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Research Article

Hierarchies of Difference Boundary Value Problems

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This paper generalises the work done in Currie and Love (2010), where we studied the effect of applying two Crum-type transformations to a weighted second-order difference equation with various combinations of Dirichlet, non-Dirichlet, and affine λ-dependent boundary conditions at the end points, where λ is the eigenparameter. We now consider general λ-dependent boundary conditions. In particular we show, using one of the Crum-type transformations, that it is possible to go up and down a hierarchy of boundary value problems keeping the form of the second-order difference equation constant but possibly increasing or decreasing the dependence on λ of the boundary conditions at each step. In addition, we show that the transformed boundary value problem either gains or loses an eigenvalue, or the number of eigenvalues remains the same as we step up or down the hierarchy.

1. Introduction

Our interest in this topic arose from the work done on transformations and factorisations of continuous Sturm-Liouville boundary value problems by Binding et al. [1] and Browne and Nillsen [2], notably. We make use of analogous ideas to those discussed in [3–5] to study difference equations in order to contribute to the development of the theory of discrete spectral problems.

Numerous efforts to develop hierarchies exist in the literature, however, they are not specifically aimed at difference equations per se and generally not for three-term recurrence relations. Ding et al., [6], derived a hierarchy of nonlinear differential-difference equations by starting with a two-parameter discrete spectral problem, as did Luo and Fan [7], whose hierarchy possessed bi-Hamiltonian structures. Clarkson et al.'s, [8], interest in hierarchies lay in the derivation of infinite sequences of systems of difference equations by using the Bäcklund transformation for the equations in the second Painleve' equation hierarchy. Wu and Geng, [9], showed early on that the hierarchy of differential-difference equations possesses Hamiltonian structures while a Darboux transformation for the discrete spectral problem is shown to exist.

In this paper, we consider a weighted second-order difference equation of the form

$$c(n)y(n+1) - b(n)y(n) + c(n-1)y(n-1) = -c(n)\lambda y(n), \tag{1.1}$$

where $c(n) > 0$ represents a weight function and $b(n)$ a potential function.

Our aim is to extend the results obtained in [10, 11] by establishing a hierarchy of difference boundary value problems. A key tool in our analysis will be the Crum-type transformation (2.1). In [10], it was shown that (2.1) leaves the form of the difference equation (1.1) unchanged. For us, the effect of (2.1) on the boundary conditions will be crucial. We consider λ (eigenparameter)-dependent boundary conditions at the end points. In particular, the eigenparameter dependence at the initial end point will be given by a positive Nevanlinna function, $N(\lambda)$ say, and at the terminal end point by a negative Nevanlinna function, $M(\lambda)$ say. The case of $N(\lambda) = M(\lambda) = 0$ was covered in [10] and the the case of $N(\lambda) = M(\lambda) =$ constant was studied in [11]. Applying transformation (2.1) to the boundary conditions results in a so-called transformed boundary value problem, where either the new boundary conditions have more λ-dependence, less λ-dependence, or the same amount of λ-dependence as the original boundary conditions. Consequently the transformed boundary value problem has either one more eigenvalue, one less eigenvalue, or the same number of eigenvalues as the original boundary value problem. Thus, it is possible to construct a chain, or hierarchy, of difference boundary value problems where the successive links in the chain are obtained by applying the variations of (2.1) given in this paper. For instance, it is possible to go from a boundary value problem with λ-dependent boundary conditions to a boundary value problem with λ-independent boundary conditions or vice versa simply by applying the correct variation of (2.1) an appropriate number of times. Moreover, at each step, we can precisely track the eigenvalues that have been lost or gained. Hence, this paper provides a significant development in the theory of three-term difference boundary value problems in regard to singularities and asymptotics in the hierarchy structure. For similar results in the continuous case, see [12].

There is an obvious connection between the three-term difference equation and orthogonal polynomials. In fact, the three-term recurrence relation satisfied by orthogonal polynomials is perhaps the most important information for the constructive and computational use of orthogonal polynomials [13].

Difference equations and operators and results concerning their existence and construction of their solutions have been discussed in [14, 15]. Difference equations arise in numerous settings and have applications in diverse areas such as quantum field theory, combinatorics, mathematical physics and biology, dynamical systems, economics, statistics, electrical circuit analysis, computer visualization, and many other fields. They are especially useful where recursive computations are required. In particular see [16] [9, Introduction] for three physical applications of the difference equation (1.1), namely, the vibrating string, electrical network theory and Markov processes, in birth and death processes and random walks.

It should be noted that G. Teschl's work, [17, Chapter 11], on spectral and inverse spectral theory of Jacobi operators, provides an alternative factorisation, to that of [10], of a second-order difference equation, where the factors are adjoints of one another.

This paper is structured as follows.

In Section 2, all the necsessary results from [10] are recalled, in particular how (1.1) transforms under (2.1). In addition, we also recap some important properties of Nevanlinna functions.

The focus of Section 3 is to show exactly the effect that (2.1) has on boundary conditions of the form

$$y(-1) = N(\lambda)y(0), \qquad y(m-1) = M(\lambda)y(m). \tag{1.2}$$

We give explicitly the new boundary conditions which are obeyed, from which it can be seen whether the λ-dependence has increased, decreased, or remained the same.

Lastly, in Section 4, we compare the spectrum of the original boundary value problem with that of the transformed boundary value problem and show under which conditions the transformed boundary value problem has one more eigenvalue, one less eigenvalue, or the same number of eigenvalues as the original boundary value problem.

2. Preliminaries

In [10], we considered (1.1) for $n = 0, \dots, m-1$, where the values of $y(-1)$ and $y(m)$ are given by boundary conditions, that is, $y(n)$ is defined for $n = -1, \dots, m$.

Let the mapping $y \mapsto w$ be defined by

$$w(n) := y(n) - y(n-1)\frac{z(n)}{z(n-1)}, \quad n = 0, \dots, m, \tag{2.1}$$

where, throughout this paper, $z(n)$ is a solution to (1.1) for $\lambda = \lambda_0$ such that $z(n) > 0$ for all $n = -1, \dots, m$. Whether or not $z(n)$ obeys the various given boundary conditions (to be specified later) is of vital importance in obtaining the results that follow.

From [10], we have the following theorem.

Theorem 2.1. *Under the mapping* (2.1), (1.1) *transforms to*

$$c_w(n)w(n+1) - b_w(n)w(n) + c_w(n-1)w(n-1) = -\lambda c_w(n)w(n), \tag{2.2}$$

where for $n = 0, \dots, m$

$$\begin{aligned} c_w(n) &= \frac{c(n-1)z(n-1)}{z(n)}, \\ b_w(n) &= \left[\frac{c(n-1)z(n-1)}{c(n)z(n)} + \frac{z(n)}{z(n-1)}\right]\frac{c(n-1)z(n-1)}{z(n)}. \end{aligned} \tag{2.3}$$

We now recall some properties of Nevanlinna functions.

(I) The inverse of a positive Nevanlinna function is a negative Nevanlinna function, that is

$$\frac{1}{N(\lambda)} = -B(\lambda), \tag{2.4}$$

where $N(\lambda)$, $B(\lambda)$ are positive Nevanlinna functions. This follows directly from the fact that $\Im(z) \geq 0$ if and only if $\Im(-1/z) \geq 0$.

(II) If

$$N(\lambda) = b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j}, \quad c_j > 0, \ b \neq 0, \tag{2.5}$$

then

$$\frac{1}{N(\lambda)} = \beta - \sum_{j=1}^{s} \frac{\sigma_j}{\lambda - \delta_j}, \quad \sigma_j > 0, \ \beta \neq 0. \tag{2.6}$$

This follows by (I) together with the fact that since $N(\lambda)$ has s zeros $1/N(\lambda)$ has s poles. Also $N(\lambda) \to b$ as $\lambda \to \pm\infty$ so $1/N(\lambda) \to 1/b := \beta$ as $\lambda \to \pm\infty$. Thus, if $N(\lambda)$ is a positive Nevanlinna function of the form (2.5), then for $b \neq 0$, $1/N(\lambda)$ is a negative Nevanlinna function of the same form.

