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Research Article

Weak Solutions of a Stochastic Model for Two-Dimensional Second Grade Fluids

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We initiate the investigation of a stochastic system of evolution partial differential equations modelling the turbulent flows of a second grade fluid filling a bounded domain of $\mathbb{R}^2$. We establish the global existence of a probabilistic weak solution.

1. Introduction

The study of turbulence either in Newtonian flows or Non-Newtonian flows is one of the greatest unsolved and still not well-understood problem in contemporary applied sciences. For indepth coverage of the deep and fascinating investigations undertaken in this field, the abundant wealth of results obtained, and remarkable advances achieved we refer to the monographs in [1–4] and references therein. The hypothesis relating the turbulence to the "randomness of the background field" is one of the motivations of the study of stochastic version of equations governing the motion of fluids flows. The introduction of random external forces of noise type reflects (small) irregularities that give birth to a new random phenomenon, making the problem more realistic. Such approach in the mathematical investigation for the understanding of the turbulence phenomenonwas pioneered by Bensoussan and Temam in [5] where they studied the Stochastic Navier-Stokes Equation (SNSE) excited by random forces. Since then, stochastic partial differential equations and stochastic models of fluid dynamics have been the object of intense investigations which have generated several important results. We refer, for instance, to [6–22]. Similar investigations for Non-Newtonian fluids have almost not been undertaken except in very few work; we refer, for instance, to [23–25] for some computational studies of stochastic models of polymeric fluids. It is worth to note that in the Non-Newtonian case the study of stochastic models is relevant not only for the analytical approach to turbulent flows but also for practical needs related to the physics of the corresponding fluids [2].

In the present work, we initiate the mathematical analysis for the stochastic model of incompressible second grade fluids. An incompressible fluid of second grade with a velocity field u is a special example of a differential Rivlin-Ericksen fluid. It was shown in [26] that its stress tensor $\mathbb{T}$ is given by

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.1}$$

where $\tilde{p}$ is the scalar pressure field, ν is the kinematic viscosity, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors defined by

$$\begin{aligned} \mathbf{A}_1 &= \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j} + \left(\frac{\partial u_j}{\partial x_i}\right)_{i,j}, \\ \mathbf{A}_2 &= \frac{DA_1}{Dt} + \mathbf{A}_1\left(\frac{\partial u_i}{\partial x_j}\right)_{i,j} + \left(\frac{\partial u_j}{\partial x_i}\right)_{i,j}\mathbf{A}_1, \end{aligned} \tag{1.2}$$

where D/Dt denotes the material derivative. The constants α_1 and α_2 represent the normal stress moduli. The incompressibility requires that

$$\operatorname{div} u = 0. \tag{1.3}$$

Taking into account some thermodynamical aspects, Dunn and Fosdick proved in [27] that the kinematic viscosity ν must be nonnegative. In addition, they found that the free energy must be a quadratic function of $\mathbf{A}_1$. This implies in particular that the Clausius-Duhem inequality is satisfied and the Helmholtz free energy is minimum at equilibrium if and only if

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0. \tag{1.4}$$

In what follows we assume that $\alpha_1 = \alpha > 0$ and $\nu > 0$. We also refer to [28, 29] for more recent works concerning those conditions.

Those thermodynamical conditions imply that the stress tensor $\mathbb{T}$ can be written in the following form:

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \nu\mathbf{A}_1 + \alpha\left(\frac{\partial}{\partial t}\mathbf{A}_1 + \frac{1}{2}\mathbf{A}_1\left(\mathbf{L} - \mathbf{L}^T\right) - \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right)\mathbf{A}_1\right), \tag{1.5}$$

where

$$\mathbf{L} = \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j}. \tag{1.6}$$

We can check that

$$\operatorname{div}\mathbb{T} = -\nabla\tilde{p} + \nu\Delta u + \alpha\frac{\partial\Delta u}{\partial t} + \alpha\left(\operatorname{curl}(\Delta u)\times u + \nabla\left(u\cdot\Delta u + \frac{1}{4}|\mathbf{A}_1|^2\right)\right). \tag{1.7}$$

For a given external force f the dynamical equation for a second grade fluid is

$$\frac{\partial u}{\partial t} + \operatorname{curl}(u)\times u + \nabla\left(\frac{1}{2}|u|^2\right) = \operatorname{div}\mathbb{T} + f. \tag{1.8}$$

Making use of the latter equation and (1.7), we obtain the system of partial differential equations

$$\begin{gathered}\frac{\partial}{\partial t}(u-\alpha\Delta u) - \nu\Delta u + \operatorname{curl}(u-\alpha\Delta u)\times u + \nabla\mathfrak{P} = f,\\ \operatorname{div} u = 0,\end{gathered} \tag{1.9}$$

where

$$\mathfrak{P} = \tilde{p} - \alpha\left(u\cdot\Delta u + \frac{1}{4}|\mathbf{A}_1|\right) + \frac{1}{2}|u|^2 \tag{1.10}$$

is the modified pressure. For a given connected and bounded domain D of $\mathbb{R}^2$ and finite time horizon $[0,T]$ we complete the above system with the initial value

$$u(0) = u_0 \quad \text{in } D, \tag{1.11}$$

and the Dirichlet boundary value condition

$$u = 0 \quad \text{on } \partial D\times(0,T]. \tag{1.12}$$

The interest in the investigation of problem (1.9) arises from the fact that it is an admissible model of slow flow fluids. Furthermore, once the above thermodynamical compatibility conditions are satisfied "the second grade fluid has general and pleasant properties such as boundedness, stability, and exponential decay" (see again [27]). It also can be taken as a generalization of the Navier-Stokes Equation (NSE). Indeed they reduce to NSE when $\alpha = 0$; moreover recent work [30] shows that it is a good approximation of the NSE. See also [31–36] for interesting discussions to their relationship with other fluid models.

Due to the above nice properties, the mathematical analysis of the second grade fluid has attracted many prominent researchers in the deterministic case. The first relevant analysis was done by Ouazar in his 1981 thesis; together with Cioranescu, they published the related result in [37, 38]. Their method was based on the Galerkin approximation scheme involving a priori estimates for the approximating solutions using a special basis consisting of eigenfunctions corresponding to the scalar product associated with the operator $\operatorname{curl}(u-\alpha\Delta u)$. They proved global existence and uniqueness without restriction on the initial data for the two-dimensional case. Cioranescu and Girault [39], as well as Bernard [40] extended this method to the three dimensional case; global existence was obtained with some reasonable restrictions on the initial data. For another approach to global existence using Schauder's fixed point technics, we refer to [41] and some relevant references therein.

As already mentioned, in this work we propose a stochastic version of the problem (1.9), (1.11)-(1.12). More precisely, we assume that a connected and bounded open set D of $\mathbb{R}^2$ with boundary ∂D of class $\mathcal{C}^3$, a finite time horizon $[0,T]$, and a nonrandom initial value u_0 are given. We consider the problem

$$\begin{gathered} d(u-\alpha\Delta u)+(-\nu\Delta u+\operatorname{curl}(u-\alpha\Delta u)\times u+\nabla\mathfrak{P})dt=F(u,t)dt+G(u,t)dW \\ \text{in } D\times(0,T), \\ \operatorname{div} u=0 \quad \text{in } D\times(0,T), \\ u=0 \quad \text{in } \partial D\times(0,T), \\ u(0)=u_0 \quad \text{in } D, \end{gathered} \tag{1.13}$$

where $u=(u_1,u_2)$ and $\mathfrak{P}$ represent the random velocity and pressure, respectively. The system is to be understood in the Ito sense. It is the equation of motion of an incompressible second grade fluid driven by random external forces $F(u,t)$ and $G(u,t)dW$, where W is a $\mathbb{R}^m$-valued standard Wiener process.

As far as we know, this paper is the first dealing with the stochastic version of the equation governing the motion of a second grade fluid filling a connected and bounded domain D of $\mathbb{R}^2$. Consequently, we could by no means exhaust the mathematical analysis of the problem; many questions are still open but we hope that this pioneering work will find its applications elsewhere. We limited ourselves to the discussion of a global existence result of a probabilistic weak solution in the two-dimensional case. In forthcoming papers we will address other questions such as the existence probabilistic strong solutions under more stringent conditions, the uniqueness of those solutions, and their behaviour when $\alpha\to 0$. It should be noted that solving this problem is not easy even in the deterministic case, the nature of the nonlinearities being one of the main difficulties in addition to the complex structure of the equations. Besides the obstacles encountered in the deterministic case, the introduction of the noise term $G(u,t)dW$ in the stochastic version induces the appearance of expressions that are very hard to control when proving some crucial estimates. Overcoming these problems will require a-tour-de force in the work.

The rest of the paper is organized as follows. In Section 2, we give some notations, necessary background of probabilistic or analytical nature. Section 3 is devoted to the formulation of the hypotheses and the main result. We introduce a Galerkin approximation of the problem and derive crucial a priori estimates for its solution in Section 4; a compactness result is also derived. We prove the main result in Section 5.

2. Notations and Preliminaries

Let us start with some informationsabout some functional spaces needed in this work. Let D be an open subset of $\mathbb{R}^2$, let $1\le p\le\infty$, and let k be a nonnegative integer. We consider the well-known Lebesgue and Sobolev spaces $L^p(D)$ and $W^{k,p}(D)$, respectively. When $p=2$, we write $W^{k,2}(D)=H^k(D)$. We denote by $W_0^{k,p}(D)$ the closure in $W^{k,p}(D)$ of $C_c^\infty(D)$ the space of infinitely differentiable function with compact support in D. If $p = 2$, we denote $W_0^{k,p}(D)$ by $H_0^k(D)$. We assume that the Hilbert space $H_0^1(D)$ is endowed with the scalar product

$$((u,v)) = \int_D \nabla u \cdot \nabla v \, dx = \sum_{i=1}^{2} \int_D \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \tag{2.1}$$

where ∇ is the gradient operator. The norm $\|\cdot\|$ generated by this scalar product is equivalent to the usual norm of $W^{1,2}(D)$ in $H_0^1(D)$. If the domain D is smooth enough and bounded, then for any m and p such that $mp > 2$ the embedding

$$W^{j+m,p} \subset W^{j,q} \tag{2.2}$$

is compact for any $1 \leq q \leq \infty$. More Sobolev embedding theorems can be found in [42] and references therein.

Next we define some probabilistic evolution spaces necessary throughout the paper. Let $(\Omega, \mathcal{F}, (\mathbb{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a given stochastic basis; that is, $(\Omega, \mathcal{F}, \mathbb{P})$ is complete probability space and $(\mathbb{F}_t)_{0\leq t\leq T}$ is an increasing sub-σ-algebras of $\mathcal{F}$ such that $\mathbb{F}_0$ contains every P-null subset of Ω. For any reflexive separable real Banach space X endowed with the norm $\|\cdot\|_X$, for any $p \geq 1$, $L^p(0,T;X)$ is the space of X-valued measurable functions u defined on $[0,T]$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_{\mathrm{X}}^p dt\right)^{1/p} < \infty. \tag{2.3}$$

For any $r, p \geq 1$ we denote by $L^p(\Omega, \mathbb{P}; L^r(0,T;\mathrm{X}))$ the space of processes $u = u(\omega, t)$ with values in X defined on $\Omega \times [0,T]$ such that

(1) u is measurable with respect to (ω, t) and, for each t, u is $\mathbb{F}^t$ measurable,

(2) $u(t,\omega) \in \mathrm{X}$ for almost all (ω, t) and

$$\|u\|_{L^p(\Omega,\mathbb{P};L^r(0,T;X))} = \left(\mathbb{E}\left(\int_0^T \|u\|_{\mathrm{X}}^r dt\right)^{p/r}\right)^{1/p} < \infty, \tag{2.4}$$

where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$.

When $r = \infty$, we write

$$\|u\|_{L^p(\Omega,\mathbb{P};L^\infty(0,T;X))} = \left(\mathbb{E} \operatorname{ess}\sup_{0\leq t\leq T} \|u\|_{\mathrm{X}}^p dt\right)^{1/p} < \infty. \tag{2.5}$$

Next we give some compactness results of probabilistic nature due to Prokhorov and Skorokhod. Let us consider Ω as a separable and complete metric space and $\mathcal{F}$ its Borel σ-field. A family $\mathfrak{P}_k$ of probability measures on $(\Omega, \mathcal{F})$ is relatively compact if every sequence of elements of $\mathfrak{P}_k$ contains a subsequence $\mathfrak{P}_{k_j}$ which converges weakly to a probability measure $\mathfrak{P}$; that is, for any ϕ bounded and continuous function on Ω,

$$\int \phi(\omega) d\mathfrak{P}_{k_j}(d\omega) \longrightarrow \int \phi(\omega) d\mathfrak{P}(d\omega). \tag{2.6}$$

The family $\mathfrak{P}_k$ is said to be tight if, for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega$ such that $\mathbb{P}(K_\varepsilon) \geq 1 - \varepsilon$, for every $\mathbb{P} \in \mathfrak{P}_k$.

We frequently use the following two theorems. We refer to [43] for their proofs.

Theorem 2.1 (see Prokhorov). *The family $\mathfrak{P}_k$ is relatively compact if and only if it is tight.*

Theorem 2.2 (see Skorokhod). *For any sequence of probability measures $\mathfrak{P}_k$ on Ω which converges to a probability measure $\mathfrak{P}$, there exist a probability space $(\Omega', \mathbb{F}', \mathbb{P}')$ and random variables X_k, X with values in Ω such that the probability law of X_k(resp., X) is $\mathfrak{P}_k$(resp., $\mathfrak{P}$) and $\lim_{k\to\infty} X_k = X\mathbb{P}'$-a.s.*

We proceed now with the definitions of additional spaces frequently used in this work. In what follows we denote by $\mathbb{X}$ the space of $\mathbb{R}^2$-valued functions such that each component belongs to X. A simply-connected bounded domain D with boundary of class $\mathcal{C}^3$ is given. We introduce the spaces

$$\begin{gathered} \mathcal{V} = \left\{ u \in [\mathcal{C}_c^\infty]^2 \text{ such that } \operatorname{div} u = 0 \right\}, \\ \mathbb{V} = \text{closure of } \mathcal{V} \quad \text{in } \mathbb{H}^1(D), \\ \mathbb{H} = \text{closure of } \mathcal{V} \quad \text{in } \mathbb{L}^2(D). \end{gathered} \tag{2.7}$$

We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbb{L}^2(D)$ on $\mathbb{H}$, respectively. The inner product and the norm induced by that of $\mathbb{H}_0^1(D)$ on $\mathbb{V}$ are denoted respectively by $((\cdot,\cdot))$ and $\|\cdot\|$. In the space $\mathbb{V}$, the latter norm is equivalent to the norm generated by the following scalar product (see, e.g., [37])

$$(u, v)_{\mathbb{V}} = (u, v) + \alpha((u, v)), \quad \text{for any } u, v \in \mathbb{V}. \tag{2.8}$$

We also introduce the following space:

$$\mathbb{W} = \left\{ u \in \mathbb{V} \text{ such that } \operatorname{curl}(u - \alpha\Delta u) \in L^2(D) \right\}. \tag{2.9}$$

The following lemma tells us that the norm generated by the scalar product

$$(u, v)_{\mathbb{W}} = (u, v)_{\mathbb{V}} + (\operatorname{curl}(u - \alpha\Delta u), \operatorname{curl}(v - \alpha\Delta v)), \tag{2.10}$$

is equivalent to the usual $\mathbb{H}^3(D)$-norm on $\mathbb{W}$. Its proof can be found, for example, in [37, 39].

Lemma 2.3. *The following (algebraic and topological) identity holds:*

$$\mathbb{W} = \widetilde{\mathbb{W}}, \tag{2.11}$$

where

$$\widetilde{\mathbb{W}} = \left\{ v \in \mathbb{H}^3(D) \text{ such that } \operatorname{div} v = 0 \text{ and } v|_{\partial D} = 0 \right\}. \tag{2.12}$$

Moreover, there is a positive constant C such that

$$|v|^2_{\mathbb{H}^3(D)} \leq C\left(|v|^2_{\mathbb{V}} + |\operatorname{curl}(v - \alpha\Delta v)|^2 \right), \tag{2.13}$$

for any $v \in \widetilde{\mathbb{W}}$.

