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Research Article

Existence of Solutions for the $p(x)$-Laplacian Problem with Singular Term

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We study the following $p(x)$-Laplacian problem with singular term: $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda |u|^{\alpha(x)-2}u + f(x,u)$, $x \in \Omega$, $u = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p^- \le p(x) \le p^+ < N$. We obtain the existence of solutions in $W_0^{1,p(x)}(\Omega)$.

1. Introduction

After Kováčik and Rákosník first discussed the $L^{p(x)}(\Omega)$ spaces and $W^{k,p(x)}(\Omega)$ spaces in [1], a lot of research has been done concerning these kinds of variable exponent spaces, for example, see [2–5] for the properties of such spaces and [6–9] for the applications of variable exponent spaces on partial differential equations. Especially in $W^{1,p(x)}(\Omega)$ spaces, there are a lot of studies on $p(x)$-Laplacian problems; see [8, 9]. In the recent years, the theory of problems with $p(x)$-growth conditions has important applications in nonlinear elastic mechanics and electrorheological fluids (see [10–14]).

In this paper, we study the existence of the weak solutions for the following $p(x)$-Laplacian problem:

$$-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = \lambda |u|^{\alpha(x)-2}u + f(x,u), \quad x \in \Omega,$$
$$u = 0, \qquad x \in \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < \lambda \in \mathbb{R}$, $p(x)$ is Lipschitz continuous on $\overline{\Omega}$, and $1 < p^- \le p(x) \le p^+ < N$.

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, \infty)$. For all $p(x) \in \mathbf{P}(\Omega)$, we denote $p^+ = \sup_{x\in\overline{\Omega}} p(x)$, $p^- = \inf_{x\in\overline{\Omega}} p(x)$, and denote by $p_1(x) \ll p_2(x)$ the fact that $\inf\{p_2(x) - p_1(x)\} > 0$.

We impose the following condition on f:

(F) $f(x,t) = b(x)g(x,t)$, $0 < b(x) \in L^{r(x)}(\Omega)$, $1 \ll r(x) \in C(\overline{\Omega})$ and for $g(x,t) \in C(\overline{\Omega} \times R)$, there exist $M > 0$ such that $|g(x,t)| \le c_1 + c_2|t|^{q(x)-1}$, whenever $|t| \ge M$.

A typical example of (1.1) is the following problem involving subcritical Sobolev-Hardy exponents of the form

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = \lambda|u|^{\alpha(x)-2}u + \frac{\widetilde{\lambda}}{|x|^{s(x)}}|u|^{q(x)-2}u, \quad x \in \Omega,$$
$$u = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where $0 < \widetilde{\lambda} \in \mathbb{R}$, $s(x), q(x) \in C(\overline{\Omega})$, $0 \le s^- \le s^+ < N$, $1 \ll q(x) < ((N - s(x))/N)p^*(x)$, for all $x \in \overline{\Omega}$. In fact, take $b(x) = \widetilde{\lambda}/|x|^{s(x)}$, $g(x,t) = |t|^{q(x)-2}t$, and $r(x) \equiv (1/2)(1 + N/s^+)$, then it is easy to verify that (F) is satisfied.

Our object is to obtain the existence of solutions in the following four cases:

(1) $\alpha(x) > p(x)$, $q(x) > p(x)$;

(2) $\alpha(x) < p(x)$, $q(x) > p(x)$;

(3) $\alpha(x) < p(x)$, $q(x) < p(x)$;

(4) $\alpha(x) > p(x)$, $q(x) < p(x)$.

When $b(x) = 1$, the solution of the p-Laplacian equations without singularity has been studied by many researchers. The study of problem (1.1) with variable exponents is a new topic.

The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge of variable exponent Lebesgue and Sobolev spaces. In Section 3, we prove our main results.

2. Preliminaries

In this section we first recall some facts on variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set; see [1–4, 8, 15] for the details.

Let $p(x) \in \mathbf{P}(\Omega)$ and

$$\|u\|_p = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of functions u such that $\int_\Omega |u(x)|^{p(x)} dx < \infty$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.1).

For a given $p(x) \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p' = \frac{p(x)}{p(x)-1}. \tag{2.2}$$

Theorem 2.1. *Let $p(x) \in \mathbf{P}(\Omega)$. Then the inequality*

$$\int_\Omega |f(x) \cdot g(x)| dx \leq r_p \|f\|_p \|g\|_{p'} \tag{2.3}$$

holds for every $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$ with the constant r_p depending on $p(x)$ and Ω only.

Theorem 2.2. *The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ if and only if $p \in L^\infty(\Omega)$. The space $L^{p(x)}(\Omega)$ is reflexive if and only if*

$$1 < p^- \leq p^+ < \infty. \tag{2.4}$$

Theorem 2.3. *Suppose that $p(x)$ satisfies (2.4). Let $\operatorname{meas}\Omega < \infty$, $p_1(x), p_2(x) \in \mathbf{P}(\Omega)$, then necessary and sufficient condition for $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$ is that for almost all $x \in \Omega$ one has $p_1(x) \leq p_2(x)$, and in this case, the imbedding is continuous.*

Theorem 2.4. *Suppose that $p(x)$ satisfies (2.4). Let $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$. If $u, u_k \in L^{p(x)}(\Omega)$, then*

(1) $\|u\|_p < 1 (= 1; > 1)$ *if and only if* $\rho(u) < 1 (= 1; > 1)$,

(2) *if* $\|u\|_p > 1$, *then* $\|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+}$,

(3) *if* $\|u\|_p < 1$, *then* $\|u\|_p^{p^+} \leq \rho(u) \leq \|u\|_p^{p^-}$,

(4) $\lim_{k\to\infty} \|u_k\|_p = 0$ *if and only if* $\lim_{k\to\infty} \rho(u_k) = 0$,

(5) $\|u_k\|_p \to \infty$ *if and only if* $\rho(u_k) \to \infty$.

We assume that k is a given positive integer.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of functions f on Ω such that $D^\alpha f \in L^{p(x)}$ for every multi-index α with $|\alpha| \leq k$, endowed with the norm

$$\|f\|_{k,p} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_p. \tag{2.5}$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.5).

In this paper we use the following equivalent norm on $W^{1,p(x)}(\Omega)$:

$$\|u\|_{1,p} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1 \right\}. \tag{2.6}$$

Then we have the inequality $(1/2)(\|\nabla u\|_p + \|u\|_p) \leq \|u\|_{1,p} \leq 2(\|\nabla u\|_p + \|u\|_p)$.

Theorem 2.5. *The spaces $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable reflexive Banach spaces, if $p(x)$ satisfies* (2.4).

Theorem 2.6. *Suppose that $p(x)$ satisfies* (2.4). *Let $I(u)=\int_\Omega |\nabla u(x)|^{p(x)}+|u(x)|^{p(x)}dx$. If $u,u_k\in W^{1,p(x)}(\Omega)$, then*

(1) $\|u\|_{1,p}<1(=1;>1)$ *if and only if* $I(u)<1(=1;>1)$,

(2) *if* $\|u\|_{1,p}>1$, *then* $\|u\|_{1,p}^{p^-}\le I(u)\le \|u\|_{1,p}^{p^+}$,

(3) *if* $\|u\|_{1,p}<1$, *then* $\|u\|_{1,p}^{p^+}\le I(u)\le \|u\|_{1,p}^{p^-}$,

(4) $\lim_{k\to\infty}\|u_k\|_{1,p}=0$ *if and only if* $\lim_{k\to\infty}I(u_k)=0$,

(5) $\|u_k\|_{1,p}\to\infty$ *if and only if* $I(u_k)\to\infty$.

We denote the dual space of $W_0^{k,p(x)}(\Omega)$ by $W^{-k,p'(x)}(\Omega)$, then we have the following.