(III) If

$$N(\lambda) = a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j}, \quad a_j, c_j > 0, \tag{2.7}$$

then

$$\frac{1}{N(\lambda)} = -\sum_{j=1}^{s+1} \frac{\sigma_j}{\lambda - \delta_j}, \quad \sigma_j > 0, \tag{2.8}$$

since $N(\lambda)$ has $s+1$ zeros so $1/N(\lambda)$ has $s+1$ poles and $N(\lambda) \to a\lambda + b \to \pm\infty$ as $\lambda \to \pm\infty$ so $1/N(\lambda) \to 1/(a\lambda + b) \to 0$ as $\lambda \to \pm\infty$.

For the remainder of the paper, $N_{s,j}^{\diamond}(\lambda)$ will denote a Nevanlinna function where

s is the number of terms in the sum;

j indicates the value of n at which the boundary condition is imposed and

$$\diamond = \begin{cases} \pm & \text{if the coefficient of } \lambda \text{ is positive or negative respectively,} \\ 0 & \text{if the coefficient of } \lambda \text{ is zero.} \end{cases} \tag{2.9}$$

3. General λ-Dependent Boundary Conditions

In this section, we show how y obeying general λ-dependent boundary conditions transforms, under (2.1), to w obeying various types of λ-dependent boundary conditions. The exact form of these boundary conditions is obtained by considering the number of zeros and poles (singularities) of the various Nevanlinna functions under discussion and these correlations are illustrated in the different graphs depicted in this section.

Lemma 3.1. *If y obeys the boundary condition*

$$y(-1) = \left[b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k}\right] y(0) := R^0_{s,-1}(\lambda) y(0), \tag{3.1}$$

then the domain of $w(n)$ may be extended from $n = 0, \dots, m$ to $n = -1, \dots, m$ by forcing the condition

$$\frac{w(-1)}{w(0)} = U, \tag{3.2}$$

where

$$U = \frac{b_w(0) - \lambda c_w(0)}{c_w(-1)} - \frac{c_w(0)}{c_w(-1)} \frac{b(0)/c(0) - \lambda - z(1)/z(0) - (c(-1)/c(0)) R^0_{s,-1}(\lambda)}{1 - R^0_{s,-1}(\lambda)(z(0)/z(-1))} \tag{3.3}$$

with $c_w(-1) = c(-1)$.

Proof. The transformed equation (2.2), for $n = 0$, together with (3.2) gives

$$c_w(0) w(1) + c_w(-1) U w(0) = [b_w(0) - \lambda c_w(0)] w(0). \tag{3.4}$$

Also the mapping (2.1), together with (3.1), yields

$$w(0) = y(0) \left[1 - R^0_{s,-1}(\lambda) \frac{z(0)}{z(-1)}\right]. \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$c_w(0) w(1) + c_w(-1) U \left[1 - R^0_{s,-1}(\lambda) \frac{z(0)}{z(-1)}\right] y(0) = [b_w(0) - \lambda c_w(0)] \left[1 - R^0_{s,-1}(\lambda) \frac{z(0)}{z(-1)}\right] y(0). \tag{3.6}$$

Now (2.1), with $n = 1$, gives

$$w(1) = y(1) - y(0) \frac{z(1)}{z(0)} \tag{3.7}$$

which when substituted into (3.6) and dividing through by $c_w(0)$ results in

$$y(1) - y(0) \frac{z(1)}{z(0)} + \frac{c_w(-1)}{c_w(0)} U \left[1 - R^0_{s,-1}(\lambda) \frac{z(0)}{z(-1)}\right] y(0) = \left[\frac{b_w(0)}{c_w(0)} - \lambda\right] \left[1 - R^0_{s,-1}(\lambda) \frac{z(0)}{z(-1)}\right] y(0). \tag{3.8}$$

This may be rewritten as

$$y(1) - y(0)\left\{\frac{z(1)}{z(0)} - \left[\frac{c_w(-1)}{c_w(0)}U - \frac{b_w(0)}{c_w(0)}\right]\left[1 - R^0_{s,-1}(\lambda)\frac{z(0)}{z(-1)}\right]\right\} = -\lambda\left[1 - R^0_{s,-1}(\lambda)\frac{z(0)}{z(-1)}\right]y(0). \tag{3.9}$$

Using (1.1), with $n = 0$, together with (3.1), gives

$$y(1) - \left[\frac{b(0)}{c(0)} - \frac{c(-1)}{c(0)}R^0_{s,-1}(\lambda)\right]y(0) = -\lambda y(0). \tag{3.10}$$

Subtracting (3.10) from (3.9) results in

$$\begin{aligned} &y(0)\left[\frac{b(0)}{c(0)} - \frac{c(-1)}{c(0)}R^0_{s,-1}(\lambda) - \frac{z(1)}{z(0)} + \left[\frac{c_w(-1)}{c_w(0)}U - \frac{b_w(0)}{c_w(0)}\right]\left[1 - R^0_{s,-1}(\lambda)\frac{z(0)}{z(-1)}\right]\right] \\ &\quad = y(0)\left(-\lambda\left[1 - R^0_{s,-1}(\lambda)\frac{z(0)}{z(-1)}\right] + \lambda\right). \end{aligned} \tag{3.11}$$

Rearranging the above equation and dividing through by $[1 - R^0_{s,-1}(\lambda)(z(0)/z(-1))](c_w(-1)/c_w(0))$ yields

$$\begin{aligned} &\frac{(c_w(0)/c_w(-1))\left[b(0)/c(0) - (c(-1)/c(0))R^0_{s,-1}(\lambda) - z(1)/z(0) - \lambda\right]}{1 - R^0_{s,-1}(\lambda)(z(0)/z(-1))} \\ &\quad + U - \frac{b_w(0)}{c_w(-1)} = -\lambda\frac{c_w(0)}{c_w(-1)} \end{aligned} \tag{3.12}$$

and hence

$$U = \frac{b_w(0) - \lambda c_w(0)}{c_w(-1)} - \frac{c_w(0)}{c_w(-1)}\frac{b(0)/c(0) - \lambda - z(1)/z(0) - (c(-1)/c(0))R^0_{s,-1}(\lambda)}{1 - R^0_{s,-1}(\lambda)(z(0)/z(-1))}. \tag{3.13}$$

Thus w obeys the equation on the extended domain. □

The remainder of this section illustrates why it is so important to distinguish between the two cases of z obeying or not obeying the boundary conditions.

Theorem 3.2. *Consider $y(n)$ obeying the boundary condition (3.1) where $R^0_{s,-1}(\lambda)$ is a positive Nevanlinna function, that is, $c_k > 0$ for $k = 1, \ldots, s$. Under the mapping (2.1), y obeying (3.1) transforms to w obeying (3.2) as follows.*

(A) *If z does not obey (3.1) then w obeys*

(i)

$$w(-1) = Uw(0) = \left[\beta - \sum_{t=1}^{s} \frac{\gamma_t}{\lambda - q_t}\right] w(0) := T^0_{s,-1}(\lambda) w(0), \quad b = 0, \tag{3.14}$$

(ii)

$$w(-1) = Uw(0) = \left[\alpha\lambda + \beta - \sum_{t=1}^{s} \frac{\gamma_t}{\lambda - q_t}\right] w(0) := T^+_{s,-1}(\lambda) w(0),$$
$$\frac{z(-1)}{z(0)} > b > 0. \tag{3.15}$$

(B) *If z does obey (3.1) for $\lambda = \lambda_0$ then w obeys*

(i)

$$w(-1) = Uw(0) = \left[\widetilde{\beta} - \sum_{t=1}^{s-1} \frac{\widetilde{\gamma}_t}{\lambda - v_t}\right] w(0) := \widetilde{T}^0_{s-1,-1}(\lambda) w(0), \quad b = 0, \tag{3.16}$$

(ii)

$$w(-1) = Uw(0) = \left[\widetilde{\alpha}\lambda + \widetilde{\beta} - \sum_{t=1}^{s-1} \frac{\widetilde{\gamma}_t}{\lambda - v_t}\right] w(0) := \widetilde{T}^+_{s-1,-1}(\lambda) w(0),$$
$$\frac{z(-1)}{z(0)} > b > 0, \tag{3.17}$$

where $\gamma_t, \widetilde{\gamma}_t, \alpha, \widetilde{\alpha} > 0$, that is, $T^0_{s,-1}(\lambda)$, $T^+_{s,-1}(\lambda)$, $\widetilde{T}^0_{s-1,-1}(\lambda)$, $\widetilde{T}^+_{s-1,-1}(\lambda)$ are positive Nevanlinna functions.