By this lemma we can endow the space $\mathbb{W}$ with norm $|\cdot|_{\mathbb{W}}$ which is generated by the scalar product (2.10).

From now on, we identify the space $\mathbb{V}$ with its dual space $\mathbb{V}^\star$ via the Riesz representation, and we have the Gelfand chain

$$\mathbb{W} \subset \mathbb{V} \subset \mathbb{W}^\star, \tag{2.14}$$

where each space is dense in the next one and the inclusions are continuous.

The following inequalities will be used frequently.

Lemma 2.4. *For any $u \in \mathbb{W}$, $v \in \mathbb{W}$, and $w \in \mathbb{W}$ one has*

$$|(\operatorname{curl}(u - \alpha\Delta u) \times v, w)| \leq C|u|_{\mathbb{H}^3}|v|_{\mathbb{V}}|w|_{\mathbb{W}}. \tag{2.15}$$

One also has

$$|(\operatorname{curl}(u - \alpha\Delta u) \times u, w)| \leq C|u|^2_{\mathbb{V}}|w|_{\mathbb{W}}, \tag{2.16}$$

for any $u \in \mathbb{W}$ and $w \in \mathbb{W}$.

Proof. We introduce the well-known trilinear form b used in the study of the Navier-Stokes equation by setting

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_D u_i \frac{\partial v_j}{\partial x_i} w_j dx. \tag{2.17}$$

We give the following identity whose proof can be found in [37, 40]. This equation is valid for any smooth (solenoidal) functions Φ, v, and w as

$$((\operatorname{curl}\Phi) \times v, w) = b(v, \Phi, w) - b(w, \Phi, v). \tag{2.18}$$

We derive from this that for any $u \in \mathbb{W}$, $v \in \mathbb{W}$, and $w \in \mathbb{W}$

$$|(\operatorname{curl}(u-\alpha\Delta u)\times v, w)| \leq C|v|_{\mathbb{L}^2(D)}|\nabla(u-\alpha\Delta u)|_{\mathbb{L}^2(D)}|w|_{\mathbb{L}^\infty(D)}, \tag{2.19}$$

where Hölder's inequality was used. The Sobolev embedding (2.2) and the equivalence of the norms $|\cdot|_{\mathbb{W}}$ and $|\cdot|^3_{\mathbb{H}}(D)$ imply (2.15).

By (2.18) we deduce

$$(\operatorname{curl}(u-\alpha\Delta u)\times u, w) = b(u,u,w) - \alpha b(u,\Delta u,w) + \alpha b(w,\Delta u,u). \tag{2.20}$$

With the help of integration by parts and using the fact that u and w are elements of $\mathbb{W}$ we derive that

$$b(u,\Delta u,w) = \sum_{j=1}^{2} b\left(\frac{\partial u}{\partial x_j}, w, \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{2} b\left(u, \frac{\partial w}{\partial x_j}, \frac{\partial u}{\partial x_j}\right), \tag{2.21}$$

$$b(w,\Delta u,u) = \sum_{j=1}^{2} b\left(\frac{\partial w}{\partial x_j}, u, \frac{\partial u}{\partial x_j}\right). \tag{2.22}$$

We use these results to derive the following estimate. For any elements $u \in \mathbb{V}$ and $w \in \mathbb{L}^4(D)$, we obtain by Hölder's inequality

$$|b(u,u,w)| \leq C|u|_{\mathbb{L}^4(D)}\|u\||w|_{\mathbb{L}^4(D)}. \tag{2.23}$$

And since the space $\mathbb{V}$ and $\mathbb{W}$ are, respectively, continuously embedded in $\mathbb{L}^4(D)$ and $\mathbb{V}$, then

$$|b(u,u,w)| \leq C|u|^2_{\mathbb{V}}|w|_{\mathbb{W}}. \tag{2.24}$$

We also have

$$\begin{aligned} |b(u,\Delta u,w)| \leq{}& |\nabla w|_{\mathbb{L}^\infty(D)} \sum_{j=1}^{2} \left|\frac{\partial u}{\partial x_j}\right|^2_{\mathbb{L}^2(D)} \\ &+ |u|_{\mathbb{L}^4(D)} \left(\sum_{j=1}^{2}\left|\frac{\partial w}{\partial x_j}\right|^2_{\mathbb{L}^4(D)}\right)^{1/2} \left(\sum_{j=1}^{2}\left|\frac{\partial u}{\partial x_j}\right|^2_{\mathbb{L}^2(D)}\right)^{1/2}. \end{aligned} \tag{2.25}$$

We derive from this and the Sobolev embedding (2.2) that

$$|b(u,\Delta u,w)| \leq C|u|^2_{\mathbb{V}}|w|_{\mathbb{W}}. \tag{2.26}$$

By an analogous argument we have

$$|b(w,\Delta u,u)| \leq C|w|_{\mathbb{W}}|u|^2_{\mathbb{V}}. \tag{2.27}$$

The estimates (2.20), (2.24)–(2.27) yield

$$|(\operatorname{curl}(u - \alpha\Delta u) \times u, w)| \leq C|u|_{\mathbb{V}}^2 |w|_{\mathbb{W}}, \tag{2.28}$$

for any $u \in \mathbb{W}$ and $w \in \mathbb{W}$. This completes the proof of the lemma. □

Next we give some results on which most of the proofs in forthcoming sections rely. We start by stating a theorem on solvability of the "generalized Stokes equations"

$$\begin{aligned} v - \alpha\Delta v + \nabla q = f \quad &\text{in } D, \\ \operatorname{div} v = 0 \quad &\text{in } D, \\ v = 0 \quad &\text{on } \partial D. \end{aligned} \tag{2.29}$$

By a solution of this system we mean a function $v \in \mathbb{V}$ which satisfies

$$(v, h) + \alpha((v, h)) = (f, h), \tag{2.30}$$

for any $h \in \mathbb{V}$.

The proof of the following result can be derived from an adaptation of the results obtained by Solonnikov in [44, 45].

Theorem 2.5. *Let D be a connected, bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ with boundary ∂D of class $\mathcal{C}^l$ and let f be a function in $\mathbb{H}^l$, $l \geq 1$. Then* (2.29) *has a unique solution v. Moreover if f is an element of $\mathbb{V}$, $v \in \mathbb{H}^{l+2} \cap \mathbb{V}$, and the following hold:*

$$\begin{aligned} (v, h)_{\mathbb{V}} &= (v, h), \\ |v|_{\mathbb{W}} &\leq C|f|_{\mathbb{V}}, \end{aligned} \tag{2.31}$$

for any $h \in \mathbb{V}$.

Next we formulate Aubin-Lions's compactness theorem; its proof can be found in [46].

Theorem 2.6. *Let X, B, Y be three Banach spaces such that the following embedding is continuous:*

$$X \subset B \subset Y. \tag{2.32}$$

Moreover, assume that the embedding $X \subset B$ is compact, then the set $\mathfrak{F}$ consisting of functions $v \in L^q(0, T; B)$, $1 \leq q \leq \infty$, such that

$$\sup_{0 \leq h \leq 1} \int_{t_1}^{t_2} |v(t + h) - v(t)|_Y^p dt \longrightarrow 0, \quad \text{as } h \longrightarrow 0, \tag{2.33}$$

for any $0 < t_1 < t_2 < T$, is compact in $L^p(0, T; B)$ for any p.

Last but not least we present the famous Kolmogorov-Čentsov continuity criterion for stochastic processes. We refer to [47, 48] for its proof and some of its extension.

Theorem 2.7 (see Kolmogorv-Čentsov). *Suppose that a real-valued process $X = \{X_t,\ 0 \le t \le T\}$ on a probability space $(\Omega, \mathbb{P})$ satisfies the condition*

$$\mathbb{E}|X_{t+h} - X_t|^{\gamma} \le Ch^{1+\beta}, \quad 0 \le t,\ h \le T, \tag{2.34}$$

for some positive constants γ, β, and C. Then there exists a continuous modification $\widetilde{X} = \{\widetilde{X}_t,\ 0 \le t \le T\}$ of X, which is locally Hölder continuous with exponent $\kappa \in (0, \beta/\gamma)$.

3. Hypotheses and the Main Result

We state on our problem the following.

3.1. Hypotheses

(1) We assume that

$$F : \mathbb{V} \times [0, T] \longrightarrow \mathbb{V} \tag{3.1}$$

is continuous in both variables. We also assume that, for any $t \in [0, T]$ and any $v \in \mathbb{V}$

$$|F(v, t)|_{\mathbb{V}} \le C(1 + |v|_{\mathbb{V}}). \tag{3.2}$$

(2) We also define a nonlinear operator G as follows:

$$G : \mathbb{V} \times [0, T] \longrightarrow \mathbb{V}^{\otimes^m} \tag{3.3}$$

is continuous in both variables. We require that, for any $t \in [0, T]$, $G(v, t)$ satisfy

$$|G(v, t)|_{\mathbb{V}^{\otimes m}} \le C(1 + |v|_{\mathbb{V}}). \tag{3.4}$$

3.2. Statement of the Main Theorem

We introduce the concept of solution of the problem (1.13) that is of interest to us.

Definition 3.1. By a solution of the problem (1.13), we mean a system

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}^t, W, u), \tag{3.5}$$

where

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space; $\mathbb{F}^t$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$,

(2) $W(t)$ is an m-dimensional $\mathbb{F}^t$ standard Wiener process,

(3) for a.e. t, $u(t) \in L^p(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{W})) \cap L^p(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{V}))$, $2 \le p < \infty$,

(4) for almost all t, $u(t)$ is $\mathbb{F}^t$ measurable,

(5) $\mathbb{P}$-a.s the following integral equation of Itô type holds:

$$\begin{aligned}(u(t) - u(0), v)_{\mathbb{V}} &+ \int_0^t [\nu((u,v)) + (\operatorname{curl}(u(s) - \alpha\Delta u(s)) \times u, v)]ds \\ &= \int_0^t (F(u(s),s),v)ds + \int_0^t (G(u(s),s),v)dW(s)\end{aligned} \tag{3.6}$$

for any $t \in [0,T]$ and $v \in \mathbb{W}$.

Remark 3.2. In the above definition the quantity $\int_0^t (G(u(s),s),v)dW(s)$ should be understood as:

$$\int_0^t (G(u(s),s),v)dW(s) = \sum_{k=1}^{m} \int_0^t (G_k(u(s),s),v)dW_k(s), \tag{3.7}$$

where G_k and W_k denote the kth component of G and W, respectively.

Now we state our main result.

Theorem 3.3. *Assume that $u_0 \in \mathbb{W}$; assume also that all the assumptions, namely,* (3.2) *and* (3.4), *on the operators F, G are satisfied; then the problem* (1.13) *has a solution in the sense of the above definition. Moreover, almost surely the paths of the process u are $\mathbb{W}$-valued weakly continuous.*

4. Auxiliary Results

In this section we derive crucial a priori estimates from the Galerkin approximation. They will serve as a toolkit for the proof of Theorem 3.3.

4.1. The Approximate Solution

The following statement is a consequence of the spectral theorem for self-adjoint compact operator stated in [49]. *The injection of $\mathbb{W}$ into $\mathbb{V}$ is compact. Let I be the isomorphism of $\mathbb{W}^\star$ onto $\mathbb{W}$, then the restriction of I to $\mathbb{V}$ is a continuous compact operator into itself. Thus, there exists a sequence (e_i) of elements of $\mathbb{W}$ which forms an orthonormal basis in $\mathbb{W}$, and an orthogonal basis in $\mathbb{V}$.* This sequence verifies:

$$\text{for any } v \in \mathbb{W} \quad (v, e_i)_{\mathbb{W}} = \lambda_i (v, e_i)_{\mathbb{V}}, \tag{4.1}$$

where $\lambda_{i+1} > \lambda_i > 0$, $i = 1, 2, \ldots$.

We have the following important result due to [39] about the regularity of the e_i-s.

Lemma 4.1. *Let D be a bounded, simply-connected open set of $\mathbb{R}^2$ with a boundary of class $\mathcal{C}^3$, then the eigenfunctions of* (4.1) *belong to* $\mathbb{H}^4(D)$.

We consider the subset $\mathbb{W}_N = \operatorname{Span}(e_1, \dots, e_N) \subset \mathbb{W}$ and we look for a finite-dimensional approximation of a solution of our problem as a vector $u^N \in \mathbb{W}_N$ that can be written as the Fourier series:

$$u^N(t) = \sum_{i=1}^{N} c_{iN}(t) e_i(x). \tag{4.2}$$

Let us consider a complete probabilistic system $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}, \overline{\mathbb{F}}^t, \overline{W})$ such that the filtration $\{\overline{\mathcal{F}}_t\}$ satisfies the usual condition and $\overline{W}$ is an m-dimensional standard Wiener process taking values in $\mathbb{R}^m$. We require u^N to satisfy the following system:

$$\begin{aligned} d\left(u^N, e_i\right)_{\mathbb{V}} &+ \nu\left(\left(u^N, e_i\right)\right)dt + b\left(u^N, u^N, e_i\right)dt - \alpha b\left(u^N, \Delta u^N, e_i\right)dt + \alpha b\left(e_i, \Delta u^N, u^N\right)dt \\ &= \left(F\left(t, u^N\right), e_i\right)dt + \left(G\left(t, u^N\right), e_i\right)d\overline{W}, \end{aligned} \tag{4.3}$$

where u_0^N as the orthogonal projection of $u(0)$ in the space $\mathbb{W}_N$ is given as

$$u_0^N\left(\text{or } u^N(0)\right) \longrightarrow u(0) \quad \text{strongly in } \mathbb{V} \tag{4.4}$$

as $N \to \infty$. The Fourier coefficients c_{iN} in (4.2) are solutions of a system of stochastic ordinary differential equations which satisfy the conditions of the existence theorem of Skorokhod [50] (see also [47]). Therefore the sequence of functions u^N exists at least on a short interval $(0, T_N)$. Global existence will follow from a priori estimates for u^N.

4.2. A Priori Estimates

From now on C is a constant depending only on the data, and may change from one line to the next one. We start by proving the following lemma.

Lemma 4.2. *For any $N \geq 1$ one has*

$$\overline{\mathbb{E}} \sup_{0 \le t \le T} \left| u^N(t) \right|_{\mathbb{V}}^2 < +\infty. \tag{4.5}$$

One also has

$$\overline{\mathbb{E}} \sup_{0 \le t \le T} \left| u^N(t) \right|_{\mathbb{W}}^2 < +\infty. \tag{4.6}$$

Proof. From now on we denote by $|v|_*$ the quantity $|\operatorname{curl}(v-\alpha\Delta v)|$ for any $v\in\mathbb{W}$. For any integer $M\geq 1$ we introduce the stopping time

$$\tau_M=\begin{cases}\inf\{0\leq t;\, |u^N(t)|_{\mathbb{V}}+|u^N(t)|_*\geq M\}\\ +\infty \quad \text{if } \{0\leq t;\, |u^N(t)|_{\mathbb{V}}+|u^N(t)|_*\geq M\}=\emptyset.\end{cases}\tag{4.7}$$

We will use a modification of the argument used in [7].

