Hindawi Publishing Corporation
Boundary Value Problems
Volume 2011, Article ID 456426, 19 pages
doi:10.1155/2011/456426

Research Article

Positive Solutions of *n*th-Order Nonlinear Impulsive Differential Equation with Nonlocal Boundary Conditions

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Received 25 March 2010; Accepted 9 May 2010

Academic Editor: Feliz Manuel Minhós

This paper is devoted to study the existence, nonexistence, and multiplicity of positive solutions for the nth-order nonlocal boundary value problem with impulse effects. The arguments are based upon fixed point theorems in a cone. An example is worked out to demonstrate the main results.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. For an introduction of the basic theory of impulsive differential equations, see Lakshmikantham et al. [1]; for an overview of existing results and of recent research areas of impulsive differential equations, see Benchohra et al. [2]. The theory of impulsive differential equations has become an important area of investigation in the recent years and is much richer than the corresponding theory of differential equations; see, for instance, [3–14] and their references.

At the same time, a class of boundary value problems with integral boundary conditions arise naturally in thermal conduction problems [15], semiconductor problems [16], hydrodynamic problems [17]. Such problems include two, three, and multipoint boundary value problems as special cases and attract much attention; see, for instance, [7, 8, 11, 18–44] and references cited therein. In particular, we would like to mention some results of Eloe and Ahmad [19] and Pang et al. [22]. In [19], by applying the fixed point theorem in cones due to the work of Krasnosel'kii and Guo, Eloe and Ahmad established the existence of positive solutions of the following nth boundary value problem:

$$\begin{gathered} x^{(n)}(t) + a(t)f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \alpha x(\eta). \end{gathered} \tag{1.1}$$

In [22], Pang et al. considered the expression and properties of Green's function for the nth-order m-point boundary value problem

$$\begin{gathered} x^{(n)}(t) + a(t)f(x(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{gathered} \tag{1.2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\beta_i > 0$, $\sum_{i=1}^{m-2} \beta_i \xi_i^{m-1} < 1$. Furthermore, they obtained the existence of positive solutions by means of fixed point index theory.

Recently, Yang and Wei [23] and the author of [24] improved and generalized the results of Pang et al. [22] by using different methods, respectively.

On the other hand, it is well known that fixed point theorem of cone expansion and compression of norm type has been applied to various boundary value problems to show the existence of positive solutions; for example, see [7, 8, 11, 19, 23, 24]. However, there are few papers investigating the existence of positive solutions of nth impulsive differential equations by using the fixed point theorem of cone expansion and compression. The objective of the present paper is to fill this gap. Being directly inspired by [19, 22], using of the fixed point theorem of cone expansion and compression, this paper is devoted to study a class of nonlocal BVPs for nth-order impulsive differential equations with fixed moments.

Consider the following nth-order impulsive differential equations with integral boundary conditions:

$$\begin{gathered} x^{(n)}(t) + f(t,x(t)) = 0, \quad t \in J,\ t \neq t_k, \\ -\Delta x^{(n-1)}\big|_{t=t_k} = I_k(x(t_k)), \quad k = 1,2,\ldots,m, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x(1) = \int_0^1 h(t)x(t)\,dt. \end{gathered} \tag{1.3}$$

Here $J = [0,1]$, $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$, and $R^+ = [0,+\infty)$, $t_k (k = 1,2,\ldots,m)$ (where m is fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < 1$, $\Delta x^{(n-1)}|_{t=t_k} = x^{(n-1)}(t_k^+) - x^{(n-1)}(t_k^-)$, where $x^{(n-1)}(t_k^+)$ and $x^{(n-1)}(t_k^-)$ represent the right-hand limit and left-hand limit of $x^{(n-1)}(t)$ at $t = t_k$, respectively, $h \in L^1[0,1]$ is nonnegative.

For the case of $h \equiv 0$, problem (1.3) reduces to the problem studied by Samoĭlenko and Perestyuk in [4]. By using the fixed point index theory in cones, the authors obtained some sufficient conditions for the existence of at least one or two positive solutions to the two-point BVPs.

Motivated by the work above, in this paper we will extend the results of [4, 19, 22–24] to problem (1.3). On the other hand, it is also interesting and important to discuss the existence of positive solutions for problem (1.3) when $I_k \neq 0$ ($k = 1, 2, \dots, m,$), $n \geq 2$, and $h \not\equiv 0$. Many difficulties occur when we deal with them; for example, the construction of cone and operator. So we need to introduce some new tools and methods to investigate the existence of positive solutions for problem (1.3). Our argument is based on fixed point theory in cones [45].

To obtain positive solutions of (1.3), the following fixed point theorem in cones is fundamental which can be found in [45, page 93].

Lemma 1.1. *Let Ω_1 and Ω_2 be two bounded open sets in Banach space E, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let P be a cone in E and let operator $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

(i) $Ax \not\geq x$, $\forall x \in P \cap \partial\Omega_1$; $Ax \not\leq x$, $\forall x \in P \cap \partial\Omega_2$;

(ii) $Ax \not\leq x$, $\forall x \in P \cap \partial\Omega_1$; $Ax \not\geq x$, $\forall x \in P \cap \partial\Omega_2$.

Then, A has at least one fixed point in $P \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

2. Preliminaries

In order to define the solution of problem (1.3), we will consider the following space.

