsilon(x)|u|^p + O(\xi) + o(1). \end{aligned} \tag{2.31}$$

In other words, $\|u_n\|_h^p \to \|u\|_h^p$. Finally, by using the Brezis-Lieb's lemma [21] and arguing as in [22, Lemma 2.4], imply $u_n \to u$ strongly in X. □

Remark 2.2. By Proposition 2.1 and the choice of V_∞ it follows that if V is coercive, namely, $V(x) \to \infty$ as $|x| \to \infty$, then $J_{h,\varepsilon}$ satisfies the Palais-Smale condition on Σ at any level. Without loss of generality, we will henceforth assume $V_\infty = \lim\inf_{|x|\to\infty} V(x) < +\infty$.

We are interested in positive solutions for $(P_{h,\varepsilon})$. Now, we state our result on the sign of solutions for $(P_{h,\varepsilon})$.

Proposition 2.3. *Suppose that assumptions* (1.1)–(1.3) *hold and*

$$\varepsilon_0 < \frac{1}{\alpha_1}\left(1 - 2^{(p-q)/q}\right). \tag{2.32}$$

Then there exists $k_2^*, h_2^* > 0$ such that, for any $0 < h < h_2^*$, every critical point u of $J_{h,\varepsilon}$ on Σ satisfying

$$J_{h,\varepsilon}(u) \le \left(m_0 + k_2^*\right)h^\theta \tag{2.33}$$

does not change sign, where θ is the same as in (2.9).

Proof. Fix $\eta_0 > 0$ such that $0 < \alpha_1(\varepsilon_0 + \eta_0) < 1 - 2^{(p-q)/q}$ and let $h_2^* \in (0, h_0^*)$ be such that $\varepsilon(h) < (\varepsilon_0 + \eta_0)h^p$ for any $0 < h < h_2^*$, where h_0^* is the same as in (2.7). Finally, choose

$$0 < k_2^* < \left(2^{(q-p)/q}\left(1 - \alpha_1(\varepsilon_0 + \eta_0)\right) - 1\right)m_0. \tag{2.34}$$

Now, let $0 < h < h_2^*$ and let $u = u^+ - u^-$ be a critical point of $J_{h,\varepsilon}$ on Σ such that $u^+, u^- \not\equiv 0$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We recall $c_h = \inf_{u\in\Sigma} J_{h,\varepsilon}(u)$. If we multiply

$$-h^p \Delta_p u + V_\varepsilon(x)|u|^{p-2}u = J_{h,\varepsilon}(u)|u|^{q-2}u \tag{2.35}$$

by u^+ and integrate on $\mathbb{R}^N$, we get

$$J_{h,\varepsilon}(u)\|u^+\|_q^q = J_{h,\varepsilon}(u^+) \ge c_h\|u^+\|_q^p, \tag{2.36}$$

thus

$$\|u^+\|_q^q \ge \left(\frac{c_h}{J_{h,\varepsilon}(u)}\right)^{q/(q-p)}. \tag{2.37}$$

Similarly, the same inequality holds for u^-, thus

$$1 = \|u^+\|_q^q + \left\|u^-\right\|_q^q \ge 2\left(\frac{c_h}{J_{h,\varepsilon}(u)}\right)^{q/(q-p)}, \tag{2.38}$$

whence

$$J_{h,\varepsilon}(u) \ge 2^{(q-p)/q}c_h. \tag{2.39}$$

Then (2.4), (2.9), (2.33), and the definition of m_0 give

$$(m_0 + k_2^*)h^\theta \geq J_{h,\varepsilon}(u) \geq 2^{(q-p)/q}(1 - \alpha_1(\varepsilon_0 + \eta_0))m_0 h^\theta, \tag{2.40}$$

if we divide by h^θ, the last inequality contradicts (2.34). This completes the proof. □