Theorem 2.7. *Let $p\in \mathbf{P}(\Omega)\cap L^\infty(\Omega)$. Then for every $G\in W^{-k,p'(x)}(\Omega)$ there exists a unique system of functions $\{g_\alpha\in L^{p'(x)}(\Omega):|\alpha|\le k\}$ such that*

$$G(f)=\sum_{|\alpha|\le k}\int_\Omega D^\alpha f(x)g_\alpha(x)dx,\quad f\in W_0^{k,p(x)}(\Omega). \tag{2.7}$$

The norm of $W^{-k,p'(x)}(\Omega)$ is defined as

$$\|G\|_{-k,p'}=\sup\left\{\frac{|G(f)|}{\|f\|_{k,p}}:f\in W_0^{k,p(x)}(\Omega)\setminus\{0\}\right\}. \tag{2.8}$$

Theorem 2.8. *Let Ω be a domain in $\mathbb{R}^n$ with cone property. If $p:\overline{\Omega}\to R$ is Lipschitz continuous and $1<p^-\le p^+<N/k$, $q(x):\overline{\Omega}\to R$ is measurable and satisfies $p(x)\le q(x)\le p^*(x):=Np(x)/(N-kp(x))$ a.e. $x\in\overline{\Omega}$, then there is a continuous embedding $W^{k,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.*

Theorem 2.9. *Let Ω be a bounded domain. If $p(x)\in L^\infty(\Omega)$ and $u\in W_0^{1,p(x)}(\Omega)$, then*

$$\|u\|_p\le C\|\nabla u\|_p, \tag{2.9}$$

where C is a constant depending on Ω.

Next let us consider the weighted variable exponent Lebesgue space. Let $a(x)\in\mathbf{P}(\Omega)$, and $a(x)>0$ for $x\in\Omega$. Define

$$L_{a(x)}^{p(x)}(\Omega)=\left\{u\in P(\Omega):\int_\Omega a(x)|u(x)|^{p(x)}dx<\infty\right\} \tag{2.10}$$

with the norm

$$|u|_{L^{p(x)}_{a(x)}(\Omega)} = \|u\|_{p,a} = \inf\left\{\lambda > 0 : \int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}, \tag{2.11}$$

then $L^{p(x)}_{a(x)}(\Omega)$ is a Banach space.

Theorem 2.10. *Suppose that $p(x)$ satisfies* (2.4). *Let $\rho(u) = \int_\Omega a(x)|u(x)|^{p(x)}dx$. If $u,u_k \in L^{p(x)}_{a(x)}(\Omega)$, then*

(1) *for $u \neq 0$, $\|u\|_{p,a} = \lambda$ if and only if $\rho(u/\lambda) = 1$,*

(2) *$\|u\|_{p,a} < 1(= 1; > 1)$ if and only if $\rho(u) < 1(= 1; > 1)$,*

(3) *if $\|u\|_{p,a} > 1$, then $\|u\|^{p^-}_{p,a} \le \rho(u) \le \|u\|^{p^+}_{p,a}$,*

(4) *if $\|u\|_{p,a} < 1$, then $\|u\|^{p^+}_{p,a} \le \rho(u) \le \|u\|^{p^-}_{p,a}$,*

(5) *$\lim_{k\to\infty}\|u_k\|_{p,a} = 0$ if and only if $\lim_{k\to\infty}\rho(u_k) = 0$,*

(6) *$\|u_k\|_{p,a} \to \infty$ if and only if $\rho(u_k) \to \infty$.*

Theorem 2.11. *Assume that the boundary of Ω possesses the cone property and $p(x) \in C(\overline{\Omega})$. Suppose that $0 < a(x) \in L^{r(x)}(\Omega)$, and $1 \ll r(x) \in C(\overline{\Omega})$, for $x \in \Omega$. If $q(x) \in C(\overline{\Omega})$ and $1 \le q(x) < ((r(x) - 1)/r(x))p^*(x)$, for all $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\Omega)$.*

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ be a measurable subset. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caracheodory function and satisfies*

$$|g(x,u)| \le a(x) + b|u|^{p_1(x)/p_2(x)} \quad \text{for any } x \in \Omega,\ t \in \mathbb{R}, \tag{2.12}$$

where $p_i(x) \ge 1$, $i = 1,2$, $a(x) \in L^{p_2(x)}(\Omega)$, $a(x) \ge 0$, $b \ge 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_g u)(x) = g(x,u(x))$ is a continuous and bounded operator.

3. Existence and Multiplicity of Solutions

Let

$$\begin{gathered} I(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) - \lambda\frac{|u|^{\alpha(x)}}{\alpha(x)} - F(x,u)dx, \\ F(x,t) = \int_0^t f(x,s)ds, \qquad G(x,t) = \int_0^t g(x,s)ds. \end{gathered} \tag{3.1}$$

The critical points of $I(u)$, that is,

$$0 = I'(u)\varphi = \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla \varphi + |u|^{p(x)-2}u\varphi - \lambda |u|^{\alpha(x)-2}u\varphi - f(x,u)\varphi \, dx \tag{3.2}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, are weak solutions of problem (1.1). So we need only to consider the existence of nontrivial critical points of $I(u)$.

Denote by c, c_i, C, C_i, K, and K_i the generic positive constants. Denote by $|\Omega|$ the Lebesgue measure of Ω.

To study the existence of solutions for problem (1.1) in the first case, we additionally impose the following conditions.

(A-1) $\alpha(x), q(x) \in C(\overline{\Omega})$ and $p(x) \ll \alpha(x) < p^*(x)$, $p(x) \ll q(x) < ((r(x)-1)/r(x))p^*(x)$.

(B-1) There exists a function $\mu(x) \in C^1(\overline{\Omega})$, such that $p(x) \ll \mu(x)$ and $0 < \mu(x)G(x,t) \le tg(x,t)$ for $x \in \Omega$, $|t| \ge M$.

(C-1) there exist $\delta > 0$ such that $|g(x,t)| \le c_3|t|^{\tilde{q}(x)-1}$ for $x \in \Omega$, $|t| \le \delta$, where $\tilde{q}(x) \in C(\overline{\Omega})$ and $p(x) \ll \tilde{q}(x) < ((r(x)-1)/r(x))p^*(x)$.

(D-1) $g(x,-t) = -g(x,t)$, for all $x \in \Omega$, $t \in \mathbb{R}$.

Theorem 3.1. *Under assumptions (F) and (A-1)–(C-1), problem* (1.1) *admits a nontrivial solution.*

Proof. First we show that any $(PS)_c$ sequence is bounded. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ and $c \in \mathbb{R}$, such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $W^{-1,p'(x)}(\Omega)$. By (A-1) and (B-1), $\inf\{1/p(x) - 1/\alpha(x)\} = a_1 > 0$, and $\inf\{1/p(x) - 1/\mu(x)\} = a_2 > 0$. Let $a_0 = \min\{a_1, a_2\}$ and $l(x) = 1/p(x) - a_0/2$, then $l(x) > 1/\alpha(x)$, $l(x) > 1/\mu(x)$, $|\nabla l(x)| < C$, $1/p(x) - l(x) = a_0/2$ and $l(x) - 1/\alpha(x) \ge a_0/2$. Let $\Omega_1 = \{x \in \Omega : |u(x)| \ge M\}$ and $\Omega_2 = \Omega \setminus \Omega_1$. Then

$$\begin{aligned} I(u_n) &- \langle I'(u_n), l(x)u_n \rangle \\ &= \int_{\Omega} \left(\frac{1}{p(x)} - l(x)\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx + \int_{\Omega} \lambda \left(l(x) - \frac{1}{\alpha(x)}\right)|u_n|^{\alpha(x)}dx \\ &\quad + \int_{\Omega_1} l(x)f(x,u_n)u_n - F(x,u_n)dx - \int_{\Omega_2} F(x,u_n)dx \\ &\quad + \int_{\Omega_2} l(x)f(x,u_n)u_n dx - \int_{\Omega} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla l \cdot u_n \, dx. \end{aligned} \tag{3.3}$$