Let $J' = J \setminus \{t_1, t_2, \dots, t_n\}$, and

$$PC^{n-1}[0,1] = \Big\{ x \in C[0,1] : x^{(n-1)}|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), \\ x^{(n-1)}(t_k^-) = x^{(n-1)}(t_k),\ \exists\ x^{(n-1)}(t_k^+) \Big\}, \quad k = 1, 2, \dots, m. \tag{2.1}$$

Then $PC^{n-1}[0,1]$ is a real Banach space with norm

$$\|x\|_{pc^{n-1}} = \max\Big\{ \|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \dots, \Big\|x^{(n-1)}\Big\|_\infty \Big\}, \tag{2.2}$$

where $\|x^{(n-1)}\|_\infty = \sup_{t \in J} |x^{(n-1)}(t)|$, $n = 1, 2, \dots$.

A function $x \in PC^{n-1}[0,1] \cap C^n(J')$ is called a solution of problem (1.3) if it satisfies (1.3).

To establish the existence of multiple positive solutions in $PC^{n-1}[0,1] \cap C^n(J')$ of problem (1.3), let us list the following assumptions:

(H_1) $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$;

(H_2) $\mu \in [0,1)$, where $\mu = \int_0^1 h(t) t^{n-1} dt$.

Lemma 2.1. *Assume that* (H_1) *and* (H_2) *hold. Then* $x \in PC^{n-1}[0,1] \cap C^n(J')$ *is a solution of problem* (1.3) *if and only if* x *is a solution of the following impulsive integral equation:*

$$x(t) = \int_0^1 H(t,s) f(s,x(s))ds + \sum_{k=1}^{m} H(t,t_k) I_k(x(t_k)), \tag{2.3}$$

where

$$H(t,s) = G_1(t,s) + G_2(t,s), \tag{2.4}$$

$$G_1(t,s) = \frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}, & 0 \le s \le t \le 1, \\ t^{n-1}(1-s)^{n-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

$$G_2(t,s) = \frac{t^{n-1}}{1 - \int_0^1 h(t) t^{n-1} dt} \int_0^1 h(t) G_1(t,s) dt. \tag{2.6}$$

Proof. First suppose that $x \in PC^{n-1}[0,1] \cap C^n(J')$ is a solution of problem (1.3). It is easy to see by integration of (1.3) that

$$\begin{aligned} x^{(n-1)}(t) &= x^{(n-1)}(0) - \int_0^t f(s,x(s))ds + \sum_{0<t_k<t} \left[x^{(n-1)}(t_k^+) - x^{(n-1)}(t_k) \right] \\ &= x^{(n-1)}(0) - \int_0^t f(s,x(s))ds - \sum_{0<t_k<t} I_k(x(t_k)). \end{aligned} \tag{2.7}$$

Integrating again and by boundary conditions, we can get

$$x^{(n-2)}(t) = x^{(n-1)}(0)t - \int_0^t (t-s) f(s,x(s))ds - \sum_{0<t_k<t} I_k(x(t_k))(t-t_k). \tag{2.8}$$

Similarly, we get

$$x(t) = -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s,x(s))ds + x^{(n-1)}(0)\frac{t^{n-1}}{(n-1)!} - \sum_{t_k<t} \frac{I_k(x(t_k))(t-t_k)^{n-1}}{(n-1)!}. \tag{2.9}$$

Letting $t=1$ in (2.9), we find

$$\begin{aligned} x^{(n-1)}(0) = (n-1)!x(1) &+ \int_0^1 (1-s)^{n-1} f(s,x(s))ds \\ &+ \sum_{t_k<1} I_k(x(t_k))(1-t_k)^{n-1}. \end{aligned} \tag{2.10}$$

Substituting $x(1) = \int_0^1 h(t)x(t)dt$ and (2.10) into (2.9), we obtain

$$\begin{aligned} x(t) &= -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s,x(s))ds + \frac{t^{n-1}}{(n-1)!}\Bigg[(n-1)!\int_0^1 h(t)x(t)dt \\ &\quad + \int_0^1 (1-s)^{n-1} f(s,x(s))ds + \sum_{t_k<1} I_k(x(t_k))(1-t_k)^{n-1}\Bigg] - \sum_{t_k<t}\frac{I_k(x(t_k))(t-t_k)^{n-1}}{(n-1)!} \\ &= \int_0^1 G_1(t,s) f(s,x(s))ds + \sum_{k=1}^m G_1(t,t_k) I_k(x(t_k)) + t^{n-1}\int_0^1 h(t)x(t)dt. \end{aligned} \tag{2.11}$$

Multiplying (2.11) with $h(t)$ and integrating it, we have

$$\begin{aligned} \int_0^1 h(t)x(t)dt &= \int_0^1 h(t)\int_0^1 G_1(t,s) f(s,x(s))ds\,dt + \int_0^1 h(t)\sum_{k=1}^m G_1(t,t_k) I_k(x(t_k))dt \\ &\quad + \int_0^1 h(t)t^{n-1}dt\int_0^1 h(t)x(t)dt, \end{aligned} \tag{2.12}$$

that is,

$$\begin{aligned} \int_0^1 h(t)x(t)dt = \frac{1}{1-\int_0^1 h(t)t^{n-1}dt}\Bigg[&\int_0^1 h(t)\int_0^1 G_1(t,s) f(s,x(s))ds\,dt \\ &+ \int_0^1 h(t)\sum_{k=1}^m G_1(t,t_k) I_k(x(t_k))dt\Bigg]. \end{aligned} \tag{2.13}$$