In (A) and (B), $b < 0$ is not possible.

Proof. The fact that $w(-1) = Uw(0)$ is by construction, see Lemma 3.1. We now examine the form of U in Lemma 3.1. Let $\Gamma_1 := b_w(0)/c_w(-1)$, $\Gamma_2 := c_w(0)/c_w(-1)$, $\Gamma_3 := b(0)/c(0) - z(1)/z(0)$ and $\Gamma_4 := c(-1)/c(0)$ then

$$\begin{aligned}\frac{w(-1)}{w(0)} = U &= \Gamma_1 - \lambda\Gamma_2 - \Gamma_2\frac{\Gamma_3 - \lambda - \Gamma_4 R^0_{s,-1}(\lambda)}{1-(z(0)/z(-1))R^0_{s,-1}(\lambda)} \\ &= \Gamma_1 - \lambda\Gamma_2 - \Gamma_2\left[\Gamma_4\frac{z(-1)}{z(0)} + \frac{\Gamma_3 - \lambda - \Gamma_4(z(-1)/z(0))}{1-(z(0)/z(-1))R^0_{s,-1}(\lambda)}\right] \\ &= \Gamma_1 - \lambda\Gamma_2 - \Gamma_2\Gamma_4\frac{z(-1)}{z(0)} + \Gamma_2\frac{(z(-1)/z(0))[\lambda - \Gamma_3 + \Gamma_4(z(-1)/z(0))]}{(z(-1)/z(0)) - R^0_{s,-1}(\lambda)}.\end{aligned} \tag{3.18}$$

But

$$\Gamma_3 - \Gamma_4\frac{z(-1)}{z(0)} = \frac{b(0)}{c(0)} - \frac{z(1)}{z(0)} - \frac{c(-1)}{c(0)} = \lambda_0 \tag{3.19}$$

thus

$$\frac{w(-1)}{w(0)} = U = \Gamma_1 - \lambda\Gamma_2 - \Gamma_2\Gamma_4\frac{z(-1)}{z(0)} + \Gamma_2\frac{(z(-1)/z(0))(\lambda-\lambda_0)}{(z(-1)/z(0)) - R^0_{s,-1}(\lambda)}. \tag{3.20}$$

Now $(\lambda-\lambda_0)/[(z(-1)/z(0)) - R^0_{s,-1}(\lambda)]$ has the expansion

$$f(\lambda) - \sum_{t=1}^{p}\frac{r_t}{\lambda - q_t}, \tag{3.21}$$

where $r_t > 0$ and the q_t's correspond to where $z(-1)/z(0) = R^0_{s,-1}(\lambda)$, that is, the singularities of (3.20).

Since $R^0_{s,-1}(\lambda)$ is a positive Nevanlinna function it has a graph of the form shown in Figure 1.

Clearly, the gradient of $R^0_{s,-1}(\lambda)$ at q_t is positive for all t, that is,

$$\frac{\partial}{\partial\lambda}R^0_{s,-1}(\lambda)\Big|_{q_t} > 0, \qquad t = 1,\ldots,p. \tag{3.22}$$

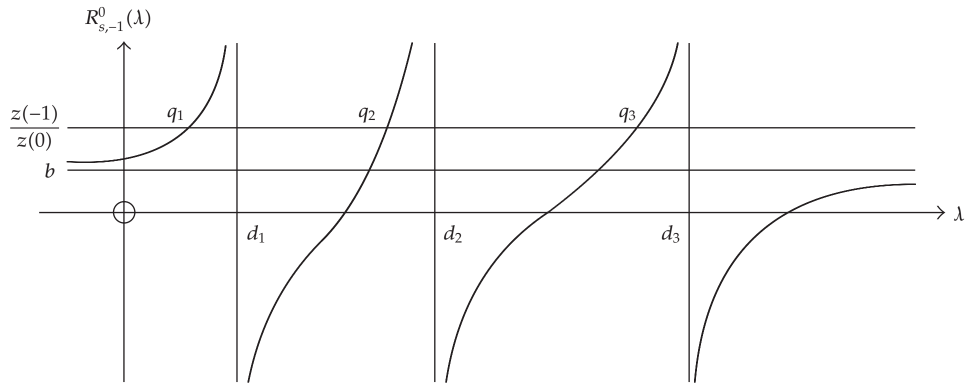

Figure 1: $R^0_{s,-1}(\lambda)$.

If z does not obey (3.1), then the zeros of

$$\frac{\lambda-\lambda_0}{(z(-1)/z(0))-R^0_{s,-1}(\lambda)} \tag{3.23}$$

are the poles of $R^0_{s,-1}(\lambda)$, that is, the d_k's and $\lambda=\lambda_0$ where $d_k\neq\lambda_0$ for $k=1,\ldots,s$. It is evident, from Figure 1, that the number of q_t's is equal to the number of d_k's, thus in (3.21), $p=s$.

We now examine the form of $f(\lambda)$ in (3.21). As $\lambda\to\pm\infty$ it follows that $R^0_{s,-1}(\lambda)\to b$. Thus

$$\frac{\lambda-\lambda_0}{(z(-1)/z(0))-R^0_{s,-1}(\lambda)}\longrightarrow\frac{\lambda-\lambda_0}{(z(-1)/z(0))-b}. \tag{3.24}$$

Therefore

$$f(\lambda)=\frac{\lambda-\lambda_0}{(z(-1)/z(0))-b}. \tag{3.25}$$

Hence, substituting into (3.20) gives

$$\begin{aligned}\frac{w(-1)}{w(0)}=U&=\Gamma_1-\lambda\Gamma_2-\Gamma_2\Gamma_4\frac{z(-1)}{z(0)}+\Gamma_2\frac{z(-1)}{z(0)}\left[f(\lambda)-\sum_{t=1}^{s}\frac{r_t}{\lambda-q_t}\right]\\&=\Gamma_1-\lambda\Gamma_2-\Gamma_2\Gamma_4\frac{z(-1)}{z(0)}+\Gamma_2\frac{z(-1)}{z(0)}\left[\frac{\lambda-\lambda_0}{(z(-1)/z(0))-b}-\sum_{t=1}^{s}\frac{r_t}{\lambda-q_t}\right]\\&=\Gamma_1-\Gamma_2\Gamma_4\frac{z(-1)}{z(0)}-\frac{\lambda_0\Gamma_2}{1-b(z(0)/z(-1))}+\lambda\left[-\Gamma_2+\frac{\Gamma_2}{1-b(z(0)/z(-1))}\right]\\&\quad-\Gamma_2\frac{z(-1)}{z(0)}\sum_{t=1}^{s}\frac{r_t}{\lambda-q_t}.\end{aligned} \tag{3.26}$$