Proof of Theorem 1.1. Let $\delta > 0$ be fixed and let $\eta : [0, +\infty) \to [0, 1]$ be a smooth, nonincreasing function, such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t \geq \delta$. Let $w = w(1; V_0)$, fix any x_0 such that $V(x_0) = V_0$ and set

$$\psi_{h,x_0}(x) = \mu_h w\left(\frac{x - x_0}{h}\right)\eta(|x - x_0|), \tag{2.41}$$

the constant μ_h is chosen in such a way that $\|\psi_{h,x_0}\|_q = 1$. Then, $\psi_{h,x_0} \in \Sigma$ and it is easy to see that

$$\begin{aligned} J_{h,\varepsilon}(\psi_{h,x_0}) \leq J_{h,0}(\psi_{h,x_0}) &= \int h^p|\nabla\psi_{h,x_0}|^p + V(x)|\psi_{h,x_0}|^p \\ &= \frac{h^N \int \left|\nabla(w(x)\eta(h|x|))\right|^p + V(hx + x_0)\left|w(x)\eta(h|x|)\right|^p}{\left(h^N \int \left|w(x)\eta(h|x|)\right|^q\right)^{p/q}} \\ &= \frac{\int |\nabla w(x)|^p + V(x_0)|w(x)| + o(1)}{\left(\int |w(x)|^q + o(1)\right)^{p/q}} h^\theta = (m_0 + o(1))h^\theta. \end{aligned} \tag{2.42}$$

As a consequence, for h small we have $c_h < (m_0 + k_1^*)h^\theta$; if $\varepsilon_0 < \varepsilon^* = 1/\alpha_1 \min\{(1 - 2^{(p-q)/q}), (1 - m_0/m_\infty)\}$, Propositions 2.1 and 2.3 apply and imply $J_{h,\varepsilon}(u) = c_h$ for some $u \in \Sigma$ and u does not change sign. We can therefore assume that u is positive and, up to a Lagrange multiplier, $(J_{h,\varepsilon}(u))^{1/(q-p)}u$ is a positive solution of $(P_{h,\varepsilon})$. □

3. Multiplicity of Solutions

We begin our discussion by giving some definitions and some known results. For any constant a, we define

$$J_{h,\varepsilon}^a = \{u \in \Sigma : J_{h,\varepsilon}(u) \leq a\}. \tag{3.1}$$

We recall that M denotes the set of global minima points of V and, for any positive δ, let $M_\delta = \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, M) \leq \delta\}$. In order to prove our multiplicity result, we need the following proposition. For the proof, based on the very definition of category and homotopical equivalence, we refer, for instance, to [23].

Proposition 3.1. *Let $a > 0$ and let J^* be a closed subset of $J_{h,\varepsilon}^a$. Let $\Phi_h : M \to J^*$, $\beta : J_{h,\varepsilon}^a \to M_\delta$ be continuous maps such that $\beta \circ \Phi_h$ is homotopically equivalent to the embedding $j : M \to M_\delta$. Then $\mathrm{cat}_{J_{h,\varepsilon}^a}(J^*) \geq \mathrm{cat}_{M_\delta}(M)$.*

In our setting, the construction of the map Φ_h is very simple. Indeed, for any $x_0 \in M$ and for any h we define $\Phi_h(x_0) = \psi_{h,x_0}$ (cf. (2.41), where ψ_{h,x_0} was introduced).

For any $\delta > 0$, let $\rho = \rho_\delta > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| < \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geq \rho$. Finally, we define the barycenter map $\beta : \Sigma \to \mathbb{R}^N$ by setting $\beta(u) = \int \chi(x)|u(x)|^q$. Since $M_\delta \subset B_\rho(0)$, we can use the definition of χ and the Lebesgue theorem to conclude that

$$\lim_{h\to 0} \beta(\Phi_h(x_0)) = x_0 \quad \text{uniformly for } x_0 \in M. \tag{3.2}$$

The content of the following proposition is that barycenters of low energy functions are close to M.

Proposition 3.2. *Suppose that assumptions* (1.1)–(1.3) *hold. For any $\delta > 0$ there exists $\varepsilon_1^*(\delta) > 0$ such that if*

$$\varepsilon_0 < \varepsilon_1^*(\delta), \tag{3.3}$$

then there exist $k_3^*, h_3^* > 0$ such that $\beta(u) \in M_\delta$ for any $u \in \Sigma$ satisfying $J_{h,\varepsilon} \leq (m_0 + k_3^*)h^\theta$ for $0 < h < h_3^*$, where θ is the same as in (2.9).

Proof. By contradiction, let us assume that for some $\delta > 0$ we can find $\varepsilon_m \geq 0$ such that $\varepsilon_m \to 0$ as $m \to \infty$, $\limsup_{h\to 0} \varepsilon(h)h^{-p} \leq \varepsilon_m$, and the claim in Proposition 3.2 does not hold.