Then we have

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s) f(s,x(s))ds + \sum_{k=1}^m G_1(t,t_k) I_k(x(t_k)) \\ &\quad + \frac{t^{n-1}}{1-\int_0^1 h(t)t^{n-1}dt}\left[\int_0^1 h(t)\int_0^1 G_1(t,s) f(s,x(s))ds\,dt + \int_0^1 h(t)\sum_{k=1}^m G_1(t,t_k) I_k(x(t_k))dt\right]. \end{aligned} \tag{2.14}$$

Then, the proof of sufficient is complete.

Conversely, if x is a solution of (2.3), direct differentiation of (2.3) implies that, for $t \neq t_k$,

$$\begin{aligned}
x'(t) &= \frac{1}{(n-2)!}\int_0^t \Big[t^{n-2}(1-s)^{n-1} - (t-s)^{n-2}\Big] f(s,x(s))ds \\
&\quad + \frac{1}{(n-2)!}\int_t^1 t^{n-2}(1-s)^{n-1} f(s,x(s))ds \\
&\quad - \frac{1}{(n-2)!}\sum_{t_k<t}\Big[t^{n-2}(1-t_k)^{n-1} - (t-t_k)^{n-2}\Big] I_k(x(t_k)) \\
&\quad + \frac{1}{(n-2)!}\sum_{t_k\ge t} t^{n-2}(1-t_k)^{n-1} I_k(x(t_k)) \\
&\quad + \frac{(n-1)t^{n-2}}{1-\int_0^1 h(t)t^{n-1}dt}\Bigg[\int_0^1 h(t)\int_0^1 G_1(t,s)f(s,x(s))ds\,dt \\
&\qquad\qquad + \int_0^1 h(t)\sum_{k=1}^m G_1(t,t_k)I_k(x(t_k))dt\Bigg], \\
&\ \ \vdots \\
x^{(n-1)}(t) &= \int_0^1 (1-s)^{n-1} f(s,x(s))ds - \int_0^t f(s,x(s))ds + \sum_{t_k<1}(1-t_k)^{n-1}I_k(x(t_k)) - \sum_{t_k<t} I_k(x(t_k)) \\
&\quad + \frac{(n-1)!}{1-\int_0^1 h(t)t^{n-1}dt}\Bigg[\int_0^1 h(t)\int_0^1 G_1(t,s)f(s,x(s))ds\,dt \\
&\qquad\qquad + \int_0^1 h(t)\sum_{k=1}^m G_1(t,t_k)I_k(x(t_k))dt\Bigg].
\end{aligned} \tag{2.15}$$

Evidently,

$$\Delta x^{(n-1)}|_{t=t_k} = -I_k(x(t_k)), \quad (k = 1,2,\dots,m), \tag{2.16}$$

$$x^{(n)}(t) = -f(t,x(t)). \tag{2.17}$$

So $x \in C^n(J')$ and $\Delta x^{(n-1)}|_{t=t_k} = -I_k(x(t_k))$, $(k = 1,2,\dots,m)$, and it is easy to verify that $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, $x(1) = \int_0^1 h(t)x(t)dt$, and the lemma is proved. □

Similar to the proof of that from [22], we can prove that $H(t,s)$, $G_1(t,s)$, and $G_2(t,s)$ have the following properties.

Proposition 2.2. *The function $G_1(t,s)$ defined by* (2.5) *satisfyong $G_1(t,s) \ge 0$ is continuous for all $t,s \in [0,1]$, $G_1(t,s) > 0$, $\forall t,s \in (0,1)$.*

Proposition 2.3. *There exists $\gamma > 0$ such that*

$$\min_{t\in[t_m,1]} G_1(t,s) \geq \gamma G_1(\tau(s),s), \quad \forall s \in [0,1], \tag{2.18}$$

where $\tau(s)$ is defined in (2.20).

Proposition 2.4. *If $\mu \in [0,1)$, then one has*

(i) $G_2(t,s) \geq 0$ *is continuous for all* $t,s \in [0,1]$, $G_2(t,s) > 0$, $\forall t,s \in (0,1)$;

(ii) $G_2(t,s) \leq (1/(1-\mu))\int_0^1 h(t)G_1(t,s)dt$, $\forall t \in [0,1]$, $s \in (0,1)$.

Proof. From the properties of $G_1(t,s)$ and the definition of $G_2(t,s)$, we can prove that the results of Proposition 2.4 hold. □

Proposition 2.5. *If $\mu \in [0,1)$, the function $H(t,s)$ defined by* (2.4) *satisfies*

(i) $H(t,s) \geq 0$ *is continuous for all* $t,s \in [0,1]$, $H(t,s) > 0$, $\forall t,s \in (0,1)$;

(ii) $H(t,s) \leq H(s) \leq H_0$ *for each* $t,s \in [0,1]$, *and*

$$\min_{t\in[t_m,1]} H(t,s) \geq \gamma^* H(s), \quad \forall s \in [0,1], \tag{2.19}$$

where $\gamma^* = \min\{\gamma, t_m^{n-1}\}$, and

$$H(s) = G_1(\tau(s),s) + G_2(1,s), \quad \tau(s) = \frac{s}{1-(1-s)^{1+1/(n-2)}}, \quad H_0 = \max_{s\in J} H(s), \tag{2.20}$$

γ is defined in Proposition 2.3.