Let

$$\beta := \Gamma_1 - \Gamma_2\Gamma_4\frac{z(-1)}{z(0)} - \frac{\lambda_0\Gamma_2}{1 - b(z(0)/z(-1))},$$

$$\alpha := -\Gamma_2 + \frac{\Gamma_2}{1 - b(z(0)/z(-1))} = \frac{\Gamma_2 b}{(z(-1)/z(0)) - b}, \tag{3.27}$$

$$\gamma_t := \Gamma_2\frac{z(-1)}{z(0)}r_t.$$

Then since $\Gamma_2 > 0$, $z(-1)/z(0) > 0$ and $r_t > 0$ we have that $\gamma_t > 0$ and clearly if $b = 0$ then $\alpha = 0$ giving (3.14), that is,

$$w(-1) = Uw(0) = \left[\beta - \sum_{t=1}^{s}\frac{\gamma_t}{\lambda - q_t}\right]w(0) := T^0_{s,-1}(\lambda)w(0). \tag{3.28}$$

If $b \neq 0$ then we want $\alpha > 0$ so that we have a positive Nevanlinna function, that is

$$\frac{\Gamma_2 b}{(z(-1)/z(0)) - b} > 0 \tag{3.29}$$

which means that either,

$$\Gamma_2 b > 0, \quad \frac{z(-1)}{z(0)} - b > 0, \tag{3.30}$$

giving that, since $\Gamma_2 > 0$,

$$b > 0, \quad \frac{z(-1)}{z(0)} > b, \tag{3.31}$$

which is as shown in Figure 1, or,

$$\Gamma_2 b < 0, \quad \frac{z(-1)}{z(0)} - b < 0, \tag{3.32}$$

giving that

$$b < 0, \quad \frac{z(-1)}{z(0)} < b, \tag{3.33}$$

but this means that $z(-1)/z(0) < 0$ which is not possible.

Thus, $\alpha > 0$ for $z(-1)/z(0) > b > 0$, that is, given b, the ratio $z(-1)/z(0)$ must be chosen suitably to ensure that $T^{+}_{s,-1}(\lambda)$ is a positive Nevanlinna function as required. Hence we obtain (3.15), that is

$$w(-1) = Uw(0) = \left[\alpha\lambda + \beta - \sum_{t=1}^{s} \frac{\gamma_t}{\lambda - q_t}\right] w(0) := T^{+}_{s,-1}(\lambda) w(0). \tag{3.34}$$

If z obeys (3.1), for $\lambda = \lambda_0$, then $z(-1)/z(0) = R^{0}_{s,-1}(\lambda_0)$. Thus in Figure 1, one of the q_t's $t = 1, \ldots, s$ is equal to λ_0 and since λ_0 is less than the least eigenvalue of the boundary value problem (1.1), (3.1) together with a boundary condition at $m-1$ (specified later) it follows that $q_1 = \lambda_0$, as $\lambda_0 < d_k$ for all $k = 1, \ldots, s$.

Now

$$\frac{\lambda - \lambda_0}{(z(-1)/z(0)) - R^{0}_{s,-1}(\lambda)} = \frac{\lambda - \lambda_0}{R^{0}_{s,-1}(\lambda_0) - R^{0}_{s,-1}(\lambda)} = \frac{1}{\left(R^{0}_{s,-1}(\lambda_0) - R^{0}_{s,-1}(\lambda)\right)/(\lambda - \lambda_0)} \tag{3.35}$$

and as $\lambda \longrightarrow \lambda_0$

$$\frac{1}{\left(R^{0}_{s,-1}(\lambda_0) - R^{0}_{s,-1}(\lambda)\right)/(\lambda - \lambda_0)} \longrightarrow -\frac{\partial}{\partial\lambda} R^{0}_{s,-1}(\lambda)\Big|_{\lambda_0} < 0. \tag{3.36}$$

Thus $\lambda = \lambda_0 = q_1$ is a removable singularity. Alternatively,

$$\begin{aligned} \frac{\lambda - \lambda_0}{R^{0}_{s,-1}(\lambda_0) - R^{0}_{s,-1}(\lambda)} &= \frac{\lambda - \lambda_0}{b - \sum_{k=1}^{s}(c_k/(\lambda_0 - d_k)) - b + \sum_{k=1}^{s}(c_k/(\lambda - d_k))} \\ &= \frac{-1}{\sum_{k=1}^{s}(c_k/((\lambda_0 - d_k)(\lambda - d_k)))}, \end{aligned} \tag{3.37}$$

which illustrates that the singularity at $\lambda = \lambda_0 = q_1$ is removable.

We now have that the number of nonremovable singularities, q_t, in (3.20) is one less than the number of d_k's $k = 1, \ldots, s$, see Figure 1. Thus (3.21) becomes

$$f(\lambda) - \sum_{t=2}^{s} \frac{r_t}{\lambda - q_t}, \quad r_t > 0 \tag{3.38}$$

which may be rewritten as

$$f(\lambda) - \sum_{t=1}^{s-1} \frac{\tilde{r}_t}{\lambda - v_t}, \quad \tilde{r}_t > 0, \tag{3.39}$$

where $v_n = q_{n+1}$, $\tilde{r}_n = r_{n+1}$ for $n = 1, \ldots, s-1$.

We now examine the form of $f(\lambda)$ in (3.39). As $\lambda \to \pm\infty$, we have that, as before, $R^0_{s,-1}(\lambda) \to b$. Thus

$$f(\lambda) = \frac{\lambda - \lambda_0}{R^0_{s,-1}(\lambda_0) - b}. \tag{3.40}$$

Hence, from (3.20),

$$\begin{aligned}
\frac{w(-1)}{w(0)} = U &= \Gamma_1 - \lambda\Gamma_2 - \Gamma_2\Gamma_4 R^0_{s,-1}(\lambda_0) + \Gamma_2 R^0_{s,-1}(\lambda_0)\left[f(\lambda) - \sum_{t=1}^{s-1}\frac{\widetilde{r}_t}{\lambda - v_t}\right] \\
&= \Gamma_1 - \lambda\Gamma_2 - \Gamma_2\Gamma_4 R^0_{s,-1}(\lambda_0) + \Gamma_2 R^0_{s,-1}(\lambda_0)\left[\frac{\lambda - \lambda_0}{R^0_{s,-1}(\lambda_0) - b} - \sum_{t=1}^{s-1}\frac{\widetilde{r}_t}{\lambda - v_t}\right] \\
&= \Gamma_1 - \Gamma_2\Gamma_4 R^0_{s,-1}(\lambda_0) - \frac{\lambda_0\Gamma_2}{1 - b\left(1/R^0_{s,-1}(\lambda_0)\right)} + \lambda\left[-\Gamma_2 + \frac{\Gamma_2}{1 - b\left(1/R^0_{s,-1}(\lambda_0)\right)}\right] \\
&\quad - \Gamma_2 R^0_{s,-1}(\lambda_0)\sum_{t=1}^{s-1}\frac{\widetilde{r}_t}{\lambda - v_t}.
\end{aligned} \tag{3.41}$$

Let

$$\begin{aligned}
\widetilde{\beta} &:= \Gamma_1 - \Gamma_2\Gamma_4 R^0_{s,-1}(\lambda_0) - \frac{\lambda_0\Gamma_2}{1 - b\left(1/R^0_{s,-1}(\lambda_0)\right)}, \\
\widetilde{\alpha} &:= -\Gamma_2 + \frac{\Gamma_2}{1 - b\left(1/R^0_{s,-1}(\lambda_0)\right)} = \frac{\Gamma_2 b}{R^0_{s,-1}(\lambda_0) - b}, \\
\widetilde{\gamma}_t &:= \Gamma_2 R^0_{s,-1}(\lambda_0)\widetilde{r}_t.
\end{aligned} \tag{3.42}$$