For h small we have $\varepsilon(h)h^{-p} < \varepsilon_m + 1/m$ and by (2.4)

$$\left(1 - \alpha_1\left(\varepsilon_m + \frac{1}{m}\right)\right) J_{h,0}(u) \leq J_{h,\varepsilon}(u). \tag{3.4}$$

Let $h_n, k_n \to 0^+$ as $n \to \infty$ and $u_n \in \Sigma$ be such that $J_{h,\varepsilon}(u_n) \leq (m_0 + k_n)h_n^\theta$ and $\beta(u_n) \notin M_\delta$. Let $v_n(x) = h_n^{N/q} u_n(h_n x)$ and from (3.4) we have

$$\int |\nabla v_n|^p + V(h_n x)|v_n|^p \leq \frac{m_0 + k_n}{1 - \alpha_1(\varepsilon_m + 1/m)}. \tag{3.5}$$

We apply Lions' lemma to the sequence of probability measures $\sigma_n = |v_n|^q$. Vanishing is easily ruled out. If dichotomy occurs, there exist $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 = 1$ such that for any $\xi > 0$ there are $y_n \in \mathbb{R}^N$, $R > 0$, $R_n \to \infty$ such that

$$\int_{|x-y_n|<R} \sigma_n \geq \delta_1 - \xi, \qquad \int_{|x-y_n|>2R_n} \sigma_n \geq \delta_2 - \xi. \tag{3.6}$$

Let us consider ζ as in the proof of Proposition 2.1 and define v_n^1, v_n^2 accordingly as in (2.20). Inequalities (3.6) give

$$\int |v_n^i|^p \geq \delta_i - \xi, \quad i = 1, 2. \tag{3.7}$$

From (3.5) and (3.7) we get

$$
\begin{aligned}
\frac{m_0 + k_n}{1 - \alpha_1(\varepsilon_m + 1/m)} &\geq \int |\nabla v_n^1|^p + V_0 |v_n^1|^p + \int |\nabla v_n^2|^p + V_0 |v_n^2|^p + O(\xi) \\
&\geq m_0 \left(\left\| v_n^1 \right\|_q^p + \left\| v_n^2 \right\|_q^p \right) + O(\xi) \\
&\geq m_0 \left((\delta_1 - \xi)^{p/q} + (\delta_2 - \xi)^{p/q} \right).
\end{aligned}
\tag{3.8}
$$

As $m, n \to \infty$ and $\xi \to 0$ we deduce $1 \geq \delta_1^{p/q} + \delta_2^{p/q}$, a contradiction. Thus $\{\sigma_n\}$ is tight; there exists $\{y_n\} \subset \mathbb{R}^N$ such that for any $\xi > 0$

$$
\int_{|x - y_n| < R} |v_n(x)|^q \geq 1 - \xi \tag{3.9}
$$

for a suitable $R > 0$. The sequence $\overline{v}_n = v_n(\cdot + y_n)$ is bounded in $W^{1,p}(\mathbb{R}^N)$, hence it weakly converges to some $\overline{v}$ in $W^{1,p}(\mathbb{R}^N)$ and, due to the compactness property (3.9), strongly in $L^q(\mathbb{R}^N)$. If the sequence $x_n \equiv h_n y_n \to \infty$ as $n \to \infty$, then (3.5) gives

$$
m_0 \geq \int |\nabla \overline{v}|^p + \liminf_{n \to \infty} \int V(h_n x + x_n) |\overline{v}_n|^p \geq \int |\nabla \overline{v}|^p + V_\infty |\overline{v}|^p \geq m_\infty, \tag{3.10}
$$

which contradicts (2.12). Thus we can assume that x_n converges to some $\overline{x}$ (up to a subsequence), and arguing as before we obtain

$$
m_0 \geq \int |\nabla \overline{v}|^p + V(\overline{x}) |\overline{v}|^p \geq m(1; V(\overline{x})) \geq m_0. \tag{3.11}
$$