Proof. (i) From Propositions 2.2 and 2.4, we obtain that $H(t,s) \geq 0$ is continuous for all $t,s \in [0,1]$, and $H(t,s) > 0$, $\forall t,s \in (0,1)$.

(ii) From (ii) of Proposition 2.2 and (ii) of Proposition 2.4, we have $H(t,s) \leq H(s)$ for each $t,s \in [0,1]$.

Now, we show that (2.19) holds.

In fact, from Proposition 2.3, we have

$$\begin{aligned}
\min_{t\in[t_m,1]} H(t,s) &\geq \gamma G_1(\tau(s),s) + \frac{t_m^{n-1}}{1-\mu}\int_0^1 h(t)G_1(t,s)dt \\
&\geq \gamma^*\left[G_1(\tau(s),s) + \frac{1}{1-\mu}\int_0^1 h(t)G_1(t,s)dt\right] \\
&= \gamma^* H(s), \quad \forall s \in [0,1].
\end{aligned} \tag{2.21}$$

Then the proof of Proposition 2.5 is completed. □

Remark 2.6. From the definition of γ^*, it is clear that $0 < \gamma^* < 1$.

Lemma 2.7. *Assume that* (H_1) *and* (H_2) *hold. Then, the solution* x *of problem* (1.3) *satisfies* $x(t) \geq 0$, $\forall t \in J$.

Proof. It is an immediate subsequence of the facts that $H(t,s) \geq 0$ on $[0,1] \times [0,1]$. □

Remark 2.8. From (ii) of Proposition 2.5, one can find that

$$\gamma^* H(s) \leq H(t,s) \leq H(s), \quad t \in [t_m, 1],\ s \in J. \tag{2.22}$$

For the sake of applying Lemma 1.1, we construct a cone K in $PC^{n-1}[0,1]$ by

$$K = \left\{ x \in PC^{n-1}[0,1] : x \geq 0, x(t) \geq \gamma^* x(s),\ t \in [t_m, 1],\ s \in J \right\}. \tag{2.23}$$

Define $T : K \to K$ by

$$(Tx)(t) = \int_0^1 H(t,s) f(s, x(s)) ds + \sum_{k=1}^{m} H(t, t_k) I_k(x(t_k)). \tag{2.24}$$

Lemma 2.9. *Assume that* (H_1) *and* (H_2) *hold. Then,* $T(K) \subset K$, *and* $T : K \to K$ *is completely continuous.*

Proof. From Proposition 2.5 and (2.24), we have

$$\begin{aligned}
\min_{t\in[t_m,1]} (Tx)(t) &= \min_{t\in[t_m,1]} \int_0^1 H(t,s) f(s,x(s)) ds + \sum_{k=1}^{m} H(t,t_k) I_k(x(t_k)) \\
&\geq \int_0^1 \min_{t\in[t_m,1]} H(t,s) f(s,x(s)) ds + \sum_{k=1}^{m} \min_{t\in[t_m,1]} H(t,t_k) I_k(x(t_k)) \\
&\geq \gamma^* \left[\int_0^1 H(s) f(s,x(s)) ds + \sum_{k=1}^{m} H(t_k) I_k(x(t_k)) \right] \\
&\geq \gamma^* \left[\int_0^1 \max_{t\in[0,1]} H(t,s) f(s,x(s)) ds + \sum_{k=1}^{m} \max_{t\in[0,1]} H(t,t_k) I_k(x(t_k)) \right] \\
&\geq \gamma^* \max_{t\in[0,1]} \left[\int_0^1 H(t,s) f(s,x(s)) ds + \sum_{k=1}^{m} H(t,t_k) I_k(x(t_k)) \right] \\
&= \gamma^* \|Tx\|, \quad \forall x \in K.
\end{aligned} \tag{2.25}$$

Thus, $T(K) \subset K$.

Next, by similar arguments to those in [8] one can prove that $T : K \to K$ is completely continuous. So it is omitted, and the lemma is proved. □

3. Main Results

Write

$$f^{\beta} = \limsup_{x \to \beta} \max_{t \in J} \frac{f(t,x)}{x}, \qquad f_{\beta} = \liminf_{x \to \beta} \min_{t \in J} \frac{f(t,x)}{x},$$
$$I_{\beta}(k) = \liminf_{x \to \beta} \frac{I_k(x)}{x}, \qquad I^{\beta}(k) = \limsup_{x \to \beta} \frac{I_k(x)}{x}, \tag{3.1}$$

where β denotes 0^+ or $+\infty$.

In this section, we apply Lemma 1.1 to establish the existence of positive solutions for BVP (1.3).