Then since $\Gamma_2 > 0$, $R^0_{s,-1}(\lambda_0) > 0$ and $\widetilde{r}_t > 0$ we have that $\widetilde{\gamma}_t > 0$ and clearly if $b = 0$ then $\widetilde{\alpha} = 0$ giving (3.16), that is,

$$w(-1) = Uw(0) = \left[\widetilde{\beta} - \sum_{t=1}^{s-1}\frac{\widetilde{\gamma}_t}{\lambda - v_t}\right]w(0) := \widetilde{T}^0_{s-1,-1}(\lambda)w(0). \tag{3.43}$$

If $b \neq 0$ then we need $\widetilde{\alpha} > 0$ so that we have a positive Nevanlinna function, that is

$$\frac{\Gamma_2 b}{R^0_{s,-1}(\lambda_0) - b} > 0 \tag{3.44}$$

which means that either

$$\Gamma_2 b > 0, \quad R^0_{s,-1}(\lambda_0) - b > 0, \tag{3.45}$$

giving that, since $\Gamma_2 > 0$,

$$b > 0, \quad R^0_{s,-1}(\lambda_0) = \frac{z(-1)}{z(0)} > b, \tag{3.46}$$

which is as shown in Figure 1, or,

$$\Gamma_2 b < 0, \quad R^0_{s,-1}(\lambda_0) - b < 0, \tag{3.47}$$

giving that

$$b < 0, \quad R^0_{s,-1}(\lambda_0) < b, \tag{3.48}$$

but this means that $R^0_{s,-1}(\lambda_0) = z(-1)/z(0) < 0$ which is not possible.

Thus, $\tilde{\alpha} > 0$ for $R^0_{s,-1}(\lambda_0) > b > 0$, that is, given b, the ratio $z(-1)/z(0) = R^0_{s,-1}(\lambda_0)$ must be chosen suitably to ensure that $\tilde{T}^+_{s-1,-1}(\lambda)$ is a positive Nevanlinna function as required. Hence, we obtain (3.17), that is,

$$w(-1) = Uw(0) = \left[\tilde{\alpha}\lambda + \tilde{\beta} - \sum_{t=1}^{s-1} \frac{\tilde{\gamma}_t}{\lambda - v_t}\right] w(0) := \tilde{T}^+_{s-1,-1}(\lambda) w(0). \tag{3.49}$$

□

In the theorem below, we increase the λ dependence by introducing a nonzero λ term in the original boundary condition. As in Theorem 3.2, the λ dependence of the transformed boundary condition depends on whether or not z obeys the given boundary condition. In addition, to ensure that the λ dependence of the transformed boundary condition is given by a positive Nevanlinna function it is necessary that the transformed boundary condition is imposed at 0 and 1 as opposed to -1 and 0. Thus the interval under consideration shrinks by one unit at the initial end point. By routine calculation it can be shown that the form of the λ dependence of the transformed boundary condition, if imposed at -1 and 0, is neither a positive Nevalinna function nor a negative Nevanlinna function.

Theorem 3.3. *Consider $y(n)$ obeying the boundary condition*

$$y(-1) = \left[a\lambda + b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k}\right] y(0) := R^+_{s,-1}(\lambda) y(0), \tag{3.50}$$

where $R^+_{s,-1}(\lambda)$ is a positive Nevanlinna function, that is, $a > 0$ and $c_k > 0$ for $k = 1, \ldots, s$. Under the mapping (2.1), *y obeying* (3.50) *transforms to w obeying the following.*

(1) *If z does not obey* (3.50) *then w obeys*

$$w(0) = \left[\widehat{\beta} - \sum_{t=1}^{s+1} \frac{\widehat{\gamma}_t}{\lambda - \widehat{\delta}_t}\right] w(1) := \widehat{T}^0_{s+1,0}(\lambda) w(1). \tag{3.51}$$

(2) *If z does obey* (3.50), *for $\lambda = \lambda_0$, then w obeys*

$$w(0) = \left[\overline{\beta} - \sum_{t=1}^{s} \frac{\overline{\gamma}_t}{\lambda - \overline{\delta}_t}\right] w(1) := \overline{T}^0_{s,0}(\lambda) w(1), \tag{3.52}$$

where $\widehat{\gamma}_t, \overline{\gamma}_t > 0$.

Proof. Since $w(0)$ and $w(1)$ are defined we do not need to extend the domain in order to impose the boundary conditions (3.51) or (3.52).

The mapping (2.1), at $n = 0$, together with (3.50) gives

$$w(0) = y(0)\left[1 - R^+_{s,-1}(\lambda)\frac{z(0)}{z(-1)}\right]. \tag{3.53}$$

Also (2.1), at $n = 1$, is

$$w(1) = y(1) - y(0)\frac{z(1)}{z(0)}. \tag{3.54}$$

Substituting in for $y(1)$ from (1.1), with $n = 1$, and using (3.50), we obtain that

$$w(1) = y(0)\left[\frac{b(0)}{c(0)} - \frac{z(1)}{z(0)} - \lambda - \frac{c(-1)}{c(0)} R^+_{s,-1}(\lambda)\right]. \tag{3.55}$$

From (3.53) and (3.55), it now follows that

$$\frac{w(0)}{w(1)} = \frac{1 - R^+_{s,-1}(\lambda)(z(0)/z(-1))}{b(0)/c(0) - z(1)/z(0) - \lambda - (c(-1)/c(0))R^+_{s,-1}(\lambda)}. \tag{3.56}$$

As in Theorem 3.2, let $\Gamma_3 = b(0)/c(0) - z(1)/z(0)$ and $\Gamma_4 = c(-1)/c(0)$. Then (3.56) becomes

$$\begin{aligned}\frac{w(0)}{w(1)} &= \frac{1 - R^+_{s,-1}(\lambda)(z(0)/z(-1))}{\Gamma_3 - \lambda - \Gamma_4 R^+_{s,-1}(\lambda)} \\ &= \frac{z(0)/z(-1)}{\left(\Gamma_3 - \lambda - \Gamma_4 R^+_{s,-1}(\lambda)\right)/\left((z(-1)/z(0)) - R^+_{s,-1}(\lambda)\right)} \\ &= \frac{z(0)/z(-1)}{\Gamma_4 - (-\Gamma_3 + \lambda + \Gamma_4(z(-1)/z(0)))/\left((z(-1)/z(0)) - R^+_{s,-1}(\lambda)\right)}.\end{aligned} \tag{3.57}$$

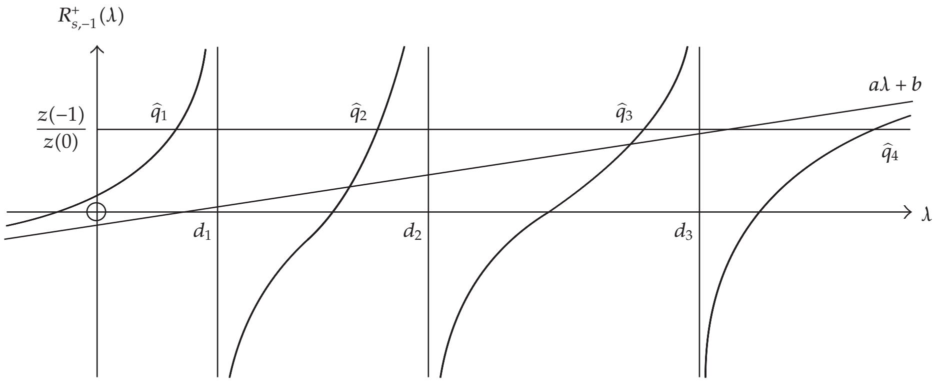

Figure 2: $R^{+}_{s,-1}(\lambda)$.