By (B-1), we get

$$\int_{\Omega_1} l(x)f(x,u_n)u_n - F(x,u_n)dx \ge \int_{\Omega_1} \left(l(x) - \frac{1}{\mu(x)}\right) f(x,u_n)u_n \, dx > 0. \tag{3.4}$$

By (F), we get $g(x,t) \in C(\overline{\Omega} \times [-M, M])$, so there exist $K > 0$ such that $|g(x,t)| \leq K$ on $\overline{\Omega} \times [-M, M]$. Note $f(x,t) = b(x)g(x,t)$, so we have

$$\begin{aligned} \left| \int_{\Omega_2} F(x, u_n) dx \right| &\leq \int_{\Omega_2} MKb(x) dx < \infty, \\ \left| \int_{\Omega_2} f(x, u_n) u_n \, dx \right| &\leq \int_{\Omega_2} MKb(x) dx < \infty. \end{aligned} \tag{3.5}$$

By Young's inequality, for $\varepsilon_1 \in (0,1)$, we get

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla l \cdot u_n \, dx \leq C\varepsilon_1 \int_{\Omega} |\nabla u_n|^{p(x)} dx + C\varepsilon_1^{1-p^+} \int_{\Omega} |u_n|^{p(x)} dx. \tag{3.6}$$

Take ε_1 sufficiently small so that $a_0/2 > C\varepsilon_1$.

Note that $p(x) \ll \alpha(x)$, by Young's inequality again and for $\varepsilon_2 \in (0,1)$, we get

$$\begin{aligned} &\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla l \cdot u_n \, dx \\ &\quad \leq C\varepsilon_1 \int_{\Omega} |\nabla u_n|^{p(x)} dx + C\varepsilon_1^{1-p^+} \varepsilon_2 \int_{\Omega} |u_n|^{\alpha(x)} dx + C\varepsilon_1^{1-p^+} \varepsilon_2^{-p^+/(\alpha-p)^-} |\Omega|. \end{aligned} \tag{3.7}$$

Take ε_2 sufficiently small so that $\lambda(a_0/2) \geq C\varepsilon_1^{1-p^+}\varepsilon_2$.

From the above remark, we have

$$I(u_n) - \left\langle I'(u_n), l(x) u_n \right\rangle \geq \int_{\Omega} \left(\frac{a_0}{2} - C\varepsilon_1 \right) \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) dx - C_1. \tag{3.8}$$

As $\|l(x)u_n\|_p \leq l^+\|u_n\|_p$, $\|l(x)\nabla u_n\|_p \leq l^+\|\nabla u_n\|_p$ and $\|\nabla l(x) \cdot u_n\|_p \leq C\|u_n\|_p$, we have $\|l(x)u_n\|_{1,p} \leq 4Cl^+\|u_n\|_{1,p}$. Since $I(u_n) \to c$ and $I'(u_n) \to 0$, by Theorem 2.6 we have

$$c + o(1) + 4Cl^+\|u_n\|_{1,p} \geq \left(\frac{a_0}{2} - C\varepsilon_1 \right) \|u_n\|_{1,p}^{p^-} - C_1, \tag{3.9}$$

when $\|u_n\|_{1,p} \geq 1$ and n is sufficiently large. Then it is easy to see that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Next we show that $\{u_n\}$ possesses a convergent subsequence (still denoted by $\{u_n\}$).

Note that

$$\begin{aligned}
&\int_{\Omega}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)(\nabla u_n-\nabla u)dx\\
&\quad\le\left|\langle I'(u_n),u_n-u\rangle\right|+\left|\langle I'(u),u_n-u\rangle\right|+\int_{\Omega}\left|\lambda\Big(|u_n|^{\alpha(x)-2}u_n-|u|^{\alpha(x)-2}\,u\Big)(u_n-u)\right|dx\\
&\qquad+\int_{\Omega}\left|(f(x,u_n)-f(x,u))(u_n-u)\right|dx\\
&\quad=I_1+I_2+I_3+I_4.
\end{aligned}\tag{3.10}$$

Because $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, there exists a subsequence $\{u_n\}$ (still denoted by $\{u_n\}$), such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. By Theorem 2.11, there are compact embeddings $W^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}_{b(x)}(\Omega)$, then $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ and $L^{q(x)}_{b(x)}(\Omega)$. So we get

$$\begin{aligned}
I_3&=\int_{\Omega}\left|\lambda\Big(|u_n|^{\alpha(x)-2}u_n-|u|^{\alpha(x)-2}u\Big)(u_n-u)\right|dx\\
&\le 2\lambda\left\||u_n|^{\alpha(x)-1}\right\|_{\alpha'}\|u_n-u\|_{\alpha}+2\lambda\left\||u|^{\alpha(x)-1}\right\|_{\alpha'}\|u_n-u\|_{\alpha}.
\end{aligned}\tag{3.11}$$

Hence $I_3 \to 0$ as $n\to\infty$.

By (F), we have

$$\begin{aligned}
\int_{\Omega}\left|f(x,u)(u_n-u)\right|dx&\le\int_{\Omega_2}C_2b(x)|u_n-u|dx+\int_{\Omega_1}b(x)|u_n-u||u|^{q(x)-1}dx\\
&\le\int_{\Omega}C_2b(x)|u_n-u|dx+\int_{\Omega}b(x)|u_n-u||u|^{q(x)-1}dx,
\end{aligned}\tag{3.12}$$

and similarly for every n,

$$\int_{\Omega}\left|f(x,u_n)(u_n-u)\right|dx\le\int_{\Omega}C_2b(x)|u_n-u|dx+\int_{\Omega}b(x)|u_n-u||u_n|^{q(x)-1}dx.\tag{3.13}$$

Since

$$\begin{aligned}
\int_{\Omega}b(x)|u_n-u||u|^{q(x)-1}dx&=\int_{\Omega}b(x)^{1/q(x)}b(x)^{1/q'(x)}|u_n-u||u|^{q(x)-1}dx\\
&\le 2\left\|b(x)^{1/q'(x)}|u|^{q(x)-1}\right\|_{q'}\left\|b(x)^{1/q(x)}|u_n-u|\right\|_{q}
\end{aligned}\tag{3.14}$$

and $u_n \to u$ in $L^1_{b(x)}(\Omega)$ and $L^{q(x)}_{b(x)}(\Omega)$, we obtain $\int_\Omega b(x)|u_n - u|dx \to 0$ and $\int_\Omega b(x)|u_n - u||u|^{q(x)-1}dx \to 0$. Similarly,

$$\int_\Omega b(x)|u_n - u||u_n|^{q(x)-1}dx \le 2\left\| b(x)^{1/q'(x)}|u_n|^{q(x)-1}\right\|_{q'}\left\| b(x)^{1/q(x)}|u_n - u|\right\|_{q}. \tag{3.15}$$

Because $\|b(x)^{1/q'(x)}|u_n|^{q(x)-1}\|_{q'}$ is bounded, we get $\int_\Omega b(x)|u_n - u||u_n|^{q(x)-1}dx \to 0$, as $n \to \infty$. From the above remark, we conclude $I_4 = \int_\Omega |(f(x,u_n) - f(x,u))(u_n - u)|dx \to 0$, as $n \to \infty$.

Thus $I_1+I_2+I_3+I_4 \to 0$, as $n \to \infty$. Then we get $\int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \to 0$. As in the proof of Theorem 3.1 in [6, 7], we divide Ω into the following two parts:

$$\Omega_3 = \{x \in \Omega : 1 < p(x) < 2\}, \qquad \Omega_4 = \{x \in \Omega : p(x) \ge 2\}. \tag{3.16}$$

On Ω_3, we have

$$\begin{aligned}
\int_{\Omega_3} |\nabla u_n - \nabla u|^{p(x)}dx &\le \int_{\Omega_3} K_1\left(\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\right)^{p(x)/2} \\
&\quad\times \left(|\nabla u_n|^{p(x)} + |\nabla u|^{p(x)}\right)^{(2-p(x))/2}dx \\
&\le 2K_1\left|\left(\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\right)^{p(x)/2}\right|_{2/p(x),\Omega_3} \\
&\quad\times \left|\left(|\nabla u_n|^{p(x)} + |\nabla u|^{p(x)}\right)^{(2-p(x))/2}\right|_{2/(2-p(x)),\Omega_3}.
\end{aligned} \tag{3.17}$$

Then $\int_{\Omega_3} |\nabla u_n - \nabla u|^{p(x)}dx \to 0$, as $n \to \infty$.