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. In addition, letting* f *and* I_k *satisfy the following conditions:*

(H_3) $f^0 = 0$ *and* $I^0(k) = 0$, $k = 1, 2, \ldots, m$;

(H_4) $f_{\infty} = \infty$ *or* $I_{\infty}(k) = \infty$, $k = 1, 2, \ldots, m$,

BVP (1.3) *has at least one positive solution.*

Proof. Considering (H_3), there exists $\eta > 0$ such that

$$f(t,x) \le \varepsilon x, \quad I_k(x) \le \varepsilon_k x, \quad k = 1, 2, \ldots, m, \quad \forall 0 \le x \le \eta,\ t \in J, \tag{3.2}$$

where $\varepsilon, \varepsilon_k > 0$ satisfy

$$\max\{H_0, 1 + G_0\} \left(\varepsilon + \sum_{k=1}^{m} \varepsilon_k \right) < 1; \tag{3.3}$$

here

$$G_0 = \max\left\{ G_0^1, G_0^2, \ldots, G_0^{n-1} \right\},$$
$$G_0^1 = \max_{t,s \in J, t \neq s} G'_{2t}(t,s) = \max_{t,s \in J, t \neq s} \frac{(n-1)t^{n-2}}{1-\mu} \int_0^1 h(t) G_1(t,s)\,dt,$$
$$G_0^2 = \max_{t,s \in J, t \neq s} G''_{2t}(t,s) = \max_{t,s \in J, t \neq s} \frac{(n-1)(n-2)t^{n-3}}{1-\mu} \int_0^1 h(t) G_1(t,s)\,dt, \tag{3.4}$$
$$\vdots$$
$$G_0^{n-1} = \max_{t,s \in J, t \neq s} G_{2t}^{(n-1)}(t,s) = \max_{t,s \in J, t \neq s} \frac{(n-1)!}{1-\mu} \int_0^1 h(t) G_1(t,s)\,dt.$$

Now, for $0 < r < \eta$, we prove that

$$Tx \ngeq x, \quad x \in K, \quad \|x\|_{pc^{n-1}} = r. \tag{3.5}$$

In fact, if there exists $x_1 \in K$, $\|x_1\|_{pc^{n-1}} = r$ such that $Tx_1 \geq x_1$. Noticing (3.2), then we have

$$\begin{aligned}
0 \leq x_1(t) &\leq \int_0^1 H(t,s) f(s, x_1(s)) ds + \sum_{k=1}^{m} H(t, t_k) I_k(x_1(t_k)) \\
&\leq \varepsilon r \int_0^1 H(s) ds + r \sum_{k=1}^{m} H(t_k) \varepsilon_k \\
&\leq r H_0 \left(\varepsilon + \sum_{k=1}^{m} \varepsilon_k \right) \\
&< r = \|x_1\|_{pc^{n-1}}, \\
\left|x_1'(t)\right| &\leq \int_0^1 \left|H_t'(t,s)\right| f(s, x_1(s)) ds + \sum_{k=1}^{m} \left|H_t'(t, t_k)\right| I_k(x_1(t_k)) \\
&\leq \int_0^1 \left(\left|G_{1t}'(t,s)\right| + \left|G_{2t}'(t,s)\right|\right) f(s, x_1(s)) ds \\
&\quad + \sum_{k=1}^{m} \left(\left|G_{1t}'(t,t_k)\right| + \left|G_{2t}'(t,t_k)\right|\right) I_k(x_1(t_k)) \\
&\leq \int_0^1 \left(1 + G_0^1\right) f(s, x_1(s)) ds + \sum_{k=1}^{m} \left(1 + G_0^1\right) I_k(x_1(t_k)) \\
&\leq r\left(1 + G_0^1\right) \left(\varepsilon + \sum_{k=1}^{m} \varepsilon_k \right) \\
&< r = \|x_1\|_{pc^{n-1}}, \\
\left|x_1''(t)\right| &\leq \int_0^1 \left|H_t''(t,s)\right| f(s, x_1(s)) ds + \sum_{k=1}^{m} \left|H_t''(t, t_k)\right| I_k(x_1(t_k)) \\
&\leq \int_0^1 \left(\left|G_{1t}''(t,s)\right| + \left|G_{2t}''(t,s)\right|\right) f(s, x_1(s)) ds \\
&\quad + \sum_{k=1}^{m} \left(\left|G_{1t}''(t,t_k)\right| + \left|G_{2t}''(t,t_k)\right|\right) I_k(x_1(t_k))
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \left(1+G_0^2\right) f(s,x_1(s))ds + \sum_{k=1}^{m}\left(1+G_0^2\right) I_k(x_1(t_k)) \\
&\leq r\left(1+G_0^2\right)\left(\varepsilon + \sum_{k=1}^{m}\varepsilon_k\right) \\
&< r = \|x_1\|_{pc^{n-1}}, \\
&\ \ \vdots \\
\left|x_1^{(n-1)}(t)\right| &\leq \int_0^1 \left|H_t^{(n-1)}(t,s)\right| f(s,x_1(s))ds + \sum_{k=1}^{m}\left|H_t^{(n-1)}(t,t_k)\right| I_k(x_1(t_k)) \\
&\leq \int_0^1 \left(\left|G_{1t}^{(n-1)}(t,s)\right| + \left|G_{2t}^{(n-1)}(t,s)\right|\right) f(s,x_1(s))ds \\
&\quad + \sum_{k=1}^{m}\left(\left|G_{1t}^{(n-1)}(t,t_k)\right| + \left|G_{2t}^{(n-1)}(t,t_k)\right|\right) I_k(x_1(t_k)) \\
&\leq \int_0^1 \left(1+G_0^n\right) f(s,x_1(s))ds + \sum_{k=1}^{m}\left(1+G_0^n\right) I_k(x_1(t_k)) \\
&\leq r\left(1+G_0^n\right)\left(\varepsilon + \sum_{k=1}^{m}\varepsilon_k\right) \\
&< r = \|x_1\|_{pc^{n-1}},
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{gathered}
G'_{1t}(t,s) = \frac{1}{(n-2)!}\begin{cases} t^{n-2}(1-s)^{n-1} - (t-s)^{n-2} & \text{if } 0 \leq s \leq t \leq 1, \\ t^{n-2}(1-s)^{n-1} & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \\
G'_{1t}(t,s) = \frac{1}{(n-3)!}\begin{cases} t^{n-3}(1-s)^{n-1} - (t-s)^{n-3} & \text{if } 0 \leq s \leq t \leq 1, \\ t^{n-3}(1-s)^{n-1} & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \\
\vdots \\
G_{1t}^{(n-1)}(t,s) = \begin{cases} (1-s)^{n-1} - 1 & \text{if } 0 \leq s \leq t \leq 1, \\ (1-s)^{n-1} & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \\
\max_{t,s\in J, t\neq s}\left|G_{1t}^{(N)}(t,s)\right| = 1, \quad N = 1,2,\ldots,n-1.
\end{gathered}
\tag{3.7}
$$