For any $0\leq s\leq t\wedge\tau_M$, $t\in(0,T_N)$, we may apply Itô's formula (see, e.g., [7, 12]) for $\phi((u^N(s),e_i)_{\mathbb{V}})=(u^N(s),e_i)_{\mathbb{V}}^2$ to (4.3) and obtain

$$\begin{aligned}&\left(u^N(s),e_i\right)_{\mathbb{V}}^2+2\int_0^s\left(u^N(r),e_i\right)_{\mathbb{V}}\left[\nu\left(\left(u^N(r),e_i\right)\right)+b\left(u^N(r),u^N(r)-\alpha\Delta u^N(r),e_i\right)\right]dr\\&\quad=2\int_0^s\left(u^N(r),e_i\right)_{\mathbb{V}}\left[-\alpha b\left(e_i,\Delta u^N(r),u^N(r)\right)+\left(F\left(r,u^N\right),e_i\right)\right]dr\\&\qquad+\int_0^s\left(G\left(r,u^N\right),e_i\right)d\overline{W}+\int_0^s\left(u^N(r),e_i\right)_{\mathbb{V}}\left(G\left(r,u^N\right),e_i\right)^2dr.\end{aligned}\tag{4.8}$$

We note that $|u^N|_{\mathbb{V}}^2=\sum_{i=1}^N\lambda_i(u^N,e_i)_{\mathbb{V}}^2$. Then, multiplying by λ_i the above equation and summing over i from 1 to N give us

$$\begin{aligned}\left|u^N(s)\right|_{\mathbb{V}}^2+2\nu\int_0^s\left\|u^N\right\|^2dr&=\left|u_0^N\right|_{\mathbb{V}}^2+2\int_0^s\left(F\left(r,u^N\right),u^N\right)dr+\sum_{i=1}^N\lambda_i\int_0^s\left(G(r,u^N),e_i\right)^2dr\\&\quad+2\int_0^s\left(G\left(r,u^N\right),u^N\right)d\overline{W},\end{aligned}\tag{4.9}$$

where we have used the fact that $b(u^N,u^N,u^N)=0$.

We obtain from (4.9) that

$$\begin{aligned}\left|u^N(s)\right|_{\mathbb{V}}^2+2\nu\int_0^s\left(\left(u^N(r),u^N(r)\right)\right)dr&\leq\left|u_0^N\right|_{\mathbb{V}}^2+\sum_{i=1}^N\lambda_i\int_0^s\left(G\left(u^N(r),r\right),e_i\right)^2dr\\&\quad+2\int_0^s\left|\left(F\left(u^N(r),r\right),u^N(r)\right)\right|dr\\&\quad+\left|2\int_0^s\left(G\left(u^N(r),r\right),u^N(r)\right)d\overline{W}\right|,\end{aligned}\tag{4.10}$$

for any $0\leq s\leq t\wedge\tau_M$, $t\in[0,T_N]$. For any $u\in\mathbb{V}$ we have

$$|u|\leq\mathcal{P}\|u\|,\tag{4.11}$$

where $\wp$ is the so called Poincaré's constant. The last inequality implies that

$$\left|\left(F\left(u^N(s),s\right),u^N\right)\right| \leq \wp^2\left\|u^N\right\|\,\left\|F\left(u^N(s),s\right)\right\|. \tag{4.12}$$

We also mention that

$$\left(\wp^2+\alpha\right)^{-1}|v|_{\mathbb{V}}^2 \leq \|v\|^2 \leq (\alpha)^{-1}|v|_{\mathbb{V}}^2, \quad \text{for any } v\in\mathbb{V}. \tag{4.13}$$

From the former equation and this one we find

$$\left|\left(F\left(u^N(s),s\right),u^N(s)\right)\right| \leq 2C\frac{\wp^2}{\alpha}\left(1+\left|u^N(s)\right|_{\mathbb{V}}^2\right). \tag{4.14}$$

To find a uniform estimate for the corrector term $\sum_{i=1}^N \lambda_i(G(u^N(s),e_i))^2$ is not straightforward; this is the difficulty already mentioned in the introduction. Since the corrector term is explicitly written as function depending on the scalar product (in $\mathbb{L}^2(D)$) $(\cdot,\cdot)$ and the e_i-s form an orthonormal basis (resp., orthogonal basis) of $\mathbb{W}$ (resp, $\mathbb{V}$), then the usual Bessel's inequality (see, e.g., [6]) does not apply anymore. To circumvent this difficulty we consider the following generalized Stokes equation:

$$\begin{gathered}\widetilde{G}-\alpha\Delta\widetilde{G}+\nabla q = G\left(u^N(s),s\right) \quad \text{in } D,\\ \operatorname{div}\widetilde{G}=0 \quad \text{in } D,\\ \widetilde{G}=0 \quad \text{on } \partial D,\end{gathered} \tag{4.15}$$

for any $s\in[0,T]$. By Theorem 2.5, (4.15) has a solution $\widetilde{G}$ in $\mathbb{W}^{\otimes m}$ when ∂D is of class $\mathcal{C}^3$ and $G(u^N(s),s)\in\mathbb{V}^{\otimes m}$. Moreover, there exists a positive constant C_0 such that

$$\left|\widetilde{G}\right|_{\mathbb{H}^3(D)^{\otimes m}} \leq C_0\left|G\left(u^N(s),s\right)\right|_{\mathbb{V}^{\otimes m}}, \tag{4.16}$$

and $(\widetilde{G},e_i)_{\mathbb{V}} = (G(u^N(s),s),e_i)$ for any $i\geq 1$.

Since the norms $|\cdot|_{\mathbb{H}^3(D)}$ and $|\cdot|_{\mathbb{W}}$ are equivalent on $\mathbb{W}$, then there exists another positive constant C_* such that

$$\left|\widetilde{G}\right|_{\mathbb{W}^{\otimes m}} \leq C_*C_0\left|G\left(u^N(s),s\right)\right|_{\mathbb{V}^{\otimes m}}. \tag{4.17}$$

Equation (4.17) implies that $\widetilde{G}$ depends continuously on the data $G(u^N(s),s)$. Therefore, we note the above $\widetilde{G}$ as $\widetilde{G}(u^N(s),s)$. We find from (4.17) that

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i\Big(G\Big(u^N(s),s\Big),e_i\Big)^2 &= \sum_{i=1}^{N}\lambda_i\Big(\widetilde{G}\Big(u^N(s),s\Big),e_i\Big)_{\mathbb{V}}^2 \\ &= \sum_{i=1}^{N}\frac{1}{\lambda_i}\Big(\widetilde{G}\Big(u^N(s),s\Big),e_i\Big)_{\mathbb{W}}^2.\end{aligned} \tag{4.18}$$

We deduce from this that

$$\sum_{i=1}^{N}\frac{1}{\lambda_i}\Big(\widetilde{G}\Big(u^N(s),s\Big),e_i\Big)_{\mathbb{W}}^2 \le \frac{1}{\lambda_1}\left|\widetilde{G}(u^N(s),s)\right|_{\mathbb{W}^{\otimes m}}^2. \tag{4.19}$$

By (4.17) and the assumption on G, we have

$$\sum_{i=1}^{N}\lambda_i\Big(G\Big(u^N(s),s\Big),e_i\Big)^2 \le C\Big(1+\left|u^N(s)\right|_{\mathbb{V}}^2\Big). \tag{4.20}$$

Collecting this information, we obtain from (4.10) that

$$\begin{aligned}&\left|u^N(s)\right|_{\mathbb{V}}^2 + 2\nu\int_0^s\left\|u^N(r)\right\|^2 dr \\ &\quad\le C + C\int_0^s\left|u^N(r)\right|_{\mathbb{V}}^2 dr + 2\left|\int_0^s\Big(G\Big(u^N(r),r\Big),u^N(r)\Big)d\overline{W}\right|.\end{aligned} \tag{4.21}$$

Taking the sup over $s \le t\wedge\tau_M$ in both sides of this inequality and passing to the mathematical expectation in the resulting relation and finally applying Burkhölder-Davis-Gundy's inequality (see, e.g., [48]) to the stochastic term, we get

$$\begin{aligned}&\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|u^N(s)\right|_{\mathbb{V}}^2 + 2\nu\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left\|u^N(s)\right\|^2 ds \\ &\quad\le C + C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|u^N(s)\right|_{\mathbb{V}}^2 ds + 2C_1\overline{\mathbb{E}}\left(\int_0^{t\wedge\tau_M}\Big(G(u^N(s),s),u^N(s)\Big)^2 ds\right)^{1/2}.\end{aligned} \tag{4.22}$$

Now, we estimate

$$\gamma = \overline{\mathbb{E}}\left(\int_0^{t\wedge\tau_M}\Big(G\Big(u^N(s),s\Big),u^N(s)\Big)^2 ds\right)^{1/2}. \tag{4.23}$$

With the same argument that we have used for the term $|(F(u^N(s),s),u^N(s))|$, we have

$$\gamma \leq C\overline{\mathbb{E}}\left[\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|_{\mathbb{V}}\left(\int_0^{t\wedge\tau_M}\left|G\left(u^N(s),s\right)\right|^2_{\mathbb{V}^{\times m}}ds\right)^{1/2}\right]. \tag{4.24}$$

By ε-Young's inequality

$$\gamma \leq C\varepsilon\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|^2_{\mathbb{V}} + C_\varepsilon\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|G(u^N(s),s)\right|^2_{\mathbb{V}^{\times m}}ds \tag{4.25}$$

Using the assumption on G one has

$$\gamma \leq C\varepsilon\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|^2_{\mathbb{V}} + C_\varepsilon\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(1+\left|u^N(s)\right|^2_{\mathbb{V}}\right). \tag{4.26}$$

With convenient choice of ε $(1-2C_1C\varepsilon = 1/2)$, the estimates (4.22) and (4.26) allow us to write

$$\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|^2_{\mathbb{V}} + 4\nu\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left\|u^N(s)\right\|^2 ds \leq C + C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|u^N(s)\right|^2_{\mathbb{V}}ds. \tag{4.27}$$

We derive from this and Gronwall's inequality that

$$\overline{\mathbb{E}}\sup_{0\leq s\leq t\wedge\tau_M}\left|u^N(s)\right|^2_{\mathbb{V}} \leq C. \tag{4.28}$$

We recall the following relationship which is very important in the sequel:

$$\lambda_i\left(G\left(u^N(s),s\right),e_i\right) = \left(\widetilde{G}\left(u^N(s),s\right),e_i\right)_{\mathbb{W}}, \quad i\geq 1, \tag{4.29}$$

where $\widetilde{G}(u^N(s),s)$ is the solution in $\mathbb{W}$ of (GS).

To alleviate notation, we only write u^N when we mean $u^N(\cdot)$. Let us set

$$\phi\left(u^N\right) = -\nu\Delta u^N + \operatorname{curl}\left(u^N - \alpha\Delta u^N\right)\times u^N - F\left(u^N,t\right). \tag{4.30}$$

By Lemma 4.1, $\phi(u^N)\in\mathbb{H}^1(D)$. We have

$$d\left(u^N,e_i\right)_{\mathbb{V}} + \left(\phi\left(u^N\right),e_i\right)dt = \left(G\left(u^N,t\right),e_i\right)d\overline{W}. \tag{4.31}$$

By Theorem 2.5 a solution $v^N \in \mathbb{W}$ of the following system exists:

$$\begin{gathered} v^N - \alpha \Delta v^N + \nabla q = \phi\left(u^N\right) \quad \text{in } D, \\ \operatorname{div} v^N = 0 \quad \text{in } D, \\ v^N = 0 \quad \text{on } \partial D. \end{gathered} \tag{4.32}$$

Moreover,

$$\left(v^N, e_i\right)_{\mathbb{V}} = \left(\phi\left(u^N\right), e_i\right), \tag{4.33}$$

for any i. Thus,

$$\begin{aligned} d\left(u^N, e_i\right)_{\mathbb{V}} + \left(\phi\left(u^N\right), e_i\right) dt &= d\left(u^N, e_i\right)_{\mathbb{V}} + \left(v^N, e_i\right)_{\mathbb{V}} dt \\ &= \left(G\left(u^N, t\right), e_i\right) d\overline{W}. \end{aligned} \tag{4.34}$$

The following follows by multiplying the latter equation by λ_i and using the relationship (4.1):

$$d\left(u^N, e_i\right)_{\mathbb{W}} + \left(v^N, e_i\right)_{\mathbb{W}} dt = \lambda_i \left(G\left(u^N, t\right), e_i\right) d\overline{W}. \tag{4.35}$$

Recalling (4.29), we obtain

$$d\left(u^N, e_i\right)_{\mathbb{W}} + \left(v^N, e_i\right)_{\mathbb{W}} dt = \left(\widetilde{G}\left(u^N, t\right), e_i\right)_{\mathbb{W}} d\overline{W}. \tag{4.36}$$

Now applying the Itô's formula (see, e.g., [7]) to $\varphi((u^N, e_i)_{\mathbb{W}}) = (u^N, e_i)^2_{\mathbb{W}}$, we have

$$d\left(u^N, e_i\right)^2_{\mathbb{W}} + 2\left(u^N, e_i\right)_{\mathbb{W}}\left(v^N, e_i\right)_{\mathbb{W}} dt = \left(\widetilde{G}\left(u^N, t\right), e_i\right)^2_{\mathbb{W}} dt + 2\left(u^N, e_i\right)_{\mathbb{W}}\left(\widetilde{G}\left(u^N, t\right), e_i\right)_{\mathbb{W}} d\overline{W}. \tag{4.37}$$

Summing both sides of the last equation from 1 to N yields

$$d\left|u^N\right|^2_{\mathbb{W}} + 2\left(u^N, v^N\right)_{\mathbb{W}} dt = \sum_{i=1}^{N} \left(\widetilde{G}\left(u^N, t\right), e_i\right)^2_{\mathbb{W}} dt + 2\left(\widetilde{G}\left(u^N, t\right), u^N\right)_{\mathbb{W}} d\overline{W}. \tag{4.38}$$

Using the definition of $|\cdot|_{\mathbb{W}}$ and the scalar product $(\cdot,\cdot)_{\mathbb{W}}$, we can rewrite the above equation in the form

$$d\left[\left|u^N\right|_{\mathbb{V}}^2+\left|u^N\right|_*^2\right]+2\left[\left(v^N,u^N\right)_{\mathbb{V}}+\left(\operatorname{curl}\left(v^N-\alpha\Delta v^N\right)\ \operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\right]dt$$
$$=2\left(\operatorname{curl}\left(\widetilde{G}\left(u^N,t\right)-\alpha\Delta\widetilde{G}\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)d\overline{W}$$
$$+\sum_{i=1}^{N}\lambda_i^2\left(\widetilde{G}\left(u^N,t\right),e_i\right)_{\mathbb{V}}^2dt+2\left(\widetilde{G}\left(u^N,t\right),u^N\right)_{\mathbb{V}}d\overline{W}. \tag{4.39}$$

In view of Remark 3.2, we have to make the convention that in the sequel

$$\left(\operatorname{curl}\left(G\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\frac{d\overline{W}}{dt}$$
$$=\sum_{k=1}^{m}\left(\operatorname{curl}\left(G_k\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\frac{d\overline{W_k}}{dt}. \tag{4.40}$$

Using the definition of v^N and $\widetilde{G}$, we obtain

$$d\left[\left|u^N\right|_{\mathbb{V}}^2+\left|u^N\right|_*^2\right]+2\left[\left(\phi\left(u^N\right),u^N\right)+\left(\operatorname{curl}\left(\phi\left(u^N\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\right]dt$$
$$=\sum_{i=1}^{N}\lambda_i^2\left(G(u^N,t),e_i\right)^2dt+2\left(G\left(u^N,t\right),u^N\right)d\overline{W}$$
$$+2\left(\operatorname{curl}\left(G\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)d\overline{W}. \tag{4.41}$$

With the help of (4.9), (4.41) can be rewritten in the following way:

$$d\left|u^N\right|_*^2+2\left(\operatorname{curl}\phi\left(u^N\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)dt$$
$$=2\left(\operatorname{curl}\left(G\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\mathrm{d}\overline{W}+\sum_{i=1}^{N}\left(\lambda_i+\lambda_i^2\right)\left(G\left(u^N,t\right),e_i\right)^2dt. \tag{4.42}$$

We infer from the definition of $\phi(u^N)$ that

$$\operatorname{curl}\phi\left(u^N\right)=-\nu\operatorname{curl}\left(\Delta u^N+F\left(u^N,t\right)\right)+\operatorname{curl}\left(\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\times u^N\right). \tag{4.43}$$

In taking advantage of the dimension we get

$$\operatorname{curl}\left(\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\times u^N\right)=\left(u^N\cdot\nabla\right)\left(\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right). \tag{4.44}$$