On Ω_4, we have

$$\int_{\Omega_4} |\nabla u_n - \nabla u|^{p(x)}dx \le K_2\int_{\Omega_4}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)dx, \tag{3.18}$$

so $\int_{\Omega_4} |\nabla u_n - \nabla u|^{p(x)}dx \to 0$, as $n \to \infty$.

Thus we get $\int_\Omega |\nabla u_n - \nabla u|^{p(x)}dx \to 0$. Then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

From (F) and (B-1) we have $G(x,t) \ge c_4|t|^{\mu(x)} - c_5$, for all $x \in \Omega$, $t \in \mathbb{R}$. So we get $F(x,t) \ge c_4 b(x)|t|^{\mu(x)} - c_5 b(x)$, for all $x \in \Omega, t \in \mathbb{R}$. for all $x_0 \in \Omega$, take $\varepsilon = (1/6)(\alpha(x_0) - p(x_0))$, then $p(x_0) < \alpha(x_0) - 3\varepsilon$. Since $p(x)$ and $\alpha(x) \in C(\overline{\Omega})$, there exists $\delta > 0$ such that $|p(x) - p(x_0)| <$

ε, and $|\alpha(x)-\alpha(x_0)|<\varepsilon$, for $x\in B(x_0,\delta)$. Let $\varphi\in C_0^\infty(B(x_0,\delta))$, such that $\varphi(x)=1$ for $x\in B(x_0,\delta/2)$, $\varphi(x)=0$ for $x\in\Omega\setminus B(x_0,\delta)$, and $0\le\varphi(x)\le 1$ in $B(x_0,\delta)$. Then we have

$$\begin{aligned} I(s\varphi) &\le \int_\Omega s^{p(x)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)}+\frac{|\varphi|^{p(x)}}{p(x)}\right)-\lambda s^{\alpha(x)}\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}-c_4b(x)s^{\mu(x)}|\varphi|^{\mu(x)}+c_5b(x)dx \\ &\le \int_{B(x_0,\delta)} s^{p(x)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)}+\frac{|\varphi|^{p(x)}}{p(x)}\right)-\lambda s^{\alpha(x)}\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}dx+c_5\int_{B(x_0,\delta)}b(x)dx \\ &\le s^{p_B^+}\int_{B(x_0,\delta)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)}+\frac{|\varphi|^{p(x)}}{p(x)}\right)dx-s^{\alpha_B^-}\int_{B(x_0,\delta)}\lambda\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}dx \\ &\quad + c_5\int_{B(x_0,\delta)}b(x)dx, \end{aligned} \tag{3.19}$$

where $p_B^+<\alpha_B^-$. So if s is sufficiently large, we obtain $I(s\varphi)<0$.

From (F) and (C-1), we have $|G(x,t)|\le c_3|t|^{\tilde{q}(x)}+c(\delta)|t|^{q(x)}$, then $|F(x,t)|\le c_3b(x)|t|^{\tilde{q}(x)}+c(\delta)b(x)|t|^{q(x)}$. So we get

$$I(u)\ge\frac{1}{p^+}\int_\Omega\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx-\frac{\lambda}{\alpha^-}\int_\Omega|u|^{\alpha(x)}-c_3b(x)|u|^{\tilde{q}(x)}-c(\delta)b(x)|u|^{q(x)}dx. \tag{3.20}$$

Let $\theta(x)=\min\{\alpha(x),\tilde{q}(x),q(x)\}$, then $\theta(x)\in C(\overline{\Omega})$. By Theorem 2.11, $\|u\|_\alpha\le c_6\|u\|_{1,p}$, $\|u\|_{\tilde{q},b}\le c_7\|u\|_{1,p}$, and $\|u\|_{q,b}\le c_8\|u\|_{1,p}$. When $\|u\|_{1,p}$ is sufficiently small, $\|u\|_\alpha<1$, $\|u\|_{\tilde{q},b}<1$ and $\|u\|_{q,b}<1$. For any $x\in\overline{\Omega}$, as $p(x),\theta(x)\in C(\overline{\Omega})$, for any $\varepsilon>0$, we can find $Q_R(x)=\{y=(y^1,\dots,y^n):|y^k-x^k|<R,\ k=1,\dots,n\}$ such that $|p(y)-p(x)|<\varepsilon$ and $|\theta(y)-\theta(x)|<\varepsilon$ whenever $y\in Q_R(x)\cap\overline{\Omega}$. Take $\varepsilon=(1/4)(\theta(x)-p(x))$, then $\sup_{y\in Q_R(x)\cap\overline{\Omega}}p(y)<\inf_{y\in Q_R(x)\cap\overline{\Omega}}\theta(y)$. $\{Q_R(x)\}_{x\in\overline{\Omega}}$ is an open covering of $\overline{\Omega}$. As $\overline{\Omega}$ is compact, we can pick a finite subcovering $\{Q_{R_k}(x_k)\}_{k=1}^m$ for $\overline{\Omega}$ from the covering $\{Q_R(x)\}_{x\in\overline{\Omega}}$. If $Q_{R_k}(x_k)\not\subset\Omega$ we define $u=0$ on $Q_{R_k}(x_k)\setminus\Omega$. We can use all the hyperplanes, for each of which there exists at least one hypersurface of some $Q_{R_k}(x_k)$ lying on it, to divide $\bigcup_{k=1}^m Q_{R_k}(x_k)$ into finite open hypercubes $\{Q_i\}_{i=1}^Q$ which mutually have no common points. It is obvious that $\overline{\Omega}\subseteq\bigcup_{i=1}^Q\overline{Q_i}$ and for each Q_i there exists at least one $Q_{R_k}(x_k)$ such that $Q_i\subseteq Q_{R_k}(x_k)$. Let $\Omega_i=Q_i\cap\Omega$, then $p_i^+=\sup_{y\in\Omega_i}p(y)<\inf_{y\in\Omega_i}\theta(y)=\theta_i^-$ and we have

$$\begin{aligned} I(u) &\ge \sum_{i=1}^Q\left(\frac{1}{p^+}\int_{\Omega_i}\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx-\frac{\lambda}{\alpha^-}\int_{\Omega_i}|u|^{\alpha(x)}-c_3b(x)|u|^{\tilde{q}(x)}-c(\delta)b(x)|u|^{q(x)}dx\right) \\ &\ge \sum_{i=1}^Q\left(\frac{1}{p^+}\|u\|_{1,p,\Omega_i}^{p_i^+}-\frac{\lambda}{\alpha^-}c_6^{\alpha_i^-}\|u\|_{1,p,\Omega_i}^{\alpha_i^-}-c_3c_7^{\tilde{q}_i^-}\|u\|_{1,p,\Omega_i}^{\tilde{q}_i^-}-c(\delta)c_8^{q_i^-}\|u\|_{1,p,\Omega_i}^{q_i^-}\right). \end{aligned} \tag{3.21}$$

If $\|u\|_{1,p} = k \geq \|u\|_{1,p,\Omega_i}$ is sufficiently small such that

$$\frac{1}{p^+}\|u\|_{1,p,\Omega_i}^{p_i^+} - \frac{\lambda}{\alpha^-}c_6^{\alpha_i^-}\|u\|_{1,p,\Omega_i}^{\alpha_i^-} - c_3 c_7^{\widetilde{q}_i^-}\|u\|_{1,p,\Omega_i}^{\widetilde{q}_i^-} - c(\delta)c_8^{q_i^-}\|u\|_{1,p,\Omega_i}^{q_i^-} > 0, \tag{3.22}$$

we have $I(u) > 0$.