Therefore, $\|x_1\|_{pc^{n-1}} < \|x_1\|_{pc^{n-1}}$, which is a contraction. Hence, (3.2) holds.

Next, turning to (H_4). Case (1). $f_\infty = \infty$. There exists $\tau > 0$ such that

$$f(t,x) \geq Mx, \quad t \in J,\ x \geq \tau, \tag{3.8}$$

where $M > [\gamma^* H_0(1-t_m)]^{-1}$. Choose

$$R > \max\left\{r, \tau\left(\gamma^*\right)^{-1}\right\}. \tag{3.9}$$

We show that

$$Tx \nleq x, \quad x \in K, \quad \|x\|_{pc^{n-1}} = R. \tag{3.10}$$

In fact, if there exists $x_0 \in K$, $\|x_0\|_{pc^{n-1}} = R$ such that $Tx_0 \leq x_0$, then

$$x_0(t) \geq \gamma^* x_0(s), \quad t \in [t_m, 1],\ s \in J. \tag{3.11}$$

This and (3.9) imply that

$$\min_{t \in [t_m, 1]} x_0(t) \geq \gamma^* \|x_0\|_{pc^{n-1}} = \gamma^* R > \tau. \tag{3.12}$$

So, we have

$$t \in J \Longrightarrow x_0(t) \geq (Tx_0)(t) \geq \min_{t \in [t_m,1]} \int_{t_m}^1 H(t,s) f(s, x_0(s)) ds \geq \gamma^* H_0 M \int_{t_m}^1 x_0(s) ds, \tag{3.13}$$

that is,

$$\int_{t_m}^1 x_0(t) dt \geq \gamma^* H_0 M (1 - t_m) \int_{t_m}^1 x_0(s) ds. \tag{3.14}$$

It is easy to see that

$$\int_{t_m}^1 x_0(s) ds > 0. \tag{3.15}$$

In fact, if $\int_{t_m}^1 x_0(s) ds = 0$, then $x_0(t) = 0$, for $t \in [t_m, 1]$. Since $x_0 \in K, x_0(s) = 0$, $\forall s \in J$. Hence, $\|x_0\|_{pc^{n-1}} = \|x_0^{(n-1)}\|_\infty = \|x_0\|_\infty = 0$, which contracts $\|x_0\|_{pc^{n-1}} = R$. So, (3.15) holds. Therefore, $M \leq [\gamma^* H_0(1-t_m)]^{-1}$, this is also a contraction. Hence, (3.10) holds.

Case (2). $I_\infty(k) = \infty$, $k = 1, 2, \ldots, m$. There exists $\tau_1 > 0$ such that

$$I_k(x) \geq M_k x, \quad x \geq \tau_1, \tag{3.16}$$

where $M_k > (\gamma^* H_0)^{-1}$, $k = 1,2,\ldots,m$. If we define $M^* = \min\{M_k : k = 1,2,\ldots,m\}$, then $M^* > (\gamma^* H_0)^{-1}$. Choose

$$R > \max\left\{r, \tau_1 (\gamma^*)^{-1}\right\}. \tag{3.17}$$

We prove that (3.10) holds.