This yields

$$\begin{aligned}&\left(\left(u^N\cdot\nabla\right)\left(\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)-\nu\operatorname{curl}\left(\Delta u^N+F\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\\&\quad=\left(\operatorname{curl}\phi\left(u^N\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right).\end{aligned}\tag{4.45}$$

Owing to Lemma 4.1 we readily check that

$$\left(\left(u^N\cdot\nabla\right)\beta,\beta\right)=0,\tag{4.46}$$

where $\beta=\operatorname{curl}(u^N-\alpha\Delta u^N)$. Consequently,

$$\begin{aligned}&\left(\nu\ \operatorname{curl}\left(\Delta u^N\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)+\left(\operatorname{curl}\left(F\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\\&\quad=-\left(\operatorname{curl}\phi\left(u^N\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right).\end{aligned}\tag{4.47}$$

Or equivalently

$$\begin{aligned}&\frac{\nu}{\alpha}\left|u^N\right|_*-\frac{\nu}{\alpha}\left(\operatorname{curl}u^N+\frac{\alpha}{\nu}\operatorname{curl}\left(F\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\\&\quad=\left(\operatorname{curl}\phi\left(u^N\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right).\end{aligned}\tag{4.48}$$

We derive from (4.42) and the last equation that

$$\begin{aligned}&\frac{d}{dt}\left|u^N\right|_*^2+\frac{2\nu}{\alpha}\left|u^N\right|_*^2-\frac{2\nu}{\alpha}\left(\operatorname{curl}u^N+\frac{\alpha}{\nu}\operatorname{curl}\left(F\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\\&\quad=\sum_{i=1}^N\left(\lambda_i+\lambda_i^2\right)\left(G\left(u^N,t\right),e_i\right)^2+2\left(\operatorname{curl}\left(G\left(u^N,t\right)\right),\operatorname{curl}\left(u^N-\alpha\Delta u^N\right)\right)\frac{d\overline{W}}{dt}.\end{aligned}\tag{4.49}$$

We argue as before in considering the stopping time τ_M. We derive from (4.49) that

$$\begin{aligned}&\left|u^N(s)\right|_*^2+\int_0^s\left(\frac{2\nu}{\alpha}\left|u^N(r)\right|_*^2-\sum_{i=1}^N\left(\lambda_i+\lambda_i^2\right)\left(G\left(u^N(r),r\right),e_i\right)^2\right)dr\\&\quad=\int_0^s\frac{2\nu}{\alpha}\left[\left(\operatorname{curl}\left(u^N(r)\right)-\frac{\alpha}{\nu}\operatorname{curl}\left(F\left(u^N(r),r\right)\right),\operatorname{curl}\left(u^N(r)-\alpha\Delta u^N(r)\right)\right)\right]dr\\&\qquad+2\int_0^s\left(\operatorname{curl}\left(G\left(u^N(r),r\right)\right),\operatorname{curl}\left(u^N(r)-\alpha\Delta u^N(r)\right)\right)\overline{W}.\end{aligned}\tag{4.50}$$

Hence,

$$\begin{aligned}
&\left|u^N(s)\right|_*^2+\int_0^s\frac{2\nu}{\alpha}\left|u^N(r)\right|_*^2dr-\sum_{i=1}^N\left(\lambda_i+\lambda_i^2\right)\int_0^s\left(G\left(u^N(r),r\right),e_i\right)^2dr\\
&\quad\le\left|u_0^N\right|_*^2+\int_0^s\frac{2\nu}{\alpha}\left|\operatorname{curl}\left(u^N(r)\right)\right|\left|u^N(r)\right|_*+\int_0^s2\left|\operatorname{curl}\left(F\left(u^N(r),r\right)\right)\right|\left|u^N(r)\right|_*dr\\
&\qquad+2\left|\int_0^s\left(\operatorname{curl}\left(G\left(u^N(r),r\right)\right),\operatorname{curl}\left(u^N(r)-\alpha\Delta u^N(r)\right)\right)d\overline{W}\right|.
\end{aligned}\tag{4.51}$$

Taking the supremum over $s\le t\wedge\tau_M$ in the last estimate, and taking the mathematical expectation in the resulting relation yields

$$\begin{aligned}
&\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|u^N(s)\right|_*^2+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\frac{2\nu}{\alpha}\left|u^N(s)\right|_*^2ds-\sum_{i=1}^N\left(\lambda_i+\lambda_i^2\right)\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(G\left(u^N(s),s\right),e_i\right)^2ds\\
&\quad\le\left|u_0^N\right|_*^2+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\frac{2\nu}{\alpha}\left|\operatorname{curl}\left(u^N(s)\right)\right|\left|u^N(s)\right|_*+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}2\left|\operatorname{curl}\left(F\left(u^N(s),s\right)\right)\right|\left|u^N(s)\right|_*ds\\
&\qquad+2\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|\int_0^{s\wedge\tau_M}\left(\operatorname{curl}\left(G\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)d\overline{W}\right|.
\end{aligned}\tag{4.52}$$

For any $\varepsilon_1\ge0$ and $\varepsilon_2\ge0$, we have

$$\begin{aligned}
&\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|u^N(s)\right|_*^2+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\frac{2\nu}{\alpha}\left|u^N(s)\right|_*^2ds-\sum_{i=1}^N\left(\lambda_i+\lambda_i^2\right)\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(G\left(u^N(s),s\right),e_i\right)^2ds\\
&\quad\le\left|u_0^N\right|_*^2+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(\frac{2\nu}{\alpha\varepsilon_1}\left|\operatorname{curl}\left(u^N(s)\right)\right|^2+\frac{2}{\varepsilon_2}\left|\operatorname{curl}\left(F\left(u^N(s),s\right)\right)\right|^2\right)ds\\
&\qquad+2\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|\int_0^{s\wedge\tau_M}\left(\operatorname{curl}\left(G\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)d\overline{W}\right|\\
&\qquad+\left(\frac{2\nu\varepsilon_1}{\alpha}+2\varepsilon_2\right)\int_0^{t\wedge\tau_M}\left|u^N(s)\right|_*^2ds.
\end{aligned}\tag{4.53}$$

We choose $\varepsilon_1=1/4$ and $\varepsilon_2=\nu/4\alpha$ and we deduce from the last inequality the following estimate,

$$\begin{aligned}
&\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|u^N(s)\right|_*^2+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\frac{\nu}{\alpha}\left|u^N(s)\right|_*^2ds-\sum_{i=1}^N\left(\lambda_i+\lambda_i^2\right)\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(G\left(u^N(s),s\right),e_i\right)^2ds\\
&\quad\le\left|u_0^N\right|_*^2+\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(\frac{8\nu}{\alpha\varepsilon_1}\left|\operatorname{curl}\left(u^N(s)\right)\right|^2+\frac{2\alpha}{\nu}\left|\operatorname{curl}\left(F\left(u^N(s),s\right)\right)\right|^2\right)ds\\
&\qquad+2\overline{\mathbb{E}}\sup_{s\le t\wedge\tau_M}\left|\int_0^{s\wedge\tau_M}\left(\operatorname{curl}\left(G\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)d\overline{W}\right|.
\end{aligned}\tag{4.54}$$

Thanks to (4.20), (4.29) and (4.17) we see that

$$\sum_{i=1}^{N}\left(\lambda_i+\lambda_i^2\right)\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left(G(u^N(s),s),e_i\right)^2 ds \leq C + C\overline{\mathbb{E}}\int_0^{s\wedge\tau_M}\left|u^N(s)\right|_{\mathbb{V}}^2 ds. \tag{4.55}$$

Let us estimate

$$\gamma = 2\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|\int_0^{t\wedge\tau_M}\left(\operatorname{curl}\left(G\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)d\overline{W}\right|. \tag{4.56}$$

By Fubini's theorem and the Burkhölder-Davis-Gundy's inequality we obtain

$$\begin{aligned}\gamma &\leq 6\overline{\mathbb{E}}\left(\int_0^{t\wedge\tau_M}\left(\operatorname{curl}\left(G\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)^2 d\overline{W}\right)^{1/2}\\ &\leq 6\overline{\mathbb{E}}\left(\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|_*\left(\int_0^{t\wedge\tau_M}\left|\operatorname{curl}\left(G\left(u^N(s),s\right)\right)\right|^2\right)^{1/2}\right).\end{aligned} \tag{4.57}$$

Making use of an ε-Young's inequality, the following holds:

$$\gamma \leq 6\varepsilon\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|_*^2 + \frac{6}{\varepsilon}\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|\operatorname{curl}\left(G\left(u^N(s),s\right)\right)\right|^2 ds. \tag{4.58}$$

Choosing $\varepsilon = 1/12$, we write

$$\gamma \leq \frac{1}{2}\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|_*^2 + 72\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|\operatorname{curl}\left(G\left(u^N(s),s\right)\right)\right|^2 ds. \tag{4.59}$$

Combining (4.54), (4.55), and (4.59), we obtain

$$\begin{aligned}&\overline{\mathbb{E}}\sup_{s\leq t\wedge\tau_M}\left|u^N(s)\right|_*^2 + \overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\frac{\nu}{\alpha}\left|u^N(s)\right|_*^2 ds\\ &\quad\leq C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|\operatorname{curl}\left(F\left(u^N(s),s\right)\right)\right|^2 + C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|\operatorname{curl}\left(G\left(u^N(s),s\right)\right)\right|^2 ds\\ &\quad\quad+C\left|u_0^N\right|_*^2 + C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|u^N(s)\right|_{\mathbb{V}}^2 ds + C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|\operatorname{curl}\left(u^N(s)\right)\right|^2 ds.\end{aligned} \tag{4.60}$$

By a straightforward calculation we have

$$\left|\operatorname{curl}(\phi)\right|^2 \leq \frac{2}{\alpha}|\phi|_{\mathbb{V}}^2 \quad \text{for any } \phi\in\mathbb{V}. \tag{4.61}$$

Owing to (4.61) and the assumptions on F and G, we derive from (4.60) that

$$\overline{\mathbb{E}} \sup_{s\le t\wedge\tau_M} \left|u^N(s)\right|_*^2 + \overline{\mathbb{E}}\int_0^{t\wedge\tau_M} \frac{\nu}{\alpha}\left|u^N(s)\right|_*^2 ds \le C + \left|u_0^N\right|_*^2 + C\overline{\mathbb{E}}\int_0^{t\wedge\tau_M}\left|u^N(s)\right|_{\mathbb{V}}^2 ds. \tag{4.62}$$

This and the estimate (4.28) imply

$$\overline{\mathbb{E}} \sup_{s\le t\wedge\tau_M} \left|u^N(s)\right|_*^2 + \overline{\mathbb{E}}\int_0^{t\wedge\tau_M} \frac{\nu}{\alpha}\left|u^N(s)\right|_*^2 ds \le C. \tag{4.63}$$

It is easy to check that, as $M \to \infty$, $t\wedge\tau_M \to t$ almost surely for any $t \in (0, T_N]$. Since the constant C is independent of N, the estimates (4.28), (4.63) and the Dominated Lebesgue's Convergence Theorem complete the proof of the lemma. □

Lemma 4.3. *For any* $4 \le p < \infty$ *one has*

$$\begin{aligned} &\overline{\mathbb{E}}\sup_{s\le T}\left|u^N(s)\right|_{\mathbb{V}}^p < \infty, \\ &\overline{\mathbb{E}}\sup_{s\le T}\left|u^N(s)\right|_{\mathbb{W}}^p < \infty. \end{aligned} \tag{4.64}$$

Proof. We recall that

$$\begin{aligned} &d\left|u^N(t)\right|_{\mathbb{V}}^2 + 2\nu\left\|u^N(t)\right\|^2 dt - 2\left(F(u^n(t),t),u^N(t)\right)dt \\ &\quad = \sum_{i=1}^N \lambda_i\left(G\left(u^N(t),t\right),e_i\right)^2 dt + 2\left(G\left(u^N(t),t\right),u^N(t)\right)d\overline{W}. \end{aligned} \tag{4.65}$$

For a fixed $p \ge 4$ the application of Itô's formula to the function $\phi(|u^N(t)|_{\mathbb{V}}^2) = |u^N(t)|_{\mathbb{V}}^{2(p/4)}$ yields

$$\begin{aligned} d\left|u^N(t)\right|_{\mathbb{V}}^{p/2} = {}& \frac{p}{2}\left|u^N(t)\right|_{\mathbb{V}}^{p/2-2}\Bigg[-\nu\left\|u^N(t)\right\|^2 + \left(F(u^n(t),t),u^N(t)\right) \\ &+ \frac{1}{2}\sum_{i=1}^N\left(G\left(u^N(t),t\right),e_i\right)^2 + \frac{p-4}{4}\frac{\left(G(u^N(t),t),u^N(t)\right)^2}{|u^N(t)|_{\mathbb{V}}^2}\Bigg]dt \\ &+ \frac{p}{2}\left|u^N(t)\right|_{\mathbb{V}}^{p/2-2}\left(G\left(u^N(t),t\right),u^N(t)\right)d\overline{W}. \end{aligned} \tag{4.66}$$

Hence

$$\begin{aligned}\left|u^N(t)\right|_{\mathbb{V}}^{p/2} = \left|u_0^N\right|_{\mathbb{V}}^{p/2} + \frac{p}{2}\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p/2-2} \\ \times \Bigg[-\nu\left\|u^N(s)\right\|^2 + \left(F(u^n(s),s),u^N(s)\right) \\ + \frac{1}{2}\sum_{i=1}^{N}\left(G\left(u^N(s),s\right),e_i\right)^2 + \frac{p-4}{4}\frac{(G(u^N(s),s),u^N(s))^2}{|u^N(s)|_{\mathbb{V}}^2}\Bigg]ds \\ + \frac{p}{2}\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p/2-2}\left(G\left(u^N(s),s\right),u^N(s)\right)d\overline{W},\end{aligned} \tag{4.67}$$

for any $t \in (0,T]$. In squaring the last equation and in making use of some elementary inequalities we obtain

$$\begin{aligned}\left|u^N(t)\right|_{\mathbb{V}}^{p} \leq C\left|u_0^N\right|_{\mathbb{V}}^{p} + C\Bigg(\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p/2-2} \\ \times \Bigg[-\nu\left\|u^N(s)\right\|^2 + \left(F(u^n(s),s),u^N(s)\right) \\ + \frac{1}{2}\sum_{i=1}^{N}(G\left(u^N(s),s\right),e_i)^2 + \frac{p-4}{4}\frac{(G(u^N(s),s),u^N(s))^2}{|u^N(s)|_{\mathbb{V}}^2}\Bigg]ds\Bigg)^2 \\ + C\left(\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p/2-2}\left(G\left(u^N(s),s\right),u^N(s)\right)d\overline{W}\right)^2.\end{aligned} \tag{4.68}$$

We deduce from (4.14), (4.20), and (4.68) that

$$\begin{aligned}\left|u^N(t)\right|_{\mathbb{V}}^{p} \leq C\left|u_0^N\right|_{\mathbb{V}}^{p} + C\left(\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p/2-2}\left(1+\left|u^N(s)\right|_{\mathbb{V}}\right)^2\right)^2 \\ + C\left(\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p/2-2}\left(G\left(u^N(s),s\right),u^N(s)\right)d\overline{W}\right)^2.\end{aligned} \tag{4.69}$$

We find from this that

$$\begin{aligned}\overline{\mathbb{E}}\sup_{s\leq t}\left|u^N(s)\right|_{\mathbb{V}}^{p} \leq C\left|u_0^N\right|_{\mathbb{V}}^{p} + C\overline{\mathbb{E}}\int_0^t \left|u^N(s)\right|_{\mathbb{V}}^{p-4}\left(1+\left|u^N(s)\right|_{\mathbb{V}}\right)^4 ds \\ + C\overline{\mathbb{E}}\sup_{s\leq t}\left(\int_0^s \left|u^N(r)\right|_{\mathbb{V}}^{p/2-2}\left(G\left(u^N(r),r\right),u^N(r)\right)d\overline{W}\right)^2.\end{aligned} \tag{4.70}$$

It is clear that

$$\left|u^N(s)\right|_{\mathbb{V}}^{p-4} \le \left(1+\left|u^N(s)\right|_{\mathbb{V}}\right)^{p-4}. \tag{4.71}$$

Hence,

$$\begin{aligned}\overline{\mathbb{E}}\sup_{s\le t}\left|u^N(s)\right|_{\mathbb{V}}^{p} &\le C\left|u_0^N\right|_{\mathbb{V}}^{p} + C\overline{\mathbb{E}}\int_0^t\left(1+\left|u^N(s)\right|_{\mathbb{V}}\right)^p ds \\ &\quad + C\overline{\mathbb{E}}\sup_{s\le t}\left(\int_0^s\left|u^N(r)\right|_{\mathbb{V}}^{p/2-2}\left(G\left(u^N(r),r\right),u^N(r)\right)d\overline{W}\right)^2.\end{aligned} \tag{4.72}$$

Now let us denote by γ_1 the stochastic term in (4.70). As before, we use the Burkhölder-Davis-Gundy's inequality and get

$$\begin{aligned}\gamma_1 &\le C\overline{\mathbb{E}}\int_0^t\left|u^N(s)\right|_{\mathbb{V}}^{p-4}\left(G\left(u^N(s),s\right),u^N(s)\right)^2 ds \\ &\le C\overline{\mathbb{E}}\int_0^t\left|u^N(s)\right|_{\mathbb{V}}^{p-4}\left|G\left(u^N(s),s\right)\right|_{\mathbb{V}}^{2}\left|u^N(s)\right|_{\mathbb{V}}^{2}.\end{aligned} \tag{4.73}$$

The following follows from the same arguments as used before and by the assumption on G:

$$\gamma_1 \le C\overline{\mathbb{E}}\int_0^t\left(1+\left|u^N(s)\right|_{\mathbb{V}}\right)^p ds. \tag{4.74}$$

This, the estimate (4.70), and Gronwall's inequality imply

$$\overline{\mathbb{E}}\sup_{s\le t}\left|u^N(s)\right|_{\mathbb{V}}^{p} < \infty, \tag{4.75}$$

which completes the proof of the first estimate of the lemma.