From Theorem 3.2, we have that $\Gamma_3 - \Gamma_4(z(-1)/z(0)) = \lambda_0$ so

$$\frac{w(0)}{w(1)} = \frac{z(0)}{z(-1)}\left[\frac{1}{\Gamma_4 - (\lambda - \lambda_0)/\left((z(-1)/z(0)) - R^{+}_{s,-1}(\lambda)\right)}\right]. \tag{3.58}$$

Also, as in Theorem 3.2,

$$\frac{\lambda - \lambda_0}{(z(-1)/z(0)) - R^{+}_{s,-1}(\lambda)}, \tag{3.59}$$

has the expansion

$$\widehat{f}(\lambda) - \sum_{t=1}^{p} \frac{\widehat{r}_t}{\lambda - \widehat{q}_t}, \quad \widehat{r}_t > 0, \tag{3.60}$$

where $\widehat{q}_t$ corresponds to $z(-1)/z(0) = R^{+}_{s,-1}(\lambda)$, that is, the singularities of (3.59). Now $R^{+}_{s,-1}(\lambda)$ is a positive Nevanlinna function with graph given in Figure 2.

Clearly, the gradient of $R^{+}_{s,-1}(\lambda)$ at $\widehat{q}(t)$ is positive for all $t = 1, \dots, p$, that is,

$$\frac{\partial}{\partial \lambda} R^{+}_{s,-1}(\lambda)\Big|_{\widehat{q}_t} > 0, \quad t = 1, \dots, p. \tag{3.61}$$

If z does not obey (3.50) then the zeros of

$$\frac{\lambda-\lambda_0}{(z(-1)/z(0))-R^+_{s,-1}(\lambda)} \tag{3.62}$$

are the poles of $R^+_{s,-1}(\lambda)$, that is, the d_k's and $\lambda=\lambda_0$ where $d_k\neq\lambda_0$ for $k=1,\dots,s$. It is evident, from Figure 2, that the number of $\hat{q}_t$'s is one more than the number of d_k's, thus in (3.60), $p=s+1$.

We now examine the form of $\hat{f}(\lambda)$ in (3.60). As $\lambda\to\pm\infty$ it follows that $R^+_{s,-1}(\lambda)\to a\lambda+b$, thus

$$\frac{\lambda-\lambda_0}{(z(-1)/z(0))-R^+_{s,-1}(\lambda)} \longrightarrow \frac{\lambda-\lambda_0}{(z(-1)/z(0))-(a\lambda+b)} \longrightarrow -\frac{1}{a}. \tag{3.63}$$

Hence, $\hat{f}(\lambda)=-1/a$.

Using (3.58) we now obtain

$$\begin{aligned}\frac{w(0)}{w(1)} &= \frac{z(0)}{z(-1)}\left[\frac{1}{\Gamma_4-\left(\hat{f}(\lambda)-\sum_{t=1}^{s+1}(\hat{r}_t/(\lambda-\hat{q}_t))\right)}\right]\\ &= \frac{z(0)}{z(-1)}\left[\frac{1}{\Gamma_4+1/a+\sum_{t=1}^{s+1}(\hat{r}_t/(\lambda-\hat{q}_t))}\right]\\ &= \frac{1}{(z(-1)/z(0))\Gamma_4+(z(-1)/z(0))(1/a)+\sum_{t=1}^{s+1}(\hat{r}_t(z(-1)/z(0))/(\lambda-\hat{q}_t))}.\end{aligned} \tag{3.64}$$

Note that $\hat{r}_t z(-1)/z(0)>0$. Let

$$\Delta := \frac{z(-1)}{z(0)}\Gamma_4+\frac{z(-1)}{z(0)}\frac{1}{a}, \tag{3.65}$$

then

$$\frac{w(0)}{w(1)} = \frac{1}{\Delta-\sum_{t=1}^{s+1}(-\hat{r}_t(z(-1)/z(0))/(\lambda-\hat{q}_t))}. \tag{3.66}$$

Now $\Delta \neq 0$ since if $\Delta = 0$ then $\Gamma = -1/a$, that is, $c(0)/c(-1) = -a$ but $a > 0$ and $c(0)/c(-1) > 0$ so this is not possible. Therefore by Section 2, Nevanlinna result (II), we have that

$$w(0) = \left[\widehat{\beta} - \sum_{t=1}^{s+1} \frac{\widehat{\gamma}_t}{\lambda - \widehat{\delta}_t}\right] w(1) := \widehat{T}^0_{s+1,0}(\lambda) w(1), \tag{3.67}$$

that is, (3.51) holds.

If z does obey (3.50) for $\lambda = \lambda_0$ then $z(-1)/z(0) = R^+_{s,-1}(\lambda_0)$. Thus, in Figure 2, one of the $\widehat{q}_t$'s, $t = 1, \ldots, p$ is equal to λ_0 and since λ_0 is less than the least eigenvalue of the boundary value problem (1.1), (3.50) together with a boundary condition at $m - 1$ (specified later) it follows that $\widehat{q}_1 = \lambda_0$, as $\lambda_0 < d_k$ for all $k = 1, \ldots, s$.

Now (3.59) can be written as

$$\frac{\lambda - \lambda_0}{R^+_{s,-1}(\lambda_0) - R^+_{s,-1}(\lambda)} = \frac{1}{\left(R^+_{s,-1}(\lambda_0) - R^+_{s,-1}(\lambda)\right)/(\lambda - \lambda_0)} \tag{3.68}$$

and as $\lambda \longrightarrow \lambda_0$

$$\frac{1}{\left(R^+_{s,-1}(\lambda_0) - R^+_{s,-1}(\lambda)\right)/(\lambda - \lambda_0)} \longrightarrow -\frac{\partial}{\partial \lambda} R^+_{s,-1}(\lambda)\Big|_{\lambda_0} < 0. \tag{3.69}$$

Thus $\lambda = \lambda_0 = \widehat{q}_1$ is a removable singularity. Alternatively, we could substitute in for $R^+_{s,-1}(\lambda)$ and $R^+_{s,-1}(\lambda_0)$ to illustrate that the singularity at $\lambda = \lambda_0 = \widehat{q}_1$ is removable, see Theorem 3.2. Hence the number of nonremovable $\widehat{q}_t$'s is the same as the number of d_k's, see Figure 2. So (3.60) becomes

$$\widehat{f}(\lambda) - \sum_{t=2}^{s+1} \frac{\widehat{r}_t}{\lambda - \widehat{q}_t}, \quad \widehat{r}_t > 0, \tag{3.70}$$

which may be rewritten as

$$\widehat{f}(\lambda) - \sum_{t=1}^{s} \frac{\overline{r}_t}{\lambda - \overline{q}_t}, \quad \overline{r}_t > 0, \tag{3.71}$$

where $\overline{r}_n = \widehat{r}_{n+1}$ and $\overline{q}_n = \widehat{q}_{n+1}$ for $n = 1, \ldots, s$.

We now examine the form of $\widehat{f}(\lambda)$ in (3.70). As $\lambda \to \pm\infty$, we have that $R^+_{s,-1}(\lambda) \to a\lambda + b$, thus

$$\frac{\lambda - \lambda_0}{R^+_{s,-1}(\lambda_0) - R^+_{s,-1}(\lambda)} \longrightarrow \frac{\lambda - \lambda_0}{R^+_{s,-1}(\lambda_0) - (a\lambda + b)} \longrightarrow -\frac{1}{a}. \tag{3.72}$$

Hence, $\widehat{f}(\lambda) = -1/a$. So, from (3.58) with $z(-1)/z(0) = R^{+}_{s,-1}(\lambda_0)$, we obtain

$$\frac{w(0)}{w(1)} = \frac{1}{R^{+}_{s,-1}(\lambda_0)\left[\Gamma_4 + 1/a + \sum_{t=1}^{s}\left(\overline{r}_t/(\lambda - \overline{q}_t)\right)\right]} = \frac{1}{\Delta - \sum_{t=1}^{s}\left(-\overline{r}_t R^{+}_{s,-1}(\lambda_0)/(\lambda - \overline{q}_t)\right)}, \tag{3.73}$$

where, as before,

$$\Delta := \frac{z(-1)}{z(0)}\Gamma_4 + \frac{z(-1)}{z(0)}\frac{1}{a} = R^{+}_{s,-1}(\lambda_0)\Gamma_4 + R^{+}_{s,-1}(\lambda_0)\frac{1}{a} \neq 0. \tag{3.74}$$

Thus, by Section 2, Nevanlinna result (II), we have that

$$w(0) = \left[\overline{\beta} - \sum_{t=1}^{s}\frac{\overline{\gamma}_t}{\lambda - \overline{\delta}_t}\right]w(1) := \overline{T}^{0}_{s,0}(\lambda)w(1), \quad \overline{\gamma}_t > 0, \tag{3.75}$$

that is, (3.52) holds. □

In Theorem 3.4, we impose a boundary condition at the terminal end point and show how it is transformed according to whether or not z obeys the given boundary condition.