The mountain pass theorem guarantees that I has a nontrivial critical point u. □

Since $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^{\infty} \subset W_0^{1,p(x)}(\Omega)$ and $\{e_n^*\}_{n=1}^{\infty} \subset W^{-1,p'(x)}(\Omega)$ such that

$$\begin{gathered}\left\langle e_i, e_j^* \right\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \\ W_0^{1,p(x)}(\Omega) = \overline{\operatorname{span}\{e_n : n = 1, 2, \ldots\}}, \\ W^{-1,p'(x)}(\Omega) = \overline{\operatorname{span}\{e_n^* : n = 1, 2, \ldots\}}.\end{gathered} \tag{3.23}$$

For $k = 1, 2, \ldots$, denote $X_k = \operatorname{span}\{e_k\}$, $Y_k = \oplus_{j=1}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Theorem 3.2. *Under assumptions (F), (A-1)–(D-1), problem* (1.1) *admits a sequence of solutions* $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ *such that* $I(u_n) \to \infty$.

Proof. Let $\varphi(u) = \int_{\Omega} (\lambda |u|^{\alpha(x)}/\alpha(x)) + F(x,u)dx$. We first show that $\varphi(u)$ is weakly strongly continuous. Let $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. By the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, we have $\int_{\Omega} |u_n - u|^{\alpha(x)}dx \to 0$ and $u_n \to u$ a.e. on Ω. By the inequality $|u_n|^{\alpha(x)} \leq 2^{\alpha^+-1}(|u_n - u|^{\alpha(x)} + |u|^{\alpha(x)})$ and the Vitali Theorem, we get $\int_{\Omega} |u_n|^{\alpha(x)}dx \to \int_{\Omega} |u|^{\alpha(x)}dx$.

Note that

$$\begin{aligned}\int_{\Omega} |F(x,u_n) - F(x,u)|dx &= \int_{\Omega} b(x)|G(x,u_n) - G(x,u)|dx \\ &\leq 2\|b(x)\|_r \|G(x,u_n) - G(x,u)\|_{r'}.\end{aligned} \tag{3.24}$$

When $|u| \geq M$,

$$\begin{aligned}|G(x,u)| &\leq c_1 \frac{|u|}{\mu(x)} + c_2 \frac{|u|^{q(x)}}{\mu(x)} \leq \frac{c_1^{q'(x)}}{q'(x)} + \frac{|u|^{q(x)}}{\mu(x)^{q(x)} q(x)} + c_2 \frac{|u|^{q(x)}}{\mu(x)} \\ &\leq C_3\left(1 + |u|^{q(x)}\right) = C_3\left(1 + |u|^{r'(x)q(x)/r'(x)}\right).\end{aligned} \tag{3.25}$$

When $|u| \leq M$, $G(x,u)$ is bounded. So we get

$$|G(x,u)| \leq C_4\left(1 + |u|^{r'(x)q(x)/r'(x)}\right). \tag{3.26}$$

By the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r'(x)q(x)}(\Omega)$, we get $u_n \to u$ in $L^{r'(x)q(x)}(\Omega)$. So by Theorem 2.12 we obtain $|G(x,u_n) - G(x,u)|_{r'} \to 0$, that is, $\int_\Omega |F(x,u_n) - F(x,u)|dx \to 0$. Hence we obtain that $\varphi(u)$ is weakly strongly continuous. By Proposition 3.5 in [8], $\beta_k = \beta_k(r) = \sup_{u\in Z_k, \|u\|_{1,p}\le r}|\varphi(u)| \to 0$ as $k \to \infty$ for $r > 0$. For all n, there exists a positive integer k_n such that $\beta_k(n) \le 1$ for all $k \ge k_n$. Assume $k_n < k_{n+1}$ for each n. Define $\{r_k : k = 1,2,\ldots\}$ in the following way:

$$r_k = \begin{cases} n, & k_n \le k < k_n + 1, \\ 1, & 1 \le k < k_1. \end{cases} \tag{3.27}$$

Note that $r_k \to \infty$ as $k \to \infty$. Hence for $u \in Z_k$ with $\|u\|_{1,p} = r_k$, we get

$$I(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \varphi(u) \ge \frac{1}{p^+}\|u\|_{1,p}^{p^-} - 1. \tag{3.28}$$

So

$$\inf_{u\in Z_k, \|u\|_{1,p}=r_k} I(u) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{3.29}$$

Note that $F(x,u) = b(x)G(x,u) \ge c_4 b(x)|u|^{\mu(x)} - c_5 b(x)$ and $p(x), \alpha(x) \in C(\overline{\Omega})$. Since the dimension of Y_k is finite, any two norms on Y_k are equivalent, then $c_9|u|_{1,p} \le |u|_\alpha \le c_{10}|u|_{1,p}$. If $|u|_{1,p} \le 1$, it is immediate that $|u|_\alpha \le c_{10}$. If $|u|_{1,p} > 1/c_9$, then $|u|_\alpha > 1$. As in the proof of Theorem 3.1 we can find hypercubes $\{Q_i\}_{i=1}^Q$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^Q \overline{Q_i}$ and $p_i^+ = \sup_{y\in\Omega_i} p(y) < \inf_{y\in\Omega_i}\alpha(y) = \alpha_i^-$, where $\Omega_i = Q_i \cap \Omega$. Then we need only to consider the case: $|u|_{1,p,\Omega_i} > \max\{1, 1/c_9\}$ for every i. We have

$$\begin{aligned} I(u) &\le \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\right) - \lambda\frac{|u|^{\alpha(x)}}{\alpha(x)} - c_4 b(x)|u|^{\mu(x)} + c_5 b(x)dx \\ &\le \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\right) - \lambda\frac{|u|^{\alpha(x)}}{\alpha(x)}dx + c_5\int_\Omega b(x)dx \\ &\le \sum_{i=1}^Q \left(\frac{1}{p^-}\|u\|_{1,p,\Omega_i}^{p_i^+} - \frac{\lambda}{\alpha^+}c_9^{\alpha_i^-}\|u\|_{1,p,\Omega_i}^{\alpha_i^-}\right) + c_5\int_\Omega b(x)dx \\ &= \sum_{i=1}^Q \frac{\lambda}{\alpha^+}c_9^{\alpha_i^-}\left(\frac{\alpha^-}{p^-\lambda c_9^{\alpha_i^-}}\|u\|_{1,p,\Omega_i}^{p_i^+} - \|u\|_{1,p,\Omega_i}^{\alpha_i^-}\right) + c_5\int_\Omega b(x)dx. \end{aligned} \tag{3.30}$$

Let $c_0 = \alpha^+/p^-\lambda c_9^{\alpha_i^-}$, $f_i(t) = t^{\alpha_i^-} - c_0 t^{p_i^+}$. Let $f_i(s_i) = \min_{t\ge0} f_i(t) < 0$ and $f_i(t_i) = 0$. Denote $t = \sum_{i=1}^Q \|u\|_{1,p,\Omega_i}$. Let t be sufficiently large such that $t > \max\{Q, Qt_i, Q(2c_0)^{1/(\alpha_i^- - p_i^+)},$