Let us now proceed to the proof of the second estimate of Lemma 4.3. We rewrite (4.49) in the form

$$\begin{aligned}d\left|u^N(s)\right|_{*}^{2} &= \sum_{i=1}^{N}\left(\lambda_i+\lambda_i^2\right)\left(G\left(u^N(s),s\right),e_i\right)^2 + \frac{2\nu}{\alpha}\left(\operatorname{curl}u^N(s),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)ds \\ &\quad + \left(-\frac{2\nu}{\alpha}\left|u^N(s)\right|_{*}^{2} + 2\left(\operatorname{curl}\left(F\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)\right) \\ &\quad + 2\left(\operatorname{curl}\left(G\left(u^N(s),s\right)\right),\operatorname{curl}\left(u^N(s)-\alpha\Delta u^N(s)\right)\right)d\overline{W}.\end{aligned} \tag{4.76}$$

Applying Itô formula to the function $\varphi(|u^N(s)|_*^2) = |u^N(s)|_*^{2(p/4)}$ we have

$$
\begin{aligned}
d\left|u^N(s)\right|_*^{p/2} &- \frac{p}{2}\left|u^N(s)\right|_*^{p/2-2} \\
&\times \Bigg(2\Big(\operatorname{curl}\Big(F\Big(u^N(s),s\Big)\Big),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big) \\
&\qquad = +\frac{1}{2}\sum_{i=1}^{N}\Big(\lambda_i+\lambda_i^2\Big)\Big(G\Big(u^N(s),s\Big),e_i\Big)^2 + \frac{2\nu}{\alpha}\Big(\operatorname{curl} u^N(s),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big) \\
&\qquad - \frac{2\nu}{\alpha}\left|u^N(s)\right|_*^2 + \frac{p-4}{4}\frac{\left(\operatorname{curl}(G(u^N(s),s),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{\left|u^N(s)\right|_*^2}\Bigg)ds \\
&+ \frac{p}{2}\left|u^N(s)\right|_*^{p/2-2}\Big(\operatorname{curl}\Big(G\Big(u^N(s),s\Big)\Big),\ \operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)d\overline{W}.
\end{aligned}
\tag{4.77}
$$

Hence,

$$
\begin{aligned}
\left|u^N(t)\right|_*^{p/2} = \left|u_0^N\right|_*^{p/2} &+ \frac{p}{2}\int_0^t\left|u^N(s)\right|_*^{p/2-2} \\
&\times \Bigg(2\Big(\operatorname{curl}\Big(F\Big(u^N(s),s\Big)\Big),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big) \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\Big(\lambda_i+\lambda_i^2\Big)\Big(G\Big(u^N(s),s\Big),e_i\Big)^2 \\
&\quad + \frac{2\nu}{\alpha}\Big(\operatorname{curl} u^N(s),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big) \\
&\quad - \frac{2\nu}{\alpha}\left|u^N(s)\right|_*^2 + \frac{p-4}{4}\frac{\left(\operatorname{curl}\left(G(u^N(s),s)\right),\ \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{\left|u^N(s)\right|_*^2}\Bigg)ds \\
&+ \frac{p}{2}\int_0^t\left|u^N(s)\right|_*^{p/2-2}\Big(\operatorname{curl}\Big(G\Big(u^N(s),s\Big)\Big),\ \operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)d\overline{W},
\end{aligned}
\tag{4.78}
$$

for any $t \in (0,T]$. The following follows in squaring both sides of the last inequality:

$$
\begin{aligned}
\left|u^N(t)\right|_*^p \le C\left|u_0^N\right|_*^p + C\Bigg[\int_0^t \left|u^N(s)\right|_*^{p/2-2} \\
&\times \Bigg(2\Big(\operatorname{curl}\Big(F\Big(u^N(s),s\Big)\Big),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big) \\
&+\frac{1}{2}\sum_{i=1}^N\Big(\lambda_i+\lambda_i^2\Big)\Big(G\Big(u^N(s),s\Big),e_i\Big)^2 \\
&+\frac{2\nu}{\alpha}\Big(\operatorname{curl} u^N(s),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)-\frac{2\nu}{\alpha}\left|u^N(s)\right|_*^2 \\
&+\frac{p-4}{4}\frac{\left(\operatorname{curl}\left(G(u^N(s),s)\right),\ \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{\left|u^N(s)\right|_*^2}\Bigg)ds\Bigg]^2 \\
&+C\left(\int_0^t\left|u^N(s)\right|_*^{p/2-2}\Big(\operatorname{curl}\Big(G\Big(u^N(s),s\Big)\Big),\ \operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)d\overline{W}\right)^2.
\end{aligned}
\tag{4.79}
$$

For almost all $s \in (0,T]$, we note that

$$
\left|\Big(\operatorname{curl} u^N(s),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)\right| \le C\Big(1+\left|u^N(s)\right|_{\mathbb{V}}\Big)\Big(1+\left|u^N(s)\right|_{\mathbb{W}}\Big). \tag{4.80}
$$

We also check readily that

$$
\left|\Big(\operatorname{curl}\Big(F\Big(u^N(s),s\Big)\Big),\operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)\right| \le C\Big(1+\left|u^N(s)\right|_{\mathbb{V}}\Big)\Big(1+\left|u^N(s)\right|_{\mathbb{W}}\Big), \tag{4.81}
$$

$$
\left|\frac{\left(\operatorname{curl}\left(G(u^N(s),s)\right),\ \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{\left|u^N(s)\right|_*^2}\right| \le C\left(1+\left|u^N(s)\right|_{\mathbb{V}}\right)^2. \tag{4.82}
$$

Thanks to the continuous injection of $\mathbb{W}$ into $\mathbb{V}$, all the above estimates still hold with $|u^N(\cdot)|_{\mathbb{V}}$ replaced by $|u^N(\cdot)|_{\mathbb{W}}$. It follows from this argument and (4.79) that

$$
\begin{aligned}
\left|u^N(t)\right|_*^p \le \left|u_0^N\right|_*^p + C\left(\int_0^t\left|u^N(s)\right|_*^{p/2-2}\Big(1+\left|u^N(s)\right|_{\mathbb{W}}\Big)^2 ds\right)^2 \\
+C\left(\int_0^t\left|u^N(s)\right|_*^{p/2-2}\Big(\operatorname{curl}\Big(G\Big(u^N(s),s\Big)\Big),\ \operatorname{curl}\Big(u^N(s)-\alpha\Delta u^N(s)\Big)\Big)d\overline{W}\right)^2.
\end{aligned}
\tag{4.83}
$$

Taking the supremum over $s \le t$ followed by the mathematical expectation yields

$$\begin{aligned}
&\overline{\mathbb{E}} \sup_{s\le t} \left| u^N(s) \right|_*^p \\
&\quad \le \left| u_0^N \right|_*^p + C\overline{\mathbb{E}} \left(\int_0^t \left| u^N(s) \right|_*^{p/2-2} \left(1 + \left| u^N(s) \right|_{\mathbb{W}} \right)^2 ds \right)^2 \\
&\qquad + C\overline{\mathbb{E}} \sup_{s\le t} \left(\int_0^s \left| u^N(r) \right|_*^{p/2-2} \left(\operatorname{curl}\left(G\left(u^N(r), r \right) \right), \operatorname{curl}\left(u^N(r) - \alpha \Delta u^N(r) \right) \right) d\overline{W} \right)^2 .
\end{aligned} \tag{4.84}$$

Applying the Martingale inequality and Hölder's inequality in the last estimate we obtain

$$\begin{aligned}
\overline{\mathbb{E}} \sup_{s\le t} \left| u^N(s) \right|_*^p \le \left| u_0^N \right|_*^p &+ C\overline{\mathbb{E}} \int_0^t \left| u^N(s) \right|_*^{p-4} \left(1 + \left| u^N(s) \right|_{\mathbb{W}} \right)^4 ds \\
&+ C\overline{\mathbb{E}} \int_0^t \left| u^N(s) \right|_*^{p-4} \left(\operatorname{curl}\left(G\left(u^N(s), s \right) \right), \ \operatorname{curl}\left(u^N(s) - \alpha \Delta u^N(s) \right) \right)^2 ds.
\end{aligned} \tag{4.85}$$

We can use the same idea we have used to find (4.81) to get an upper bound of the form $C(1+|u^N(s)|_{\mathbb{W}})^4$ for $|(\operatorname{curl}(G(u^N(s), s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))^2|$. Then, we derive from (4.85) that

$$\overline{\mathbb{E}} \sup_{s\le t} \left| u^N(s) \right|_*^p \le C \left| u_0^N \right|_*^p + C \int_0^t \left(1 + \left| u^N(s) \right|_{\mathbb{W}} \right)^p ds. \tag{4.86}$$

We obviously have

$$\left| u^N(s) \right|_{\mathbb{W}} \le C \left(\left| u^N(s) \right|_{\mathbb{V}}^p + \left| u^N(s) \right|_*^p \right). \tag{4.87}$$

Finally, using a previous result concerning $\overline{\mathbb{E}} \sup_{s\le t} |u^N(s)|_{\mathbb{V}}^p$, (4.86) and Gronwall's inequality we obtain

$$\overline{\mathbb{E}} \sup_{s\le t} \left| u^N(s) \right|_*^p < \infty. \tag{4.88}$$

This completes the proof of the lemma. □

Remark 4.4. Lemmas 4.2 and 4.3 imply in particular that

$$\begin{aligned}
&\overline{\mathbb{E}} \sup_{t\le T} \left| u^N(t) \right|_{\mathbb{V}}^p < \infty, \\
&\overline{\mathbb{E}} \sup_{t\le T} \left| u^N(t) \right|_{\mathbb{W}}^p < \infty,
\end{aligned} \tag{4.89}$$

for any $1 \le p < \infty$.

The following result is central in the proof of the forthcoming crucial estimate of the finite difference of our approximating solution.

Lemma 4.5. *Let $t, s \in [0,T]$ such that $s \le t$. For a fixed $t \in [0,T]$, let*

$$v^N(t) = \sum_{i=1}^{N} \lambda_i \left(v^N(t), e_i\right)_{\mathbb{V}} e_i \tag{4.90}$$

be an element of $\mathbb{W}_N$ which satisfies Lemmas 4.2 and 4.3. The following holds:

$$\begin{aligned}
&\left|u^N(t) - v^N(s)\right|_{\mathbb{V}}^2 - \left|u^N(s) - v^N(s)\right|_{\mathbb{V}}^2 + 2\int_s^t \nu\left[\left\|u^N(r)\right\|^2 - \left(\left(u^N(r), v^N(r)\right)\right)\right] dr \\
&\quad = 2\int_s^t \left(G\left(u^N(r), r\right), u^N(r) - v^N(s)\right) d\overline{W} + \sum_{i=1}^{N} \lambda_i \int_s^t \left(G\left(u^N(r), r\right), e_i\right)^2 dr \\
&\qquad + 2\int_s^t b\left(v^N(s), \Delta u^N(r), u^N(r)\right) dr + 2\int_s^t \left(F\left(u^N(r), r\right), u^N(r) - v^N(s)\right) dr \\
&\qquad - 2\int_s^t b\left(u^N(r), u^N(r), v^N(s)\right) dr - 2\int_s^t b\left(u^N(r), \Delta u^N(r), v^N(s)\right) dr.
\end{aligned} \tag{4.91}$$

Proof. For v^N, for any s, t such that $0 \le s \le t \le T$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(u^N(t) - v^N(s), e_i\right)_{\mathbb{V}} + \nu\left(\left(u^N(t), e_i\right)\right) + b\left(u^N(t), u^N(t), e_i\right) \\
&\quad - \alpha b\left(u^N(t), \Delta u^N(t), e_i\right) + \alpha b\left(e_i, \Delta u^N(t), u^N(t)\right) \\
&\quad = \left(F\left(u^N(t), t\right), e_i\right) + \left(G\left(u^N(t), t\right), e_i\right) \frac{d}{dt}\overline{W}, \quad 1 \le i \le N.
\end{aligned} \tag{4.92}$$

This relation can be rewritten as the following Itô equation:

$$\begin{aligned}
&d\left(u^N(t) - v^N(s), e_i\right)_{\mathbb{V}} + \nu\left(\left(u^N(t), e_i\right)\right) dt + b\left(u^N(t), u^N(t), e_i\right) dt \\
&\quad - \alpha b\left(u^N(t), \Delta u^N(t), e_i\right) dt + \alpha b\left(e_i, \Delta u^N(t), u^N(t)\right) dt \\
&\quad = \left(F\left(u^N(t), t\right), e_i\right) dt + \left(G\left(u^N(t), t\right), e_i\right) d\overline{W}.
\end{aligned} \tag{4.93}$$

Applying Itô's formula to the function $(u^N(t), v^N(s), e_i)^2_{\mathbb{V}}$, multiplying the result by λ_i, and then summing over i from 1 to N yield

$$\begin{aligned}
&d|w|^2_{\mathbb{V}} + 2\nu\Big(\Big(u^N(t), w\Big)\Big)dt + 2b\Big(u^N(t), u^N(t), w\Big)dt - 2\alpha b\Big(u^N(t), \Delta u^N(t), w\Big)dt \\
&\quad + 2\alpha b\Big(w, \Delta u^N(t), u^N(t)\Big)dt - 2\Big(F\Big(u^N(t), t\Big), w\Big)dt \\
&\quad = \sum_{i=1}^{N}\lambda_i\Big(G\Big(u^N(t), t\Big), e_i\Big)^2 + 2\Big(G\Big(u^N(t), t\Big), w\Big)d\overline{W},
\end{aligned} \tag{4.94}$$

where $w = u^N(t) - v^N(s)$. Using the trilinearity of b and the well-known identity $b(u, u, u) = 0$, $u \in \mathbb{V}$, we find that

$$\begin{aligned}
&b\Big(u^N(t), u^N(t), w\Big) - \alpha b\Big(u^N(t), \Delta u^N(t), w\Big) + \alpha b\Big(w, \Delta u^N(t), u^N(t)\Big) \\
&\quad = b\Big(u^N(t), u^N(t), v^N(s)\Big) - \alpha b\Big(u^N(t), \Delta u^N(t), v^N(s)\Big) + \alpha b\Big(v^N(s), \Delta u^N(t), u^N(t)\Big).
\end{aligned} \tag{4.95}$$

The lemma follows in combining this relation with (4.94), and integrating the resulting equation between s and t.