From this we have $V(\overline{x}) = V_0$ and $\int |\nabla \overline{v}|^p + V_0(\overline{x}) |\overline{v}|^p = m_0$, hence $m_0 = m(1; V_0)$ is achieved by $\overline{v} \in \Sigma$. Furthermore, since $\int |\nabla \overline{v}_n|^p + V_0 |\overline{v}_n|^p \geq m_0$, from (3.5) we get $\int |\nabla \overline{v}_n|^p + V_0 |\overline{v}_n|^p \to m_0 = \int |\nabla \overline{v}|^p + V_0 |\overline{v}|^p$ as $n \to \infty$. By using the Brezis-Lieb's lemma [21] and as in [22, Lemma 2.4], we get that $\overline{v}_n$ converges to $\overline{v}$ strongly in $W^{1,p}(\mathbb{R}^N)$. Finally, let $\delta > 0$ be fixed and let $\eta : [0, +\infty) \to [0, 1]$ be a smooth, nonincreasing function, such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t \geq \delta$. Set

$$
\psi_n(x) = \mu_n \overline{v}\left(\frac{x - x_n}{h_n} \right) \eta(|x - x_n|), \tag{3.12}
$$

where the constant μ_n is chosen in such a way that $\|\psi_n\|_q = 1$. Then, $\psi_n \in \Sigma$ and it is easy to see that

$$
\left| \beta(u_n) - \beta(\psi_n) \right| \leq \rho \int \left| |\overline{v}_n|^q - |\overline{v}|^q \right| = o(1). \tag{3.13}
$$

By $x_n \rightarrow \overline{x} \in M$ and the fact $M_\delta \subset B_\rho(0)$ and Lebesgue theorem, it follows that $|\beta(\psi_n) - x_n| = o(1)$. Therefore, $|\beta(u_n) - x_n| = o(1)$, which contradicts $\beta(u_n) \notin M_\delta$. This completes the proof. □

Proof of Theorem 1.2. Let $\delta > 0$ be fixed and let $\varepsilon_1^*(\delta)$ be as in Proposition 3.2. Let

$$\varepsilon^*(\delta) = \min\left\{ \frac{1}{\alpha_1}\left(1 - 2^{(p-q)/q}\right), \frac{1}{\alpha_1}\left(1 - \frac{m_0}{m_\infty}\right), \varepsilon_1^*(\delta) \right\}, \tag{3.14}$$

and assume $\varepsilon_0 < \varepsilon^*(\delta)$. Let $0 < h^* \leq \min\{h_i^* : i = 1,2,3\}$ and $k^* = \min\{k_i^* : i = 1,2,3\}$, with the constants h_i^*, k_i^* being defined in Propositions 2.1, 2.3, and 3.2. Let $0 < h < h^*$; we can assume that $a(h) \equiv (m_0 + k^*)h^\theta$ is not a critical value for $J_{h,\varepsilon}$ on Σ. For convenience, we set $\Sigma_h = \{u \in \Sigma : J_{h,\varepsilon}(u) \leq a(h)\}, \Sigma_h^+ = \{u \in \Sigma_h : u \geq 0\}$, and $\Sigma_h^- = \{u \in \Sigma_h : u \leq 0\}$.

If h is small enough, (2.42) gives $J_{h,\varepsilon}(\Phi_h(x_0)) \leq (m_0 + k^*)h^\theta$ for any $x_0 \in M$. In other words, $\Phi_h(x_0) \in \Sigma_h^+$ for any $x_0 \in M$. Furthermore, Proposition 3.2 implies $\beta(u) \in M_\delta$ for any $u \in \Sigma_h$. Finally, as a consequence of (3.2) it is easy to see that $\beta \circ \Phi_h$ is homotopically equivalent to the embedding $j : M \rightarrow M_\delta$. Thus Proposition 3.1 gives $\text{cat}_{\Sigma_h}(\Sigma_h^+) \geq \text{cat}_{M_\delta}(M)$. If we use the map $-\Phi_h$ we also get $\text{cat}_{\Sigma_h}(\Sigma_h^-) \geq \text{cat}_{M_\delta}(M)$, whence $\text{cat}(\Sigma_h) \geq 2\text{cat}_{M_\delta}(M)$, for h small.

Proposition 2.1 guarantees that the Palais-Smale condition holds in a sublevel containing Σ_h. Thus Ljusternik-Schnirelman theory applies and we deduce that $J_{h,\varepsilon}$ has at least $2\text{cat}_{M_\delta}(M)$ critical points on Σ, satisfying $J_{h,\varepsilon}(u) \leq a(h) < (m_0 + k_1^*)h^\theta$. Therefore, by Proposition 2.3 they do not change sign and we can assume that at least $\text{cat}_{M_\delta}(M)$ critical points are positive. □

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