$i = 1, 2, \ldots, Q\}$. There at least exists one i_0 such that $\|u\|_{1,p,\Omega_{i_0}} \geq (1/Q)\sum_{i=1}^{Q}\|u\|_{1,p,\Omega_i} = (t/Q)$. We have

$$\begin{aligned}
\sum_{i=1}^{Q} f_i\left(\|u\|_{1,p,\Omega_i}\right) &= \sum_{i=1}^{Q}\|u\|_{1,p,\Omega_i}^{p_i^+}\left(\|u\|_{1,p,\Omega_i}^{\alpha_i^- - p_i^+} - c_0\right) \\
&= \sum_{i \neq i_0}\|u\|_{1,p,\Omega_i}^{p_i^+}\left(\|u\|_{1,p,\Omega_i}^{\alpha_i^- - p_i^+} - c_0\right) + \|u\|_{1,p,\Omega_{i_0}}^{p_{i_0}^+}\left(\|u\|_{1,p,\Omega_{i_0}}^{\alpha_{i_0}^- - p_{i_0}^+} - c_0\right) \\
&\geq \sum_{i \neq i_0} f_i(s_i) + \left(\frac{t}{Q}\right)^{p_{i_0}^+}\left(\left(\frac{t}{Q}\right)^{\alpha_{i_0}^- - p_{i_0}^+} - c_0\right) \\
&\geq \sum_{i=1}^{Q} f_i(s_i) + \frac{c_0}{Q}t,
\end{aligned} \tag{3.31}$$

and $\sum_{i=1}^{Q} f_i(\|u\|_{1,p,\Omega_i}) \to \infty$ as $t \to \infty$. Hence we obtain that $I(u) \to -\infty$ as $\|u\|_{1,p} \to \infty$. Thus for each k, there exists $\rho_k > r_k$ such that $I(u) < 0$ for $u \in Y_K \cap S_{\rho_k}$. From Theorem 3.1 $I(u)$ satisfies $(PS)_c$ condition. In view of (D-1), by Fountain Theorem (see [16]), we conclude the result. □

In the second case, we additionally impose the following condition:

(A-2) $\alpha(x), q(x) \in C(\overline{\Omega})$ and $1 \leq \alpha(x) \ll p(x) \ll q(x) < ((r(x)-1)/r(x))p^*(x)$.

Theorem 3.3. *Under assumptions (F), (A-2), (B-1), and (C-1) there exist $\lambda^* > 0$ such that when $\lambda \in (0, \lambda^*)$, problem* (1.1) *admits a nontrivial solution.*

Proof. It is obvious that $\alpha^- < p^-$. Let $\varepsilon_0 > 0$ be such that $\alpha^- + \varepsilon_0 < p^-$. Since $\alpha(x) \in C(\overline{\Omega})$, there exists $\delta > 0$ and $x_0 \in \Omega$ such that $|\alpha(x) - \alpha^-| < \varepsilon_0$, for all $x \in B(x_0, \delta)$. Thus $\alpha(x) \leq \alpha^- + \varepsilon_0$ for all $x \in B(x_0, \delta)$. Let φ be as defined in Theorem 3.1. By (C-1), $|G(x,t)| \leq c_3|t|^{\tilde{q}(x)}$, and $|F(x,t)| \leq c_3 b(x)|t|^{\tilde{q}(x)}$, when $t < \delta$. Then for any $t \in (0,1)$ and $t < \delta$, we have

$$\begin{aligned}
I(t\varphi) &= \int_{\Omega} t^{p(x)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)} + \frac{|\varphi|^{p(x)}}{p(x)}\right) - \lambda t^{\alpha(x)}\frac{|\varphi|^{\alpha(x)}}{\alpha(x)} - F(x, t\varphi)dx \\
&\leq t^{p^-}\int_{B(x_0,\delta)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)} + \frac{|\varphi|^{p(x)}}{p(x)}\right)dx - t^{\alpha^+}\int_{B(x_0,\delta)}\lambda\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}dx \\
&\quad + t^{\tilde{q}^-}\int_{B(x_0,\delta)} c_3 b(x)|\varphi|^{\tilde{q}(x)}dx \\
&\leq t^{p^-}\int_{B(x_0,\delta)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)} + \frac{|\varphi|^{p(x)}}{p(x)}\right)dx - t^{\alpha^- + \varepsilon_0}\int_{B(x_0,\delta)}\lambda\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}dx \\
&\quad + t^{\tilde{q}^-}\int_{B(x_0,\delta)} c_3 b(x)|\varphi|^{\tilde{q}(x)}dx.
\end{aligned} \tag{3.32}$$

If t is sufficiently small, $I(t\varphi) < 0$.

From (F) and (C-1), we have $|G(x,t)| \le c_3|t|^{\tilde{q}(x)} + c(\delta)|t|^{q(x)}$ and $|F(x,t)| \le c_3 b(x)|t|^{\tilde{q}(x)} + c(\delta)b(x)|t|^{q(x)}$. By Theorems 2.8 and 2.11, there exist positive constants $k_1,\ k_2,\ k_3$ such that $|u|_\alpha \le k_1|u|_{1,p}$, $|u|_{q,b} \le k_2|u|_{1,p}$, $|u|_{\tilde{q},b} \le k_3|u|_{1,p}$. When $\|u\|_{1,p}$ is sufficiently small, we have $\|u\|_\alpha < 1$, $\|u\|_{\tilde{q},b} < 1$, and $\|u\|_{q,b} < 1$. As in the proof of Theorem 3.1 we can find hypercubes $\{Q_i\}_{i=1}^{Q}$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^{Q} \overline{Q_i}$, $p_i^+ = \sup_{y\in\Omega_i} p(y) < \inf_{y\in\Omega_i} q(y) = q_i^-$ and $p_i^+ < \tilde{q}_i^-$, where $\Omega_i = Q_i \cap \Omega$. Then

$$I(u) \ge \sum_{i=1}^{Q}\left(\int_{\Omega_i} \frac{1}{p^+}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) - c_3 b(x)|u|^{\tilde{q}(x)} - c(\delta)b(x)|u|^{q(x)}dx\right) - \frac{\lambda}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx$$
$$\ge \sum_{i=1}^{Q}\left(\frac{1}{p^+}\|u\|_{1,p,\Omega_i}^{p_i^+} - c_3 k_3^{\tilde{q}_i^-}\|u\|_{1,p,\Omega_i}^{\tilde{q}_i^-} - c(\delta)k_2^{q_i^-}\|u\|_{1,p,\Omega_i}^{q_i^-}\right) - \frac{\lambda}{\alpha^-}k_1^{\alpha^-}\|u\|_{1,p}^{\alpha^-}. \tag{3.33}$$

Since $\|u\|_{1,p} \ge \|u\|_{1,p,\Omega_i}$, for all i, when $\|u\|_{1,p} = \rho$, $\|u\|_{1,p,\Omega_i} = \rho_i < \rho$. Fix ρ such that $(1/p^+)\rho_i^{p_i^+} - c_3 k_3^{\tilde{q}_i^-}\rho_i^{\tilde{q}_i^-} - c(\delta)k_2^{q_i^-}\rho_i^{q_i^-} > 0$. Then we have

$$I(u) \ge \sum_{i=1}^{N}\left(\frac{1}{p^+}\rho_i^{p_i^+} - c_3 k_3^{\tilde{q}_i^-}\rho_i^{\tilde{q}_i^-} - c(\delta)k_2^{q_i^-}\rho_i^{q_i^-}\right) - \frac{\lambda}{\alpha^-}k_1^{\alpha^-}\rho^{\alpha^-} = l_1(\rho) - \lambda l_2(\rho). \tag{3.34}$$

Let $\lambda^* = l_1(\rho)/2l_2(\rho)$. When $\lambda \in (0,\lambda^*)$, $I(u) \ge l_1/2 > 0$. As in the proof of Theorem 2.1 in [17], denote $B_\rho(0) = \{u \in W_0^{1,p(x)}(\Omega) : \|u\|_{1,p} < \rho\}$, we have $\inf_{\partial B_\rho(0)} I(u) > 0$ and $-\infty < c = \inf_{\overline{B_\rho(0)}} I(u) < 0$. Let $0 < \varepsilon < \inf_{\partial B_\rho(0)} I(u) - \inf_{B_\rho(0)} I(u)$. Applying Ekeland's variational principle to the functional $I(u) : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that $I(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} I(u) + \varepsilon$, $I(u_\varepsilon) < I(u) + \varepsilon\|u - u_\varepsilon\|_{1,p}$, $u_\varepsilon \ne u \in \overline{B_\rho(0)}$, and $\|I'(u_\varepsilon)\|_{-1,p'} \le \varepsilon$. Thus we get a sequence $\{u_n\} \subset B_\rho(0)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$. It is clear that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. As in the proof of Theorem 3.1, we get a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. So $I(u) = c < 0$ and $I'(u) = 0$. □

Theorem 3.4. *Under assumptions (F), (A-2), and (B-1)–(D-1), problem* (1.1) *has a sequence of solutions* $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ *such that* $I(u_n) \to \infty$.