The following result can be proved by a similar argument used in [15], but we prefer to give our own proof which is interesting in itself. □

Lemma 4.6. *There exists a positive constant $C > 0$ such that for all $0 \le \delta < 1$ and $N \in \mathbb{N}$, the following inequality holds:*

$$\overline{\mathbb{E}} \sup_{|\theta| \le \delta} \int_0^{T-\delta} \Big| u^N(s+\theta) - u^N(s) \Big|^2_{\mathbb{W}^*} \le C\delta^{1/2}. \tag{4.96}$$

Proof. Since $u^N(s) \in \mathbb{W}_N$, $s \in (0, T)$ and it satisfies Lemmas 4.2 and 4.3 then we can take $v^N(s) = u^N(s)$ and $t = s + \theta$, $0 \le \theta \le \delta \le 1$ and apply Lemma 4.5. We obtain

$$\begin{aligned}
&\Big| u^N(s+\theta) - u^N(s) \Big|^2_{\mathbb{V}} + 2\int_s^{s+\theta} \nu\Big[\Big\| u^N(r) \Big\|^2 - \Big(\Big(u^N(r), u^N(r)\Big)\Big)\Big]dr \\
&\quad = 2\int_s^{s+\theta} \Big(G\Big(u^N(r), r\Big), u^N(r) - u^N(s)\Big)d\overline{W} + \sum_{i=1}^{N}\lambda_i \int_s^{s+\theta} \Big(G\Big(u^N(r), r\Big), e_i\Big)^2 dr \\
&\qquad + 2\int_s^{s+\theta} b\Big(u^N(s), \Delta u^N(r), u^N(r)\Big)dr + 2\int_s^{s+\theta} \Big(F\Big(u^N(r), r\Big), u^N(r) - u^N(s)\Big)dr \\
&\qquad - 2\int_s^{s+\theta} b\Big(u^N(r), u^N(r), u^N(s)\Big)dr - 2\int_s^{s+\theta} b\Big(u^N(r), \Delta u^N(r), u^N(s)\Big)dr.
\end{aligned} \tag{4.97}$$

We derive that

$$\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}^2 \le I_1+I_2+I_3+I_4+I_5+I_6+I_7, \tag{4.98}$$

where

$$\begin{aligned}
I_1 &= 2\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_s^{s+\theta}\left(G\left(u^N(r),r\right),u^N(r)-u^N(s)\right)d\overline{W}\right|,\\
I_2 &= \overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\sum_{i=1}^N\lambda_i\int_s^{s+\theta}\left(G\left(u^N(r),r\right),e_i\right)^2dr\right|,\\
I_3 &= \overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|2\int_s^{s+\theta}\nu\left[\left\|u^N(r)\right\|+\left(\left(u^N(r),u^N(r)\right)\right)\right]dr\right|,\\
I_4 &= 2\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_s^{s+\theta}b\left(u^N(s),\Delta u^N(r),u^N(r)\right)dr\right|,\\
I_5 &= 2\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_s^{s+\theta}b\left(u^N(r),\Delta u^N(r),u^N(s)\right)dr\right|,\\
I_6 &= 2\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_s^{s+\theta}b\left(u^N(r),u^N(r),u^N(s)\right)dr\right|,\\
I_7 &= 2\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_s^{s+\theta}\left(F\left(u^N(r),r\right),u^N(r)-u^N(s)\right)dr\right|.
\end{aligned} \tag{4.99}$$

The proof of the lemma will consist of the following five steps.

Step 1 (estimate of I_3). Owing to the equivalence of the two norms $\|\cdot\|$ and $|\cdot|_{\mathbb{V}}$ we see that I_3 is dominated by

$$2C\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}\left(\overline{\mathbb{E}}\int_s^{s+\delta}\left|u^N(r)\right|_{\mathbb{V}}dr\right)ds. \tag{4.100}$$

We find from this and by a successive application of Cauchy-Schwarz's inequality that

$$I_3\le C\delta\left(\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}^2ds\right)^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\le r\le T}\left|u^N(r)\right|_{\mathbb{V}}^2dr\right)^{1/2}. \tag{4.101}$$

Since $s+\theta\in[0,T]$ for any $s\in[0,T-\delta]$ and any $0\le\theta\le\delta$, we get using the triangle inequality and Lemma 4.2 that

$$I_3\le C\delta. \tag{4.102}$$

Step 2 (estimates for I_4, I_5 and I_6). The estimate (2.16) and a successive application of Cauchy-Schwarz's inequality yield

$$I_4 \leq C\delta^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\left|u^N(s)\right|_{\mathbb{W}}^2 ds\right)^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\int_s^{s+\delta}\left|u^N(r)\right|_{\mathbb{V}}^4 dr ds\right)^{1/2}. \tag{4.103}$$

By Cauchy's inequality, we have

$$I_4 \leq C\delta^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\left|u^N(s)\right|_{\mathbb{W}}^2 ds + \overline{\mathbb{E}}\int_0^{T-\delta}\int_s^{s+\delta}\left|u^N(r)\right|_{\mathbb{V}}^4 dr ds\right). \tag{4.104}$$

Thanks to Lemmas 4.2 and 4.3, we derive from the latter estimate that

$$I_4 \leq C\delta^{1/2}. \tag{4.105}$$

Similar estimates hold for I_5 and I_6.

Step 3 (estimate for I_7). Thanks to the idea used in the proof of the estimate (4.102), we have that the quantity

$$C\delta^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\leq\theta\leq\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}^2 ds\right)^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\int_s^{s+\delta}\left|F\left(u^N(r),r\right)\right|_{\mathbb{V}}^2 dr ds\right)^{1/2}, \tag{4.106}$$

dominates I_7. By the assumption on F and the argument used in deriving (4.102) we have

$$I_7 \leq C\delta. \tag{4.107}$$

Step 4 (estimate for I_2). We use the same argument as used in the proof of Lemma 4.2 to get an estimate of the form

$$\sum_{i=1}^N \lambda_i\left(G\left(u^N(r),r\right),e_i\right) \leq C\left(1+\left|u^N(r)\right|_{\mathbb{V}}^2\right). \tag{4.108}$$

We derive from the definition of I_2, the latter estimate and Lemma 4.2 that

$$I_2 \leq C\delta. \tag{4.109}$$

Step 5 (estimate for I_1). Thanks to Fubini's Theorem and the Burkhölder-Davis-Gundy inequality we have

$$I_1 \leq 6\int_0^{T-\delta}\overline{\mathbb{E}}\left(\int_s^{s+\delta}\left(G\left(u^N(r),r\right),u^N(r)-u^N(s)\right)^2 dr\right)^{1/2} ds. \tag{4.110}$$

By a sequence of Cauchy-Schwarz's inequality we find that I_1 is bounded from above by

$$C\left(\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\leq\theta\leq\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}^2 ds\right)^{1/2}\left(\overline{\mathbb{E}}\int_0^{T-\delta}\int_s^{s+\delta}\left|G\left(u^N(r),r\right)\right|_{\mathbb{V}}^2 dr ds\right)^{1/2}. \tag{4.111}$$

By the assumption on G and by Lemma 4.2 we get from the latter equation that

$$I_1 \leq C\delta^{1/2}. \tag{4.112}$$

Combining all the estimates in Steps 1–5 we get,

$$\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\leq\theta\leq\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}^2 ds \leq C\delta^{1/2}. \tag{4.113}$$

Since $\mathbb{V}$ is continuously embedded in $\mathbb{W}^*$,

$$\begin{aligned}\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\leq\theta\leq\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{W}^*}^2 ds &\leq C\overline{\mathbb{E}}\int_0^{T-\delta}\sup_{0\leq\theta\leq\delta}\left|u^N(s+\theta)-u^N(s)\right|_{\mathbb{V}}^2 ds\\ &\leq C\delta^{1/2}.\end{aligned} \tag{4.114}$$

The lemma follows readily from this last inequality and noting that a similar argument can be carried out to find a similar estimate for negative values of θ. □

4.3. *Tightness Property and Application of Prokhorov's and Skorokhod's Theorems*

We denote by $\mathfrak{Z}$ the following subset of $L^2(0,T;\mathbb{V})$:

$$\mathfrak{Z} = \left\{z \in L^\infty(0,T;\mathbb{W}) \cap L^\infty(0,T;\mathbb{V});\ \sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|z(t+\theta)-z(t)|_{\mathbb{W}^*}^2 \leq C\nu_M\right\}, \tag{4.115}$$

for any sequences ν_M, μ_M such that $\nu_M, \mu_M \to 0$ as $M \to \infty$. The following result is a version of Theorem 2.6 due to Bensoussan [51].

Lemma 4.7. *The set $\mathfrak{Z}$ is compact in $L^2(0,T;\mathbb{V})$.*

Next we consider the space $\mathfrak{S} = C(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{V})$ endowed with its Borel σ-algebra $\mathcal{B}(\mathfrak{S})$ and the family of probability measures $\mathfrak{P}^N$ on $\mathfrak{S}$, which is the probability measure induced by the following mapping:

$$\phi : w \longmapsto \left(\overline{W}(w,\cdot), u^N(w,\cdot)\right), \tag{4.116}$$

that is, for any $A \in \mathcal{B}(\mathfrak{S})$, $\mathfrak{P}^N(A) = \overline{\mathbb{P}}(\phi^{-1}(A))$.

We have the following lemma.

Lemma 4.8. *The family* $(\mathfrak{P}^N)_{N\geq 1}$ *is tight.*

Proof. For any $\varepsilon > 0$ and $M \geq 1$, we claim that there exists a compact subset $\mathfrak{K}_\varepsilon$ of $\mathfrak{S}$ such that $\mathfrak{P}^N(\mathfrak{K}_\varepsilon) \geq 1 - \varepsilon$. To prove our claim we define the sets

$$\mathfrak{W}_\varepsilon = \left\{ W : \sup_{\substack{t,s\in[0,T]\\ |t-s|<T/2^M}} 2^{M/8}|W(t) - W(s)| \leq J_\varepsilon,\ \forall M \right\},$$

$$\mathfrak{Z}_\varepsilon = \left\{ z; \sup_{t\leq T}|z(t)|^2_{\mathbb{V}} \leq K_\varepsilon,\ \sup_{t\leq T}|z(t)|^2_{\mathbb{W}} \leq L_\varepsilon,\ \sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|z(t+\theta) - z(t)|^2_{\mathbb{W}^*} \leq R_\varepsilon\nu_M \right\}, \tag{4.117}$$

where the sequences ν_M and μ_M are chosen so that they are independent of ε, $\nu_M, \mu_M \to 0$ as $M \to \infty$ and $\sum_M \sqrt{\mu_M}/\nu_M < \infty$. It is clear by Ascoli-Arzela's Theorem that $\mathfrak{W}_\varepsilon$ is a compact subset of $C(0,T;\mathbb{R}^m)$, and by Lemma 4.7, $\mathfrak{Z}_\varepsilon$ is a compact subset of $L^2(0,T;\mathbb{V})$. We have to show that $\mathfrak{A}_\varepsilon = \mathfrak{P}^N((\overline{W}, u^N) \notin \mathfrak{W}_\varepsilon \times \mathfrak{Z}_\varepsilon) < \varepsilon$. Indeed, we have

$$\begin{aligned}\mathfrak{A}_\varepsilon \leq{}& \overline{\mathbb{P}}\left[\bigcup_{M=1}^{\infty}\bigcup_{j=1}^{2^M}\left(\sup_{t,s\in I_j}\left|\overline{W}(t) - \overline{W}(s)\right| \geq J_\varepsilon\frac{1}{2^{M/8}}\right)\right] + \overline{\mathbb{P}}\left(\sup_{t\leq T}\left|u^N(t)\right|^2_{\mathbb{V}} \geq K_\varepsilon\right)\\ &+ \overline{\mathbb{P}}\left(\bigcup_M\left\{\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}\left|u^N(t+\theta) - u^N(t)\right|^2_{\mathbb{W}^*} \geq R_\varepsilon\nu_M\right\}\right) + \overline{\mathbb{P}}\left(\sup_{t\leq T}\left|u^N(t)\right|^2_{\mathbb{W}} \geq L_\varepsilon\right),\end{aligned} \tag{4.118}$$

where $\{I_j : 1 \leq j \leq 2^M\}$ is a family of intervals of length $T/2^M$ which forms a partition of the interval $[0,T]$. It is well-known that for any Wiener process B

$$\overline{\mathbb{E}}|B(t) - B(s)|^{2m} = C_m|t-s|^m \quad \text{for any } m \geq 1, \tag{4.119}$$

where C_m is a constant depending only on m. From this and Markov's Inequality

$$\overline{\mathbb{P}}(\omega : \zeta(w) \geq \alpha) \leq \frac{1}{\alpha^k}\overline{\mathbb{E}}\left(|\zeta(\omega)|^k\right), \tag{4.120}$$

where $\zeta(\omega)$ is a random variable on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ and positive numbers k and α, we obtain

$$\begin{aligned}\mathfrak{A}_\varepsilon \leq{}& \sum_{M=1}^{\infty}\sum_{j=1}^{2^M} C_m\left(2^{M/8}\right)^{2m}\frac{1}{J_\varepsilon^{2m}}\left(\frac{T}{2^M}\right)^m + \frac{1}{K_\varepsilon}\overline{\mathbb{E}}\sup_{t\leq T}\left|u^N(t)\right|^2_{\mathbb{V}} + \frac{1}{L_\varepsilon}\overline{\mathbb{E}}\sup_{t\leq T}\left|u^N(t)\right|^2_{\mathbb{W}}\\ &+ \sum_M\frac{1}{R_\varepsilon\nu_M}\overline{\mathbb{E}}\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}\left|u^N(t+\theta) - u^N(t)\right|^2_{\mathbb{W}^*}.\end{aligned} \tag{4.121}$$

Owing to Lemmas 4.2, 4.6 and by choosing $m = 2$, we have

$$\begin{aligned}\mathfrak{A}_\varepsilon &\le \frac{C_2T^2}{L_\varepsilon^4}\sum_{M=1}^{\infty}2^{-(1/2)M} + C\left(\frac{1}{K_\varepsilon} + \frac{1}{L_\varepsilon} + \frac{1}{R_\varepsilon}\sum_M \frac{\sqrt{\mu_M}}{\nu_M}\right) \\ &\le \frac{C_2T^2}{J_\varepsilon^4}\left(2+\sqrt{2}\right) + C\left(\frac{1}{K_\varepsilon} + \frac{1}{L_\varepsilon} + \frac{1}{R_\varepsilon}\sum_M \frac{\sqrt{\mu_M}}{\nu_M}\right).\end{aligned} \tag{4.122}$$

A convenient choice of J_ε, K_ε, L_ε, R_ε completes the proof of the claim, and hence the proof of the lemma. □

It follows by Prokhorov's Theorem (Theorem 2.1) that the family $(\mathfrak{P}^N)_{N\ge1}$ is relatively compact in the set of probability measures (equipped with the weak convergence topology) on $\mathfrak{S}$. Then, we can extract a subsequence $\mathfrak{P}^{N_\mu}$ that weakly converges to a probability measure $\mathfrak{P}$. By Skorokhod's Theorem (Theorem 2.2), there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables (W^{N_μ}, u^{N_μ}) and (W, u) on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathfrak{S}$ such that

$$W^{N_\mu} \longrightarrow W \quad \text{in } C(0,T;\mathbb{R}^m)\mathbb{P}\text{-a.s,} \tag{4.123}$$

$$u^{N_\mu} \longrightarrow u \quad \text{in } L^2(0,T;\mathbb{V})\mathbb{P}\text{-a.s.} \tag{4.124}$$

Moreover,

$$\text{the probability law of } \left(W^{N_\mu}, u^{N_\mu}\right) \text{ is } \mathfrak{P}^{N_\mu} \text{ and that of } (W,u) \text{ is } \mathfrak{P}. \tag{4.125}$$

For the filtration $\mathbb{F}^t$, it is enough to choose $\mathbb{F}^t = \sigma(W(s), u(s) : 0 \le s \le t), t \in (0, T]$.