Theorem 3.4. *Consider y obeying the boundary condition at $n = m$ given by*

$$y(m-1) = y(m)\left[g\lambda + h - \sum_{k=1}^{l}\frac{s_k}{\lambda - p_k}\right] := y(m)R^{-}_{l,m}(\lambda), \tag{3.76}$$

where $R^{-}_{l,m}(\lambda)$ is a negative Nevanlinna function, that is, $g < 0$ and $s_k < 0$ for $k = 1, \ldots, l$. Under the mapping (2.1), *y obeying* (3.76) *transforms to w obeying the following.*

(I) *If z does not obey* (3.76) *then w obeys*

$$w(m-1) = w(m)\left[\phi\lambda + \varphi - \sum_{t=1}^{l+1}\frac{\epsilon_t}{\lambda - \sigma_t}\right] := w(m)T^{-}_{l+1,m}(\lambda). \tag{3.77}$$

(II) *If z does obey* (3.76) *then w obeys*

$$w(m-1) = w(m)\left[\widetilde{\phi}\lambda + \widetilde{\varphi} - \sum_{t=1}^{l}\frac{\widetilde{\epsilon}_t}{\lambda - \widetilde{\sigma}_t}\right] := w(m)\widetilde{T}^{-}_{l,m}(\lambda), \tag{3.78}$$

where $\phi, \widetilde{\phi}, \epsilon_k, \widetilde{\epsilon}_k < 0$.

Proof. Since $w(m-1)$ and $w(m)$ are defined we do not need to extend the domain of w in order to impose the boundary conditions (3.77) or (3.78).

The mapping (2.1), at $n = m-1$, gives

$$w(m-1) = y(m-1) - y(m-2)\frac{z(m-1)}{z(m-2)}. \tag{3.79}$$

From (1.1), with $n = m-1$, we can substitute in for $y(m-2)$ in the above equation to get

$$w(m-1) = y(m-1)\left[1 + \lambda\frac{z(m-1)c(m-1)}{z(m-2)c(m-2)} - \frac{z(m-1)b(m-1)}{z(m-2)c(m-2)}\right] + \frac{z(m-1)c(m-1)}{z(m-2)c(m-2)}y(m). \tag{3.80}$$

Using (3.76), we obtain

$$w(m-1) = y(m)\left\{R^-_{l,m}(\lambda)\left[1 + \lambda\frac{z(m-1)c(m-1)}{z(m-2)c(m-2)} - \frac{z(m-1)b(m-1)}{z(m-2)c(m-2)}\right] + \frac{z(m-1)c(m-1)}{z(m-2)c(m-2)}\right\}. \tag{3.81}$$

But z obeys (1.1) at $n = m-1$, for $\lambda = \lambda_0$, so that (3.81) becomes

$$w(m-1) = y(m)\frac{z(m-1)c(m-1)}{z(m-2)c(m-2)}\left\{R^-_{l,m}(\lambda)(\lambda-\lambda_0) - R^-_{l,m}(\lambda)\frac{z(m)}{z(m-1)} + 1\right\}. \tag{3.82}$$

Also, for $n = m$, (2.1) together with (3.76) yields

$$w(m) = y(m)\left[1 - R^-_{l,m}(\lambda)\frac{z(m)}{z(m-1)}\right]. \tag{3.83}$$

Therefore,

$$\frac{w(m-1)}{w(m)} = \frac{z(m-1)c(m-1)}{z(m-2)c(m-2)}\left[\frac{(\lambda-\lambda_0)R^-_{l,m}(\lambda)}{1 - R^-_{l,m}(\lambda)(z(m)/z(m-1))} + 1\right]. \tag{3.84}$$

Let $\Omega := (z(m-1)c(m-1))/(z(m-2)c(m-2)) > 0$, then (3.84) may be rewritten as

$$\frac{w(m-1)}{w(m)} = \Omega - \Omega\left[\frac{\lambda-\lambda_0}{(z(m)/z(m-1)) - 1/R^-_{l,m}(\lambda)}\right]. \tag{3.85}$$

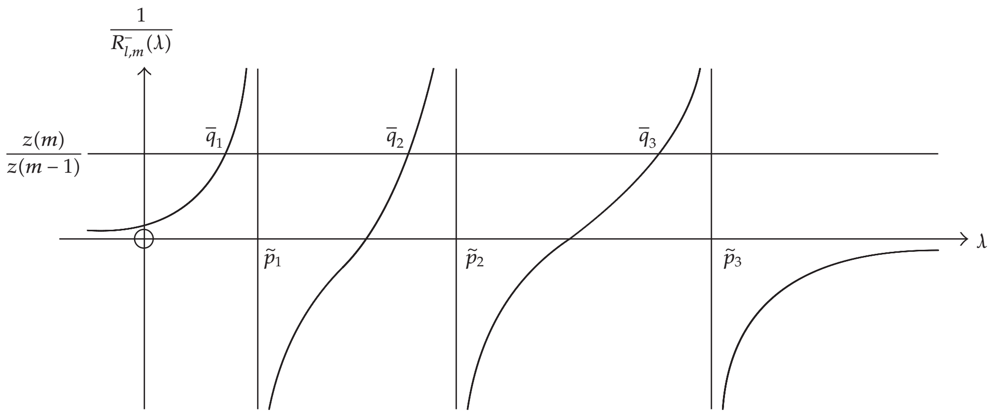

Figure 3: $1/R^-_{l,m}(\lambda)$.

By Section 2, Nevanlinna result (I), since $R^-_{l,m}(\lambda)$ is a negative Nevanlinna function it follows that $1/R^-_{l,m}(\lambda)$ is a positive Nevanlinna function, which has the form

$$\frac{1}{R^-_{l,m}(\lambda)} = -\sum_{k=1}^{l+1} \frac{\tilde{s}_k}{\lambda - \tilde{p}_k}, \quad \tilde{s}_k > 0, \tag{3.86}$$

by Section 2, Nevanlinna result (III).

As before $(\lambda - \lambda_0)/((z(m)/z(m-1)) - 1/R^-_{l,m}(\lambda))$ has expansion

$$\overline{f}(\lambda) - \sum_{t=1}^{p} \frac{\overline{r}_t}{\lambda - \overline{q}_t}, \quad \overline{r}_t > 0, \tag{3.87}$$

where $\overline{q}_t$, $t = 1, \dots, p$, corresponds to the singularities of (3.85), that is, where $z(m-1)/z(m) = R^-_{l,m}(\lambda)$. The graph of $1/R^-_{l,m}(\lambda)$ is as shown in Figure 3.

As before, the gradient of $1/R^-_{l,m}(\lambda)$ at $\overline{q}(t)$ is positive for all $t = 1, \dots, p$, that is

$$\frac{\partial}{\partial \lambda} \frac{1}{R^-_{l,m}(\lambda)}\bigg|_{\overline{q}_t} > 0, \quad t = 1, \dots, p. \tag{3.88}$$

If z does not obey (3.76) then the zeros of

$$\frac{\lambda - \lambda_0}{(z(m)/z(m-1)) - 1/R^-_{l,m}(\lambda)} \tag{3.89}$$

are the poles of $1/R^-_{l,m}(\lambda)$, that is, the $\tilde{p}_k$'s and $\lambda = \lambda_0$ where $\tilde{p}_k \neq \lambda_0$ for $k = 1, \dots, l+1$. Clearly, from Figure 3, the number of $\overline{q}_t$'s is the same as the the number of $\tilde{p}_k$'s, thus in (3.87), $p = l + 1$.