Proof. First we show that any $(PS)_c$ sequence is bounded. Let $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ and $c \in \mathbb{R}$, such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $W^{-1,p'(x)}(\Omega)$. By (B-1), there exist $\sigma > 0$ such that $\inf\{1/p(x) - (1+\sigma)/\mu(x)\} = \mu_1 > 0$. From (F), (A-2), and (B-1)–(D-1), we have

$$\int_{\Omega_1} \frac{1}{\mu(x)} f(x,u_n)u_n - F(x,u_n)dx > 0,$$
$$\int_{\Omega_2} \left|\frac{1}{\mu(x)} f(x,u_n)u_n - F(x,u_n)\right| dx < C_1, \tag{3.35}$$
$$\int_\Omega \frac{1}{\mu(x)} f(x,u_n)u_n dx + C_2 \ge \int_\Omega F(x,u_n)dx \ge \int_\Omega c_4 b(x)|u_n|^{\mu(x)}dx - \int_\Omega c_5 b(x)dx.$$

Thus we have

$$\begin{aligned}
I(u_n) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) - \lambda\frac{|u_n|^{\alpha(x)}}{\alpha(x)} - F(x,u_n)dx \\
&= \int_\Omega \left(\frac{1}{p(x)} - \frac{1+\sigma}{\mu(x)}\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) - \lambda\frac{|u_n|^{\alpha(x)}}{\alpha(x)} - F(x,u_n)dx \\
&\quad + \int_\Omega \frac{1+\sigma}{\mu(x)}\lambda|u_n|^{\alpha(x)}dx + \int_\Omega \frac{1+\sigma}{\mu(x)}f(x,u_n)u_n + \frac{1+\sigma}{\mu(x)}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) \\
&\quad - \frac{1+\sigma}{\mu(x)}\lambda|u_n|^{\alpha(x)} - \frac{1+\sigma}{\mu(x)}f(x,u_n)u_n\,dx \\
&\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1+\sigma}{\mu(x)}\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\left(\frac{1+\sigma}{\mu(x)}\right)u_n\,dx \\
&\quad - \int_\Omega \lambda\frac{|u_n|^{\alpha(x)}}{\alpha(x)}dx + \int_\Omega \sigma c_4 b(x)|u_n|^{\mu(x)}dx - \int_\Omega \sigma c_5 b(x)dx \\
&\quad + \left\langle I'(u_n), \frac{1+\sigma}{\mu(x)}u_n\right\rangle - C_3.
\end{aligned} \tag{3.36}$$

By Young's inequality, for $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0,1)$, we get

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\left(\frac{1+\sigma}{\mu(x)}\right)\cdot u_n\,dx \\
&\quad\leq C_1\varepsilon_1 \int_\Omega |\nabla u_n|^{p(x)}dx + C_1\varepsilon_1^{1-p^+}\varepsilon_2 \int_\Omega |u_n|^{\mu(x)}dx + C_1\varepsilon_1^{1-p^+}\varepsilon_2^{-p^+/(\mu-p)^-}|\Omega|, \\
&\qquad\qquad \int_\Omega \frac{|u_n|^{\alpha(x)}}{\alpha(x)}dx \leq \varepsilon_3\int_\Omega |u_n|^{p(x)}dx + \varepsilon_3^{-\alpha^+/(p-\alpha)^-}|\Omega|.
\end{aligned} \tag{3.37}$$

Take ε_1, ε_2, and ε_3 sufficiently small so that $\mu_1 - C_1\varepsilon_1 > 0$, $\mu_1 - \lambda\varepsilon_3 \geq 0$ and $\sigma c_4 b(x) - C_1\varepsilon_1^{1-p^+}\varepsilon_2 \geq 0$, then

$$I(u_n) - \left\langle I'(u_n), \frac{1+\sigma}{\mu(x)}u_n\right\rangle \geq \int_\Omega C_4|\nabla u_n|^{p(x)}dx - C_5. \tag{3.38}$$

Therefore by Theorems 2.6 and 2.9, we get that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Then as in the proof of Theorem 3.1$\{u_n\}$ possesses a convergent subsequence $\{u_n\}$ (still denoted by $\{u_n\}$). By Theorem 3.2, we can also have

$$\inf_{u\in Z_k, \|u\|_{1,p}=r_k} I(u) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{3.39}$$

As in the proof of Theorem 3.1 we can find hypercubes $\{Q_i\}_{i=1}^{Q}$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^{Q} \overline{Q_i}$ and $p_i^+ = \sup_{y\in\Omega_i} p(y) < \inf_{y\in\Omega_i} \mu(y) = \mu_i^-$, where $\Omega_i = Q_i \cap \Omega$. Since the dimension of Y_k is finite, any two norms on Y_k are equivalent. Then we need only to consider the cases $|u|_{1,p,\Omega_i} > 1$ and $\|u\|_{\mu,b,\Omega_i} > 1$ for every i. We have

$$\begin{aligned} I(u) &\le \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\right) - \lambda \frac{|u|^{\alpha(x)}}{\alpha(x)} - c_4 b(x)|u|^{\mu(x)} + c_5 b(x)dx \\ &\le \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\right) - c_4 b(x)|u|^{\mu(x)}dx + c_5\int_\Omega b(x)dx \\ &\le \sum_{i=1}^{Q}\left(\frac{1}{p^+}\|u\|_{1,p,\Omega_i}^{p_i^+} - C_4\|u\|_{1,p,\Omega_i}^{\mu_i^-}\right) + c_5\int_\Omega b(x)dx. \end{aligned} \tag{3.40}$$

Hence $I(u) \to -\infty$ as $\|u\|_{1,p} \to \infty$. As in the proof of Theorem 3.2, we complete the proof. □

In the third case, we additionally impose the following condition:

(A-3) $\alpha(x), q(x) \in C(\overline{\Omega})$, and $1 \le \alpha(x),\ q(x) \ll p(x)$,

(B-3) there exist $\theta > 0$ such that $G(x,t) \ge 0$ for $t \in (0,\theta)$.

Theorem 3.5. *Under assumptions (F), (A-3), and (B-3), problem* (1.1) *admits a nontrivial solution.*

Proof. By Young's inequality, for $\varepsilon \in (0,1)$, we get $|u_n|^{\alpha(x)}/\alpha(x) \le \varepsilon|u_n|^{p(x)} + \varepsilon^{-\alpha^+/(p-\alpha)^-}$. By (F), we have $F(x,u) \le Cb(x)|u|^{q(x)} + C_1 b(x)$. Thus

$$\begin{aligned} I(u) &= \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\right) - \lambda \frac{|u|^{\alpha(x)}}{\alpha(x)} - F(x,u)dx \\ &\ge \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\right)dx - \lambda\varepsilon\int_\Omega |u|^{p(x)}dx \\ &\quad - C\int_\Omega b(x)|u|^{q(x)}dx - \lambda\varepsilon^{-\alpha^+/(p-\alpha)^-}|\Omega| - C_1\int_\Omega b(x)dx \\ &\ge \int_\Omega \left(\frac{1}{p^+} - \lambda\varepsilon\right)\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - C\int_\Omega b(x)|u|^{q(x)}dx - C_2. \end{aligned} \tag{3.41}$$

Take ε sufficiently small so that $1/p^+ - \lambda\varepsilon > 0$. From Theorem 2.11, $\|u\|_{q,b} \le c\|u\|_{1,p}$. If $\|u\|_{1,p} \le 1$, $\|u\|_{q,b}$ is bounded. Then we need only to consider the case $\|u\|_{1,p} > 1$. As in the proof of