It remains to prove that the limit process W is a Wiener process. To fix this, it is sufficient to show that for any $0 < t_1 < t_2 < \cdots < t_m = T$, the increments process $(W(t_j) - W(t_{j-1}))$ are independent with respect to $\mathbb{F}^{t_{j-1}}$, distributed normally with mean 0 and variance $t_j - t_{j-1}$. That is, to show that for any $\lambda_j \in \mathbb{R}^m$ and $i^2 = -1$

$$\mathbb{E}\exp\left(i\sum_{j=1}^m \lambda_j(W(t_j) - W(t_{j-1}))\right) = \prod_{j=1}^m \exp\left(-\frac{1}{2}\lambda_j^2(t_j - t_{j-1})\right). \tag{4.126}$$

Equation (4.126) will follow if we have

$$\mathbb{E}\left[\exp\frac{(i\lambda(W(t+\theta) - W(t)))}{\mathbb{F}^t}\right] = \exp\left(-\frac{\lambda\theta^2}{2}\right). \tag{4.127}$$

We rely on the fact that for any random variables X and Y on any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that X is $\mathcal{F}$-measurable and $\mathbb{E}|Y| < \infty$, $\mathbb{E}|XY| < \infty$, we have

$$\mathbb{E}\left(\frac{XY}{\mathcal{F}}\right) = X\mathbb{E}\left(\frac{Y}{\mathcal{F}}\right), \qquad \mathbb{E}\mathbb{E}\left(\frac{Y}{\mathcal{F}}\right) = \mathbb{E}(Y), \tag{4.128}$$

that is,

$$\mathbb{E}(XY) = \mathbb{E}\left(X\mathbb{E}\left(\frac{Y}{\mathcal{F}}\right)\right). \tag{4.129}$$

Now, let us consider an arbitrary bounded continuous functional $\vartheta_t(W, v)$ on $\mathfrak{S}$ depending only on the values of W and v on $(0, T)$. Owing to the independence of $\overline{W}(t)$ to $\vartheta_t(\overline{W}, v)$ and the fact that $\overline{W}$ is a Wiener process, we have

$$\begin{aligned}
&\overline{\mathbb{E}}\left[\exp\left(i\lambda\left(\overline{W}(t+\theta) - \overline{W}(t)\right)\right)\vartheta_t\left(\overline{W}, v\right)\right] \\
&\quad = \overline{\mathbb{E}}\left[\exp\left(i\lambda\left(\overline{W}(t+\theta) - \overline{W}(t)\right)\right)\right]\overline{\mathbb{E}}\left[\vartheta_t\left(\overline{W}, v\right)\right] \\
&\quad = \exp\left(-\frac{\lambda\theta^2}{2}\right)\overline{\mathbb{E}}\left[\vartheta_t\left(\overline{W}, v\right)\right].
\end{aligned} \tag{4.130}$$

In view of (4.125), this implies that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\lambda\left(W^{N_\mu}(t+\theta) - W^{N_\mu}(t)\right)\right)\vartheta_t\left(W^{N_\mu}, v\right)\right] \\
&\quad = \mathbb{E}\left[\exp\left(i\lambda\left(W^{N_\mu}(t+\theta) - W^{N_\mu}(t)\right)\right)\right]\mathbb{E}\left[\vartheta_t\left(W^{N_\mu}, v\right)\right] \\
&\quad = \exp\left(-\frac{\lambda\theta^2}{2}\right)\mathbb{E}\left[\vartheta_t\left(W^{N_\mu}, v\right)\right].
\end{aligned} \tag{4.131}$$

Now, the convergences (4.123) and (4.124) and the continuity of ϑ allow us to pass to the limit in this latter equation and obtain

$$\mathbb{E}\left[\exp(i\lambda(W(t+\theta) - W(t)))\vartheta_t(W, v)\right] = \exp\left(-\frac{\lambda\theta^2}{2}\right)\mathbb{E}[\vartheta_t(W, v)], \tag{4.132}$$

which, in view of (4.129), implies (4.127). The choice of the above filtration implies then that W is a $\mathbb{F}^t$-standard m-dimensional Wiener process.

Theorem 4.9. *The pair u^{N_μ}, W^{N_μ} satisfies the equation*

$$\begin{aligned}
&\left(u^{N_\mu}(s), e_i\right)_{\mathbb{V}} + \nu\int_0^t\left(\left(u^{N_\mu}(s), e_i\right)\right)ds + \int_0^t\left(\operatorname{curl}\left(u^{N_\mu}(s) - \alpha\Delta u^{N_\mu}(s)\right) \times u^{N_\mu}(s), e_i\right)ds \\
&\quad = \left(u_0^{N_\mu}, e_i\right)_{\mathbb{V}} + \int_0^t\left(F\left(u^{N_\mu}(s), s\right), e_i\right)ds + \int_0^t\left(G\left(u^{N_\mu}(s), s\right), e_i\right)dW^{N_\mu},
\end{aligned} \tag{4.133}$$

for any $i \geq 1$.

Proof. Let $i \geq 1$ be an arbitrary fixed integer. We set

$$\mathfrak{X}^N = \int_0^T \left| \left(u^N(s), e_i\right)_{\mathbb{V}} - \left(u_0^N, e_i\right)_{\mathbb{V}} + \nu \int_0^t \left(\left(u^N(s), e_i\right)\right) ds - \int_0^t \left(F\left(u^N(s), s\right), e_i\right) ds \right.$$
$$\left. + \int_0^t \left(\operatorname{curl}\left(u^N(s) - \alpha \Delta u^N(s)\right) \times u^N(s), e_i\right) ds - \int_0^t \left(G\left(u^N(s), s\right), e_i\right) d\overline{W} \right|^2 dt. \tag{4.134}$$

Obviously

$$\mathfrak{X}^N = 0 \quad \mathbb{P}\text{-a.s}, \tag{4.135}$$

which implies in particular that

$$\overline{\mathbb{E}} \frac{\mathfrak{X}^N}{1 + \mathfrak{X}^N} = 0. \tag{4.136}$$

Now we let

$$\mathfrak{Y}^{N_\mu} = \int_0^T \left| \left(u^{N_\mu}(s), e_i\right)_{\mathbb{V}} + \nu \int_0^t \left(\left(u^{N_\mu}(s), e_i\right)\right) ds - \int_0^t \left(F\left(u^{N_\mu}(s), s\right), e_i\right) ds \right.$$
$$- \left(u_0^{N_\mu}, e_i\right)_{\mathbb{V}} + \int_0^t \left(\operatorname{curl}\left(u^{N_\mu}(s) - \alpha \Delta u^{N_\mu}(s)\right) \times u^N(s), e_i\right) ds \tag{4.137}$$
$$\left. + \int_0^t \left(G\left(u^{N_\mu}(s), s\right), e_i\right) dW^{N_\mu} \right|^2 dt.$$

We will prove that

$$\mathbb{E} \frac{\mathfrak{Y}^{N_\mu}}{1 + \mathfrak{Y}^{N_\mu}} = 0. \tag{4.138}$$

The difficulty we encounter is that $\mathfrak{X}^N$ is not a deterministic functional of u^N and $\overline{W}$ because of the stochastic term. To overcome this obstacle we introduce

$$G^\varepsilon(u(t), t) = \frac{1}{\varepsilon} \int_0^T \phi\left(-\frac{t-s}{\varepsilon}\right) G(u(s), s) ds, \tag{4.139}$$

where ϕ is a mollifier. It is clear that

$$\mathbb{E} \int_0^T |G^\varepsilon(u(t), t)|^2 dt \leq \mathbb{E} \int_0^T |G(u(t), t)|^2 dt. \tag{4.140}$$

Moreover,

$$G^{\varepsilon}(u(\cdot),\cdot) \longrightarrow G(u(\cdot),\cdot) \quad \text{in } L^2\left(\overline{\Omega},\overline{\mathbb{P}};L^2(0,T;\mathbb{V})\right), \tag{4.141}$$

which implies in particular that

$$(G^{\varepsilon}(u(\cdot),\cdot),e_i) \longrightarrow (G(u(\cdot),\cdot),e_i) \quad \text{in } L^2\left(\overline{\Omega},\overline{\mathbb{P}};L^2(0,\mathrm{T})\right) \text{ for any } i \geq 1. \tag{4.142}$$

Let us denote by $\mathfrak{X}^{N,\varepsilon}$ and $\mathfrak{Y}^{N_\mu,\varepsilon}$ the analog of $\mathfrak{X}^{N}$ and $\mathfrak{Y}^{N_\mu}$ with G replaced by G^{ε}. Introduce the mapping

$$\begin{aligned} \varphi_{N,\varepsilon} &: C(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{V}) \longrightarrow \left(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}}\right), \\ \varphi_{N,\varepsilon}\left(\overline{W},v^N\right) &= \frac{\mathfrak{X}^{N,\varepsilon}}{1+\mathfrak{X}^{N,\varepsilon}}. \end{aligned} \tag{4.143}$$

Now, it is seen that $\varphi_{N,\varepsilon}$ is a bounded continuous functional on $\mathfrak{S}$. Next, let us define

$$\varphi_{N_\mu,\varepsilon}\left(\overline{W},u^{N_\mu}\right) = \frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}. \tag{4.144}$$

We have

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} = \mathbb{E}\varphi_{N_\mu,\varepsilon}\left(W^{N_\mu},u^{N_\mu}\right). \tag{4.145}$$

Since $\varphi_{N_\mu,\varepsilon}(W^{N_\mu},u^{N_\mu})$ is a bounded functional on $\mathfrak{S}$ and since the law of W^{N_μ},u^{N_μ} is $\mathfrak{P}^{N_\mu}$ (see (4.125)), then

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} = \int_{\mathfrak{S}}\varphi(w,v)\,d\mathfrak{P}^{N_\mu}. \tag{4.146}$$

We note that law $(\overline{W},u^{N_\mu}) = \mathfrak{P}^{N_\mu}$, so

$$\begin{aligned} \int_{\mathfrak{S}}\varphi(w,v)\,d\mathfrak{P}^{N_\mu} &= \overline{\mathbb{E}}\varphi\left(\overline{W},u^{N_\mu}\right) \\ &= \overline{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}. \end{aligned} \tag{4.147}$$

That is,

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} = \overline{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}. \tag{4.148}$$

Note that

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \overline{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}} = \mathbb{E}\left(\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}}\right) + \mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} - \overline{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}} + \overline{\mathbb{E}}\left(\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}} - \frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}}\right). \tag{4.149}$$

We can check that

$$\mathbb{E}\left|\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}}\right| = \mathbb{E}\left|\frac{\mathfrak{Y}^{N_\mu} - \mathfrak{Y}^{N_\mu,\varepsilon}}{(1+\mathfrak{Y}^{N_\mu})(1+\mathfrak{Y}^{N_\mu,\varepsilon})}\right|, \tag{4.150}$$

and it implies that

$$\mathbb{E}\left|\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}}\right| \le C\left(\mathbb{E}\int_0^T \left|\left(G^\varepsilon\left(u^{N_\mu}(t),t\right) - G\left(u^{N_\mu}(t),t\right),e_i\right)\right|^2 dt\right)^{1/2}. \tag{4.151}$$

We also have

$$\mathbb{E}\left|\frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}} - \frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}\right| \le C\left(\overline{\mathbb{E}}\int_0^T \left|\left(G^\varepsilon\left(u^{N_\mu}(t),t\right) - G\left(u^{N_\mu}(t),t\right),e_i\right)\right|^2 dt\right)^{1/2}. \tag{4.152}$$

The above estimates and (4.145) yield

$$\left|\left|\mathbb{E}\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}}\right| - \left|\overline{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}}\right|\right| \le C\left(\overline{\mathbb{E}}\int_0^T \left|\left(G^\varepsilon\left(u^{N_\mu}(t),t\right) - G\left(u^{N_\mu}(t),t\right),e_i\right)\right|^2 dt\right)^{1/2}. \tag{4.153}$$

Passing to limit to the above relation implies (4.138) and, hence, (4.133). □

5. Proof of the Main Result

5.1. Passage to the Limits

From the tightness property we have

$$u^{N_\mu} \longrightarrow u \quad \text{in } L^2(0,T;\mathbb{V}) \quad \mathbb{P}\text{-a.s.} \tag{5.1}$$

Since u^{N_μ} satisfies the two equivalent equations (4.133), then it verifies the same estimates as u^N.

Let us consider the positive nondecreasing function $\varphi(x) = x^p$, $p \geq 4$ defined on $\mathbb{R}_+$. We have

$$\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty. \tag{5.2}$$

Thanks to the estimate $\mathbb{E}\sup_{t\in[0,T]} |u^{N_\mu}|_{\mathbb{V}}^p \leq C$, we have

$$\mathbb{P}\left(\phi\left(\left|u^{N_\mu}\right|_{L^2(0,T;\mathbb{V})}\right)\right) < \infty. \tag{5.3}$$

Thanks to the uniform integrability criteria in [52] we see that $|u^{N_\mu}|_{L^2(0,T;\mathbb{V})}$ is uniform integrable with respect to the probability measure.

We can deduce from Vitali's Theorem that

$$u^{N_\mu} \longrightarrow u \quad \text{in } L^2\left(\Omega, \mathbb{P}, L^2(0,T;\mathbb{V})\right). \tag{5.4}$$

This implies in particular that

$$u^{N_\mu} \longrightarrow u \quad \text{in } L^2\left(\Omega, \mathbb{P}, L^2\left(0,T;L^2(D)\right)\right), \tag{5.5}$$

$$\frac{\partial u^{N_\mu}}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \quad \text{in } L^2\left(\Omega, \mathbb{P}, L^2\left(0,T;L^2(D)\right)\right), \quad i = 1,2. \tag{5.6}$$

Thanks to (5.4), we can still extract a new subsequence from u^{N_μ} denoted again by u^{N_μ} so that

$$u^{N_\mu} \longrightarrow u \quad \text{in } \mathbb{V}\, dt \times d\mathbb{P}\text{-almost everywhere}. \tag{5.7}$$

It is readily seen that

$$\left(\left(u^{N_\mu}, e_i\right)\right) \longrightarrow ((u, e_i)) \quad \text{strongly in } L^2\left(\Omega, \mathbb{P}; L^2(0,T)\right). \tag{5.8}$$

Let χ be an element of $L^\infty(\Omega \times [0,T], d\mathbb{P} \otimes dt)$.

Since $e_i \in \mathbb{H}^3(D) \subset \mathbb{L}^\infty(D)$, then $\chi e_i \in L^\infty(\Omega \times (0,T] \times D, d\mathbb{P} \otimes dt \otimes dx)$. Thanks to (5.5), (5.6) we have that

$$u_j^{N_\mu} \frac{\partial u_k^{N_\mu}}{\partial x_j} (\chi e_i)_k \longrightarrow u_j \frac{\partial u_k}{\partial x_j} (\chi e_i)_k \quad \text{in } L^1(\Omega \times (0,T] \times D), \tag{5.9}$$

which implies that

$$\mathbb{E}\int_0^T b\left(u^{N_\mu}, u^{N_\mu}, \chi e_i\right) dt \longrightarrow \mathbb{E}\int_0^T b(u, u, \chi e_i) dt \quad \text{for any } i. \tag{5.10}$$

Since in view of Lemma 4.1 $e_i \in \mathbb{H}^4(D)$, then

$$\frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \in L^\infty\Big(\Omega, \mathbb{P}, L^\infty\Big(0,T;\mathbb{H}^2(D)\Big)\Big). \tag{5.11}$$

Since $\mathbb{H}^2(D) \subset \mathbb{L}^\infty(D)$, then

$$\frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \in L^\infty(\Omega, \mathbb{P}, L^\infty(0,T;\mathbb{L}^\infty(D))). \tag{5.12}$$

With the help of (5.5) we obtain

$$\Big(u^{N_\mu}\Big)_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \longrightarrow u_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \quad \text{in } L^2\Big(\Omega, \mathbb{P}; L^2\Big(0,T;\mathbb{L}^2(D)\Big)\Big). \tag{5.13}$$

We derive from this and (5.6) that

$$\mathbb{E}\int_0^T\int_D \Big(u^{N_\mu}\Big)_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \left(\frac{\partial u^{N_\mu}}{\partial x_j}\right)_l dx\,dt \longrightarrow \mathbb{E}\int_0^T\int_D u_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \left(\frac{\partial u}{\partial x_j}\right)_l dx\,dt, \tag{5.14}$$

for any i, j, k, l. In the above equations $(f)_k$ denotes the kth component of the vector function f.