In fact, if there exists $x_{00} \in K$, $\|x_{00}\|_{pc^{n-1}} = R$ such that $Tx_{00} \leq x_{00}$, then

$$x_{00}(t) \geq \gamma^* x_{00}(s), \quad t \in [t_m, 1],\ s \in J. \tag{3.18}$$

This and (3.17) imply that

$$\min_{t\in[t_m,1]} x_{00}(t) \geq \gamma^* \|x_{00}\|_{pc^{n-1}} = \gamma^* R > \tau_1. \tag{3.19}$$

So, we have

$$\begin{aligned} t \in J \Longrightarrow x_{00}(t) \geq (Tx_{00})(t) &\geq \min_{t\in[t_m,1]} \sum_{k=1}^{m} H(t,t_k) I_k(x_{00}(t_k)) \\ &\geq \gamma^* H_0 \sum_{k=1}^{m} M_k x_{00}(t_k) \\ &\geq \gamma^* H_0 M^* \sum_{k=1}^{m} x_{00}(t_k). \end{aligned} \tag{3.20}$$

From (3.20), we obtain that

$$\begin{aligned} x_{00}(t_1) &\geq \gamma^* H_0 M^* \sum_{k=1}^{m} x_{00}(t_k), \\ x_{00}(t_2) &\geq \gamma^* H_0 M^* \sum_{k=1}^{m} x_{00}(t_k), \\ &\vdots \\ x_{00}(t_k) &\geq \gamma^* H_0 M^* \sum_{k=1}^{m} x_{00}(t_k). \end{aligned} \tag{3.21}$$

So, we have

$$\sum_{k=1}^{m} x_{00}(t_k) \geq m\gamma^* H_0 M^* \sum_{k=1}^{m} x_{00}(t_k). \tag{3.22}$$

From the definition of M^*, we can find that

$$\sum_{k=1}^{m} x_{00}(t_k) > m\sum_{k=1}^{m} x_{00}(t_k), \quad x_{00} \in K, \ \|x_{00}\|_{pc^{n-1}} = R. \tag{3.23}$$

Similar to the proof in case (1), we can show that $\sum_{k=1}^{m} x_{00}(t_k) > 0$. Then, from (3.23), we have $m < 1$, which is a contraction. Hence, (3.10) holds.

Applying (i) of Lemma 1.1 to (3.2) and (3.10) yields that T has a fixed point $x \in \overline{K}_{r,R} = \{x : r \le \|x\|_{pc^{n-1}} \le R\}$. Thus, it follows that BVP (1.3) has at least one positive solution, and the theorem is proved. □

Theorem 3.2. *Assume that (H_1) and (H_2) hold. In addition, letting f and I_k satisfy the following conditions:*

(H_5) $f^{\infty} = 0$ *and* $I^{\infty}(k) = 0$, $k = 1, 2, \ldots, m$;

(H_6) $f_0 = \infty$ *or* $I_0(k) = \infty$, $k = 1, 2, \ldots, m$,

BVP (1.3) *has at least one positive solution.*

Proof. Considering (H_5), there exists $\overline{r} > 0$ such that $f(t, x) \le \overline{\varepsilon r}$, $I_k(x) \le \overline{\varepsilon}_k\overline{r}$, and $k = 1, 2, \ldots, m$, for $x \ge \overline{r}$, $t \in J$, where $\overline{\varepsilon}, \overline{\varepsilon}_k > 0$ satisfy $\max\{H_0, 1 + G_0\}(\varepsilon + \sum_{k=1}^{m} \varepsilon_k) < 1$.

Similar to the proof of (3.2), we can show that

$$Tx \ngeq x, \quad x \in K, \quad \|x\|_{pc^1} = \overline{r}. \tag{3.24}$$

Next, turning to (H_6). Under condition (H_6), similar to the proof of (3.10), we can also show that

$$Tx \nleq x, \quad x \in K, \quad \|x\|_{pc^1} = \overline{R}. \tag{3.25}$$

Applying (i) of Lemma 1.1 to (3.24) and (3.25) yields that T has a fixed point $\overline{x} \in \overline{K}_{\overline{r},\overline{R}} = \{x : \overline{r} \le \|x\|_{pc^{n-1}} \le \overline{R}\}$. Thus, it follows that BVP (1.3) has one positive solution, and the theorem is proved. □

Theorem 3.3. *Assume that (H_1), (H_2), (H_3), and (H_5) hold. In addition, letting f and I_k satisfy the following condition:*

(H_7) *there is a $\varsigma > 0$ such that $\gamma^*\varsigma \le x \le \varsigma$ and $t \in J$ implies*

$$f(t, x) \ge \tau\varsigma, \qquad I_k(x) \ge \tau_k\varsigma, \quad k = 1, 2, \ldots, \tag{3.26}$$

where $\tau, \tau_k \ge 0$ satisfy $\tau + \sum_{k=1}^{m} \tau_k > 0, \tau\int_{t_m}^{1} H(1/2, s)ds + \sum_{k=1}^{m} \tau_k H(1/2, t_k) > 1$, BVP (1.3) *has at least two positive solutions x^* and x^{**} with $0 < \|x^*\|_{pc^{n-1}} < \varsigma < \|x^{**}\|_{pc^{n-1}}$.*