Theorem 3.1 we can find hypercubes $\{Q_i\}_{i=1}^{Q}$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^{Q} \overline{Q_i}$ and $q_i^+ = \sup_{y\in\Omega_i} q(y) < \inf_{y\in\Omega_i} p(y) = p_i^-$, where $\Omega_i = Q_i \cap \Omega$. Then we have

$$\begin{aligned} I(u) &\geq \sum_{i=1}^{M}\left(c_1\|u\|_{1,p,\Omega_i}^{p_i^-} - c_2\|u\|_{1,p,\Omega_i}^{q_i^+}\right) + \sum_{j=1}^{J}\left(c_1\|u\|_{1,p,\Omega_j}^{p_j^-} - c_3\|u\|_{1,p,\Omega_j}^{q_j^-}\right) + C_3 \\ &\geq \sum_{i=1}^{M} c_1\left(\|u\|_{1,p,\Omega_i}^{p_i^-} - c_4\|u\|_{1,p,\Omega_i}^{q_i^+}\right) + C_4, \end{aligned} \tag{3.42}$$

where $\int_{\Omega_i} b(x)|u|^{q(x)}dx > 1,\ i = 1,2,\ldots M$, and $\int_{\Omega_j} b(x)|u|^{q(x)}dx \leq 1,\ i = 1,2,\ldots J$, $M + J = Q$. As in the proof of Theorem 3.2, we obtain that $I(u)$ is coercive, that is, $I(u) \to \infty$ as $\|u\|_{1,p} \to \infty$. Thus I has a critical point u such that $I(u) = \inf_{u\in W_0^{1,p(x)}(\Omega)} I(u)$ and further u is a weak solution of (1.1).

Next we show that u is nontrivial. Let $B(x_0, \delta)$ be the same as that in Theorem 3.3. By (B-3), $F(x, t\varphi) \geq 0$. Then

$$\begin{aligned} I(t\varphi) &= \int_\Omega t^{p(x)}\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)} + \frac{|\varphi|^{p(x)}}{p(x)}\right) - \lambda t^{\alpha(x)}\frac{|\varphi|^{\alpha(x)}}{\alpha(x)} - F(x,t\varphi)dx \\ &\leq t^{p^-}\int_\Omega\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)} + \frac{|\varphi|^{p(x)}}{p(x)}\right)dx - t^{\alpha^+}\int_{\Omega_0}\lambda\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}dx \\ &\leq t^{p^-}\int_\Omega\left(\frac{|\nabla\varphi|^{p(x)}}{p(x)} + \frac{|\varphi|^{p(x)}}{p(x)}\right)dx - t^{\alpha^-+\varepsilon_0}\int_{\Omega_0}\lambda\frac{|\varphi|^{\alpha(x)}}{\alpha(x)}dx. \end{aligned} \tag{3.43}$$

If t is sufficiently small, $I(t\varphi) < 0$. □

In the fourth case, we additionally impose the following condition:

(A-4) $\alpha(x), q(x) \in C(\overline{\Omega})$ and $1 \leq q(x) \ll p(x) \ll \alpha(x) < p^*(x)$.

Theorem 3.6. *Under assumptions (F), (A-4), and (D-1), problem* (1.1) *admits a sequence of solutions* $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ *such that* $I(u_n) \to \infty$.

Proof. First we show that any $(PS)_c$ sequence is bounded. Let $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ and $c \in \mathbb{R}$, such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $W^{-1,p'(x)}(\Omega)$. Denote $\inf\{1/p(x) - 1/\alpha(x)\} = a > 0$ and $l(x) = 1/p(x) - a/2$. We have $1/p(x) - l(x) = a/2$ and $l(x) - 1/\alpha(x) \geq a/2$.

We can get

$$
\begin{aligned}
& I(u_n) - \langle I'(u_n), l(x)u_n \rangle \\
& \quad = \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) - \lambda \frac{|u_n|^{\alpha(x)}}{\alpha(x)} - F(x,u_n)dx \\
& \quad = \int_\Omega \left(\frac{1}{p(x)} - l(x)\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) - \lambda \frac{|u_n|^{\alpha(x)}}{\alpha(x)} - F(x,u_n)dx \\
& \qquad + \int_\Omega l(x)\lambda |u_n|^{\alpha(x)}dx + \int_\Omega l(x)f(x,u_n)u_n - |\nabla u_n|^{p(x)-2}\nabla u_n \nabla l(x)\cdot u_n\,dx \\
& \quad \geq \int_\Omega \frac{a}{2}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla l(x)\cdot u_n\,dx \\
& \qquad + \int_\Omega \left(l(x) - \frac{1}{\alpha(x)}\right)\lambda |u_n|^{\alpha(x)} + \left(l(x)f(x,u_n)u_n - F(x,u_n)\right)dx.
\end{aligned}
\tag{3.44}
$$

By Young's inequality, for $\varepsilon_1, \varepsilon_2 \in (0,1)$, we get

$$
\begin{aligned}
& \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla l(x)\cdot u_n\,dx \\
& \qquad \leq C\varepsilon_1 \int_\Omega |\nabla u_n|^{p(x)}dx + C\varepsilon_1^{1-p^+}\varepsilon_2 \int_\Omega |u_n|^{\alpha(x)}dx + C\varepsilon_1^{1-p^+}\varepsilon_2^{-p^+/(\alpha-p)^-}|\Omega|.
\end{aligned}
\tag{3.45}
$$

Take ε_1 and ε_2 sufficiently small so that $C\varepsilon_1 < a/2$ and $C\varepsilon_1^{1-p^+}\varepsilon_2 \leq a/2$. Then

$$
\begin{aligned}
& +\infty > I(u_n) - \langle I'(u_n), l(x)u_n \rangle \\
& \quad \geq \int_\Omega \left(\frac{a}{2} - C\varepsilon_1\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) - \left(\left|l(x)f(x,u_n)u_n\right| + |F(x,u_n)|\right)dx - C_1 \\
& \quad \geq \int_\Omega \left(\frac{a}{2} - C\varepsilon_1\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - C_2 \int_\Omega b(x)|u_n|^{q(x)}dx - C_3.
\end{aligned}
\tag{3.46}
$$

As in the proof of Theorem 3.5, $I(u_n) - \langle I'(u_n), l(x)u_n \rangle \longrightarrow +\infty$, when $\|u_n\|_{1,p} \longrightarrow \infty$. Thus, we conclude that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Then as in the proof of Theorem 3.1$\{u_n\}$ possesses a convergent subsequence $\{u_n\}$ (still denoted by $\{u_n\}$). By Theorem 3.2, we can also get

$$
\inf_{u \in Z_k, \|u\|_{1,p} = r_k} I(u) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.
\tag{3.47}
$$

As in the proof of Theorem 3.1 we can find hypercubes $\{Q_i\}_{i=1}^{Q}$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^{Q} \overline{Q_i}$ and $p_i^+ = \sup_{y\in\Omega_i} p(y) < \inf_{y\in\Omega_i} \alpha(y) = \alpha_i^-$, where

$\Omega_i = Q_i \cap \Omega$. Since the dimension of Y_k is finite, any two norms on Y_k are equivalent. Then we need only to consider the cases $|u|_{1,p,\Omega_i} > 1|u|_{\alpha,\Omega_i} > 1$ and $|u|_{q,b,\Omega_i} > 1$ for every i. We have

$$\begin{aligned} I(u) &\le \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)} \right) - \lambda \frac{|u|^{\alpha(x)}}{\alpha(x)} dx + C_4 \int_\Omega b(x) |u_n|^{q(x)} dx + C_5 \\ &\le \sum_{i=1}^{Q} \left(c_0 \left(\|u\|_{1,p,\Omega_i}^{p_i^+} + \|u\|_{1,p,\Omega_i}^{q_i^+} \right) - c \|u\|_{1,p,\Omega_i}^{\alpha_i^-} \right) + C_6. \end{aligned} \tag{3.48}$$

Hence we obtain $I(u) \to -\infty$ as $\|u\|_{1,p} \to \infty$. As in the proof of Theorem 3.2, we complete the proof. □

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