We can use the same argument to show that

$$\mathbb{E}\int_0^T\int_D \left(\frac{\partial u^{N_\mu}}{\partial x_j}\right)_k \chi \frac{\partial (e_i)_l}{\partial x_j} \left(\frac{\partial u^{N_\mu}}{\partial x_j}\right)_l dx\,dt \longrightarrow \mathbb{E}\int_0^T\int_D \left(\frac{\partial u}{\partial x_j}\right)_k \chi \frac{\partial (e_i)_l}{\partial x_j} \left(\frac{\partial u}{\partial x_j}\right)_l dx\,dt, \tag{5.15}$$

for any i, j, k, l.

With the help of (2.21), and (5.14), (5.15), we have

$$\mathbb{E}\int_0^T b\Big(u^{N_\mu}, \Delta u^{N_\mu}, e_i\Big)\chi\,dt \longrightarrow \mathbb{E}\int_0^T b(u, \Delta, e_i)\chi\,dt, \quad \forall i. \tag{5.16}$$

Thanks to density of $L^\infty(\Omega \times [0,T], d\mathbb{P} \otimes dt)$ in $L^2(\Omega \times [0,T], d\mathbb{P} \otimes dt)$ and by taking $\chi \in L^\infty(\Omega \times [0,T], d\mathbb{P} \otimes dt)$ as a test function, we deduce from (5.16) that

$$b\Big(u^{N_\mu}, \Delta u^{N_\mu}, e_i\Big) \rightharpoonup b(u, \Delta u, e_i) \quad \text{weakly in } L^2\Big(\Omega, \mathbb{P}; L^2(0,T)\Big), \tag{5.17}$$

for any i.

Using (2.22), we can imitate the argument used above to show that

$$b\left(e_i, \Delta u^{N_\mu}, u^{N_\mu}\right) \rightharpoonup b(e_i, \Delta u, u) \quad \text{weakly in } L^2\left(\Omega, \mathbb{P}; L^2(0,T)\right), \tag{5.18}$$

for any i.

We conclude with (2.18), (5.10), (5.17) and (5.18) that

$$\left(\operatorname{curl}\left(u^{N_\mu} - \alpha\Delta u^{N_\mu}\right) \times u^{N_\mu}, e_i\right) \rightharpoonup (\operatorname{curl}(u - \alpha\Delta u) \times u, e_i), \tag{5.19}$$

weakly in $L^2(\Omega, \mathbb{P}; L^2(0,T))$ for any i.

It follows from (5.7), the Lemma 4.3, the assumption on F, and Vitali's theorem that

$$F\left(u^{N_\mu}(\cdot), \cdot\right) \longrightarrow F(u(\cdot), \cdot) \quad \text{strongly in } L^2\left(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V})\right). \tag{5.20}$$

This implies in particular that

$$\left(F\left(u^{N_\mu}(\cdot), \cdot\right), e_i\right) \longrightarrow (F(u(\cdot), \cdot), e_i) \quad \text{strongly in } L^2\left(\Omega, \mathbb{P}; L^2(0,T)\right), \tag{5.21}$$

for any i.

It remains to prove that

$$\int_0^t \left(G\left(u^{N_\mu}, s\right), e_i\right) dW^{N_\mu} \rightharpoonup \int_0^t (G(u,s), e_i)\, dW \quad \text{weakly-}\star \text{ in } L^2(\Omega, \mathbb{P}; L^\infty(0,T)), \tag{5.22}$$

for any $t \in (0,T)$ and i as $\mu \to \infty$. Using a similar argument as in [53], we will just show that

$$\int_0^T \left(G\left(u^{N_\mu}, s\right), e_i\right) dW^{N_\mu} \rightharpoonup \int_0^T (G(u,s), e_i)\, dW \quad \text{weakly in } L^2(\Omega, \mathbb{P}), \tag{5.23}$$

from (5.22) follows. From now on we fix $i \geq 1$. First, Lemma 4.3, the convergence (5.7), the assumption on G, and Vitali's theorem imply that

$$\left(G\left(u^{N_\mu}, \cdot\right), e_i\right) \longrightarrow (G(u, \cdot), e_i) \quad \text{in } L^2\left(\Omega, \mathbb{P}; L^2(0,T)\right) \tag{5.24}$$

as $\mu \to \infty$. We consider the already introduced regularized function $G^\varepsilon(u(\cdot), \cdot)$ in (4.139). We readily check that

$$(G^\varepsilon(u(\cdot), \cdot), e_i) \longrightarrow (G(u(\cdot), \cdot), e_i) \quad \text{in } L^2\left(\Omega, \mathbb{P}; L^2(0,T)\right), \tag{5.25}$$

as $\varepsilon \to 0$. We also have

$$\mathbb{E}\int_0^T \left| \left(G^\varepsilon\left(u^{N_\mu}, t\right) - G^\varepsilon(u,t), e_i\right) \right|^2 dt \leq \mathbb{E}\int_0^T \left| \left(G\left(u^{N_\mu}, t\right) - G(u,t), e_i\right) \right|^2 dt. \tag{5.26}$$

The crucial point is to show that

$$\int_0^T \left(G^\varepsilon\left(u^{N_\mu},t\right),e_i\right)dW^{N_\mu} \rightharpoonup \int_0^T (G^\varepsilon(u,t),e_i)dW \quad \text{weakly in } L^2(\Omega,\mathbb{P}). \tag{5.27}$$

Since

$$\mathbb{E}\left|\int_0^T \left(G^\varepsilon\left(u^{N_\mu},t\right),e_i\right)dW^{N_\mu}\right|^2 = \mathbb{E}\int_0^T \left(G^\varepsilon\left(u^{N_\mu},t\right),e_i\right)^2 dt < \infty, \tag{5.28}$$

then $\int_0^T (G^\varepsilon(u^{N_\mu},t),e_i)dW^{N_\mu}$ weakly converges to a certain β in $L^2(\Omega,\mathbb{P})$. An integration-by-parts yields

$$\int_0^T \left(G^\varepsilon\left(u^{N_\mu},t\right),e_i\right)dW^{N_\mu} = \left(G^\varepsilon\left(u^{N_\mu}(T),T\right),e_i\right) - \int_0^T W^{N_\mu}(t)\frac{d}{dt}\left(G^\varepsilon\left(u^{N_\mu}(t),t\right),e_i\right)dt, \tag{5.29}$$

where

$$\frac{d}{dt}\left(G^\varepsilon\left(u^{N_\mu}(t),t\right),e_i\right) = \frac{1}{\varepsilon}\int_0^T \frac{d}{dt}\phi\left(-\frac{t-s}{\varepsilon}\right)\left(G\left(v^{N_\mu}(s),s\right),e_i\right)ds. \tag{5.30}$$

By virtue of the convergence

$$\left(u^{N_\mu},W^{N_\mu}\right) \longrightarrow (u,W) \quad \text{in } C(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{V}) \tag{5.31}$$

$\mathbb{P}$-almost surely, we have

$$\int_0^T \left(G^\varepsilon\left(u^{N_\mu},t\right),e_i\right)dW^{N_\mu} \longrightarrow (G^\varepsilon(u(T),T),e_i) - \int_0^T W(t)\frac{d}{dt}(G^\varepsilon(u(t),t),e_i)dt \tag{5.32}$$

for almost all $\omega\in\Omega$. The term in the left-hand side of (5.32) is equal to

$$\int_0^T (G^\varepsilon(u(t),t),e_i)dW. \tag{5.33}$$

Now let us pick an element $\zeta\in L^\infty(\Omega,\mathbb{P})$. We have

$$\mathbb{E}\int_0^T \left(G^\varepsilon\left(u^{N_\mu}(t),t\right),\zeta e_i\right)dW^{N_\mu} \longrightarrow \mathbb{E}\int_0^T (G^\varepsilon(u(t),t),\zeta e_i)dW, \tag{5.34}$$

that is

$$\beta = \int_0^T (G^\varepsilon(u(t),t),e_i)dW. \tag{5.35}$$

Indeed, thanks to the estimate (4.140), Lemma 4.2 the sequence of random variables $\int_0^T (G^\varepsilon(u^{N_\mu}(t),t),\zeta e_i)dW^{N_\mu}$ is uniformly integrable. Owing to the convergence (5.32) and the applicability of the Vitali's Theorem, we get (5.34). We also have (5.27) since $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ is dense in $L^2(\Omega,\mathbb{P})$.

Let $\zeta \in L^\infty(\Omega,\mathbb{P})$; we write

$$\left|\mathbb{E}\int_0^T \left(G\left(u^{N_\mu}(t),t\right),\zeta e_i\right)dW^{N_\mu} - \mathbb{E}\int_0^T (G(u(t),t),\zeta e_i)dW\right| \leq J_1 + J_2 + J_3, \tag{5.36}$$

where

$$\begin{aligned} J_1 &= \left|\mathbb{E}\int_0^T \left(G^\varepsilon\left(u^{N_\mu}(t),t\right),\zeta e_i\right)dW^{N_\mu} - \mathbb{E}\int_0^T \left(G\left(u^{N_\mu}(t),t\right),\zeta e_i\right)dW^{N_\mu}\right|, \\ J_2 &= \left|\mathbb{E}\int_0^T \left(G^\varepsilon\left(u^{N_\mu}(t),t\right),\zeta e_i\right)dW^{N_\mu} - \mathbb{E}\int_0^T (G^\varepsilon(u(t),t),\zeta e_i)dW\right|, \\ J_3 &= \left|\mathbb{E}\int_0^T (G^\varepsilon(u(t),t),\zeta e_i)dW - \mathbb{E}\int_0^T (G(u(t),t),\zeta e_i)dW\right|. \end{aligned} \tag{5.37}$$

By Cauchy-Schwarz's inequality and owing to (5.25), the term J_3 of the RHS of (5.36) converges to zero as $\varepsilon \to 0$.

By (5.34), the term J_2 in the RHS of (5.36) converges to zero as $\mu \to \infty$.

By Cauchy-Schwarz's inequality again, some simple calculations, and making use of the estimate (5.26) and the convergence (5.24) and (5.25) we see that J_1 converges to zero as $\varepsilon \to 0$ and $\mu \to \infty$.

In view of these convergences passing to the limit as $\varepsilon \to 0$ and $\mu \to \infty$ in (5.36) we get (5.22).

Combining all those results and passing to the limit in (4.133), we see that u satisfies (3.6). This proves the first part of Theorem 3.3. The next subsection addresses the continuity in time of our solution.

5.2. Proof of Continuity of the Paths of u

We have already shown that for any $i \geq 1$ the equation

$$\begin{aligned} (u(t),e_i)_{\mathbb{V}} = (u_0,e_i)_{\mathbb{V}} &+ \int_0^t ((F(u(s),s) - \operatorname{curl}(u-\alpha\Delta u)\times u,e_i) - \nu((u(s),e_i)))ds \\ &+ \int_0^t (G(u(s),s),e_i)dW \end{aligned} \tag{5.38}$$

holds almost surely for any $t \in [0,T]$.

For any $i \geq 1$ let φ be the mapping

$$\begin{aligned} [0,T] &\longrightarrow \mathbb{R}, \\ t &\longmapsto \varphi(t) = (u(t), e_i)_{\mathbb{V}}. \end{aligned} \tag{5.39}$$

Let $\theta > 0$. We have

$$\begin{aligned} \left|\varphi(t) - \varphi(t+\theta)\right| \leq{}& \left|\int_t^{t+\theta} ((F(u(s),s) - \operatorname{curl}(u - \alpha\Delta u) \times u, e_i) - \nu((u(s), e_i)))ds\right| \\ &+ \left|\int_t^{t+\theta} (G(u(s),s), e_i)dW\right|. \end{aligned} \tag{5.40}$$

Let $p > 4$, we obtain by raising both sides of the last inequality to the power $p/2$

$$\begin{aligned} \left|\varphi(t) - \varphi(t+\theta)\right|^{p/2} \leq{}& C\left|\int_t^{t+\theta} (G(u(s),s), e_i)dW\right|^{p/2} + C\left(\int_t^{t+\theta} |(F(u(s),s), e_i)|ds\right)^{p/2} \\ &+ C\left(\int_t^{t+\theta} |(\operatorname{curl}(u - \alpha\Delta u) \times u, e_i)|ds\right)^{p/2} + C\left(\nu\int_t^{t+\theta} |((u(s), e_i))|ds\right)^{p/2}. \end{aligned} \tag{5.41}$$

We infer from this that

$$\begin{aligned} \mathbb{E}\left|\varphi(t) - \varphi(t+\theta)\right|^{p/2} \leq{}& \mathbb{E}\left(\int_t^{t+\delta} |(\operatorname{curl}(u - \alpha\Delta u) \times u, e_i)|ds\right)^{p/2} \\ &+ C\mathbb{E}\sup_{0\leq\delta\theta}\left|\int_t^{t+\delta} (G(u(s),s), e_i)dW\right|^{p/2} \\ &+ C\mathbb{E}\left(\int_t^{t+\delta} |(F(u(s),s), e_i)|ds\right)^{p/2} \\ &+ C\mathbb{E}\left(\int_t^{t+\delta} \nu|((u(s), e_i))|ds\right)^{p/2}, \end{aligned} \tag{5.42}$$

which implies by the help of martingale inequality that

$$\begin{aligned}\mathbb{E}\left|\varphi(t)-\varphi(t+\theta)\right|^{p/2} \leq\ & C\theta^{(p-2)/2}\int_t^{t+\delta}|(\operatorname{curl}(u-\alpha\Delta u)\times u, e_i)|^{p/2}ds \\ &+ C\theta^{(p-2)/2}\int_t^{t+\delta}|(F(u(s),s),e_i)|^{p/2}ds \\ &+ C\theta^{(p-2)/2}\nu\int_t^{t+\delta}|((u(s),e_i))|^{p/2}ds \\ &+ C\mathbb{E}\left(\int_t^{t+\theta}|(G(u(s),s),e_i)|^2\right)^{p/2}.\end{aligned} \tag{5.43}$$

Using previous estimates and some elementary inequalities, the following holds:

$$\mathbb{E}\left|\varphi(t)-\varphi(t+\theta)\right|^{p/2} \leq C\left(\theta^{1+(p-2)/2}+\theta^{1+(p-4)/4}\right), \tag{5.44}$$

for any $\theta > 0$. We conclude from Kolmogorov-Čentsov Theorem that the stochastic process $\varphi(\cdot) = (u(\cdot), e_i)_{\mathbb{V}}$ has almost surely a continuous modification with respect to the time variable t. Identifying u with this modification, we see that u has almost surely continuous paths taking values in $\mathbb{V}$-weak. Since u is also in the class $L^{p/2}(\Omega,\mathbb{P};L^\infty(0,T;\mathbb{W}))$, then $u(\cdot)$ also has almost surely continuous paths with respect to t taking values in $\mathbb{W}$-weak (see [54] for justification). It follows that the initial condition $u(x,0) = u_0 \in \mathbb{W}$ in (1.13) makes sense.

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