Proof. We choose ρ, ξ with $0 < \rho < \varsigma < \xi$. If (H_3) holds, similar to the proof of (3.2), we can prove that

$$Tx \ngeq x, \quad x \in K, \quad \|x\|_{pc^1} = \rho. \tag{3.27}$$

If (H_5) holds, similar to the proof of (3.24), we have

$$Tx \not\geq x, \quad x \in K, \quad \|x\|_{pc^{n-1}} = \xi. \tag{3.28}$$

Finally, we show that

$$Tx \not\leq x, \quad x \in K, \quad \|x\|_{pc^{n-1}} = \varsigma. \tag{3.29}$$

In fact, if there exists $x_2 \in K$ with $\|x_2\|_{pc^{n-1}} = \varsigma$, then by (2.23), we have

$$x_2(t) \geq \gamma^* \|x_2\|_{pc^{n-1}} = \gamma^* \varsigma, \tag{3.30}$$

and it follows from (H_7) that

$$\begin{aligned} x_2(t) &\geq \int_{t_m}^{1} H\left(\frac{1}{2}, s\right) f(s, x_2(s)) ds + \sum_{k=1}^{m} H\left(\frac{1}{2}, t_k\right) I_k(x_2(t_k)) \\ &\geq \varsigma \left[\tau \int_{t_m}^{1} H\left(\frac{1}{2}, s\right) ds + \sum_{k=1}^{m} \tau_k H\left(\frac{1}{2}, t_k\right) \right] \\ &> \varsigma = \|x_2\|_{pc^{n-1}}, \end{aligned} \tag{3.31}$$

that is, $\|x_2\|_{pc^{n-1}} > \|x_2\|_{pc^{n-1}}$, which is a contraction. Hence, (3.29) holds.

Applying Lemma 1.1 to (3.27), (3.28), and (3.29) yields that T has two fixed points x^*, x^{**} with $x^* \in \overline{K}_{\rho,\varsigma}$, $x^{**} \in \overline{K}_{\varsigma,\xi}$. Thus it follows that BVP (1.3) has two positive solutions x^*, x^{**} with $0 < \|x^*\|_{pc^{n-1}} < \varsigma < \|x^{**}\|_{pc^{n-1}}$. The proof is complete. □

Our last results corresponds to the case when problem (1.3) has no positive solution. Write

$$\Delta = H_0(1 + m). \tag{3.32}$$

Theorem 3.4. *Assume* (H_1), (H_2), $f(t, x) < \Delta^{-1} x$, $t \in J$, $x > 0$, *and* $I_k(x) < \Delta^{-1} x$, $\forall x > 0$, *then problem* (1.3) *has no positive solution.*

Proof. Assume to the contrary that problem (1.3) has a positive solution, that is, T has a fixed point y. Then $y \in K$, $y > 0$ for $t \in (0,1)$, and

$$\begin{aligned}
\|y\|_\infty &\leq \int_0^1 H(s) f(s, y(s)) ds + \sum_{k=1}^m H(t_k) I_k(y(t_k)) \\
&< \int_0^1 H(s) \Delta^{-1} y(s) ds + \sum_{k=1}^m H(t_k) \Delta^{-1} \|y\|_\infty \\
&\leq H_0 \Delta^{-1} \|y\|_\infty + \sum_{k=1}^m H_0 \Delta^{-1} \|y\|_\infty \\
&= H_0 \Delta^{-1} (1+m) \|y\|_\infty \\
&= \|y\|_\infty,
\end{aligned} \tag{3.33}$$

which is a contradiction, and this completes the proof. □

To illustrate how our main results can be used in practice we present an example.

Example 3.5. Consider the following boundary value problem:

$$\begin{gathered}
-x^{(4)}(t) = \sqrt[3]{t^5 + 1} x^5 \tanh x, \quad t \in J, \, t \neq \frac{1}{2}, \\
-\Delta x^{(3)}|_{t_1 = 1/2} = x^3 \left(\frac{1}{2}\right), \\
x(0) = x'(0) = x''(0) = 0, \quad x(1) = \int_0^1 t x(t) dt.
\end{gathered} \tag{3.34}$$

Conclusion. BVP (3.34) has at least one positive solution.

Proof. BVP (3.34) can be regarded as a BVP of the form (1.3), where

$$\begin{gathered}
h(t) = t, \quad \mu = \int_0^1 t \cdot t^3 dt = \frac{1}{5}, \quad t_1 = \frac{1}{2}, \quad f(t,x) = \sqrt[3]{t^5 + 1} x^5 \tanh x, \quad I_1(x) = x^3, \\
G_1(t,s) = \frac{1}{6} \begin{cases} t^3 (1-s)^3 - (t-s)^3, & 0 \leq s \leq t \leq 1, \\ t^3 (1-s)^3, & 0 \leq t \leq s \leq 1, \end{cases} \\
G_2(t,s) = \frac{1}{24} t^3 \left(\frac{3}{4} s - 2s^2 + \frac{3}{2} s^3 - \frac{1}{4} s^5\right).
\end{gathered} \tag{3.35}$$

It is not difficult to see that conditions (H_1) and (H_2) hold. In addition,

$$f^0 = \limsup_{x\to 0} \max_{t\in J} \frac{f(t,x)}{x} = 0, \qquad I^0(k) = \limsup_{x\to 0} \frac{I_k(x)}{x} = 0,$$
$$f_\infty = \liminf_{x\to\infty} \min_{t\in J} \frac{f(t,x)}{x} = \infty. \tag{3.36}$$

□

Then, conditions (H_3) and (H_4) of Theorem 3.1 hold. Hence, by Theorem 3.1, the conclusion follows, and the proof is complete.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (10771065), the Natural Sciences Foundation of Heibei Province (A2007001027), the Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (PHR201008430), the Scientific Research Common Program of Beijing Municipal Commission of Education(KM201010772018) and Beijing Municipal Education Commission(71D0911003). The authors thank the referee for his/her careful reading of the paper and useful suggestions.

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