Next, we examine the form of $\overline{f}(\lambda)$ in (3.87). As $\lambda \to \pm\infty$ it follows that $1/R^{-}_{l,m}(\lambda) \to 1/(g\lambda + h) \to 0$. Thus

$$\frac{\lambda - \lambda_0}{(z(m)/z(m-1)) - 1/R^{-}_{l,m}(\lambda)} \longrightarrow (\lambda - \lambda_0)\frac{z(m-1)}{z(m)}. \tag{3.90}$$

Therefore, $\overline{f}(\lambda) = (\lambda - \lambda_0)(z(m-1)/z(m))$. Hence,

$$\begin{aligned}
\frac{w(m-1)}{w(m)} &= \Omega - \Omega\left[\overline{f}(\lambda) - \sum_{t=1}^{l+1}\frac{\overline{r}_t}{\lambda - \overline{q}_t}\right] \\
&= \Omega - \Omega\left[(\lambda - \lambda_0)\frac{z(m-1)}{z(m)} - \sum_{t=1}^{l+1}\frac{\overline{r}_t}{\lambda - \overline{q}_t}\right] \\
&= \phi\lambda + \varphi - \sum_{t=1}^{l+1}\frac{\epsilon_t}{\lambda - \sigma_t} := T^{-}_{l+1,m}(\lambda),
\end{aligned} \tag{3.91}$$

where $\varphi := \Omega + \Omega(z(m-1)/z(m))\lambda_0$, $\phi := -\Omega(z(m-1)/z(m)) < 0$, $\epsilon_t := -\Omega\overline{r}_t < 0$ and $\sigma_t := \overline{q}_t$ for $t = 1, \ldots, l+1$, which is precisely (3.77).

If z does obey (3.76) for $\lambda = \lambda_0$ then $z(m-1)/z(m) = R^{-}_{l,m}(\lambda_0)$. Thus in Figure 3, one of the $\overline{q}_t$'s, $t = 1, \ldots, p$ is equal to λ_0 and since λ_0 is less than the least eigenvalue of the boundary value problem (1.1), (3.76) together with a boundary condition at -1 (as given in Theorems 3.2 or 3.3) it follows that $\overline{q}_1 = \lambda_0$, as $\lambda_0 < \widetilde{p}_k$ for all $k = 1, \ldots, l+1$.

Now

$$\begin{aligned}
\frac{\lambda - \lambda_0}{(z(m)/z(m-1)) - 1/R^{-}_{l,m}(\lambda)} &= \frac{\lambda - \lambda_0}{1/R^{-}_{l,m}(\lambda_0) - 1/R^{-}_{l,m}(\lambda)} \\
&= \frac{R^{-}_{l,m}(\lambda_0)R^{-}_{l,m}(\lambda)}{\left(R^{-}_{l,m}(\lambda) - R^{-}_{l,m}(\lambda_0)\right)/(\lambda - \lambda_0)}
\end{aligned} \tag{3.92}$$

and as $\lambda \to \lambda_0$

$$\frac{R^{-}_{l,m}(\lambda_0)R^{-}_{l,m}(\lambda)}{\left(R^{-}_{l,m}(\lambda) - R^{-}_{l,m}(\lambda_0)\right)/(\lambda - \lambda_0)} \longrightarrow \frac{R^{-}_{l,m}(\lambda_0)^2}{(\partial/\partial\lambda)\, R^{-}_{l,m}(\lambda)\big|_{\lambda_0}} > 0. \tag{3.93}$$

Thus $\lambda = \lambda_0 = \overline{q}_1$ is a removable singularity. Again, alternatively, we could have substituted in for $R^{-}_{l,m}(\lambda)$ and $R^{-}_{l,m}(\lambda_0)$ to illustrate that the singularity at $\lambda = \lambda_0 = \overline{q}_1$ is removable, see Theorem 3.2. Hence the number of nonremovable $\overline{q}_t$'s is one less than the number of $\widetilde{p}_k$'s, see Figure 3.

So (3.87) becomes

$$\overline{f}(\lambda) - \sum_{t=2}^{l+1} \frac{\overline{r}_t}{\lambda - \overline{q}_t}, \quad \overline{r}_t > 0, \tag{3.94}$$

which may be rewritten as

$$\overline{f}(\lambda) - \sum_{t=1}^{l} \frac{\underline{r}_t}{\lambda - \underline{q}_t}, \quad \underline{r}_t > 0, \tag{3.95}$$

where $\underline{r}_n = \overline{r}_{n+1}$ and $\underline{q}_n = \overline{q}_{n+1}$ for $n = 1, \ldots, l$.

Now as $\lambda \to \pm\infty$,

$$\frac{\lambda - \lambda_0}{1/R^{-}_{l,m}(\lambda_0) - 1/R^{-}_{l,m}(\lambda)} \longrightarrow R^{-}_{l,m}(\lambda_0)(\lambda - \lambda_0). \tag{3.96}$$

So, we obtain

$$\begin{aligned} \frac{w(m-1)}{w(m)} &= \Omega - \Omega\left[\overline{f}(\lambda) - \sum_{t=1}^{l} \frac{\underline{r}_t}{\lambda - \underline{q}_t}\right] \\ &= \Omega - \Omega\left[(\lambda - \lambda_0)R^{-}_{l,m}(\lambda_0) - \sum_{t=1}^{l} \frac{\underline{r}_t}{\lambda - \underline{q}_t}\right] \\ &= \widetilde{\phi}\lambda + \widetilde{\varphi} - \sum_{t=1}^{l} \frac{\widetilde{e}_t}{\lambda - \widetilde{\sigma}_t} := \widetilde{T}^{-}_{l,m}(\lambda), \end{aligned} \tag{3.97}$$

where $\widetilde{\varphi} := \Omega + \Omega\lambda_0 R^{-}_{l,m}(\lambda_0)$, $\widetilde{\phi} := -\Omega R^{-}_{l,m}(\lambda_0) < 0$, $\widetilde{e}_t := -\Omega\underline{r}_t < 0$, and $\widetilde{\sigma}_t := \underline{q}_t$ for all $t = 1, \ldots, l$, that is, we obtain (3.78). □

4. Comparison of the Spectra

In this section, we investigate how the spectrum of the original boundary value problem compares to the spectrum of the transformed boundary value problem. This is done by considering the degree of the eigenparameter polynomial for the various eigenconditions.

Lemma 4.1. *Consider the boundary value problem given by* (1.1) *for* $n = 0, \ldots, r-1$ *together with boundary conditions*

$$y(-1) = \left[a\lambda + b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k}\right] y(0), \quad a > 0,\ c_k > 0, \tag{4.1}$$

$$y(r-1) = \left[\alpha\lambda + \beta - \sum_{j=1}^{p} \frac{\gamma_j}{\lambda - \sigma_j}\right] y(r), \quad \alpha < 0,\ \gamma_j < 0. \tag{4.2}$$

Then the boundary value problem (1.1), (4.1), (4.2) *has* $s+p+r+1$ *eigenvalues. (Note that the number of unit intervals considered is* $r+1$*.)*

Proof. From (1.1), with $n = 0$, we obtain

$$y(1) = -\frac{c(-1)y(-1)}{c(0)} + \left(\frac{b(0)}{c(0)} - \lambda\right) y(0). \tag{4.3}$$

Substituting in for $y(-1)$ from (4.1) yields

$$y(1) = \left[-\frac{c(-1)}{c(0)}\left(a\lambda + b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k}\right) + \left(\frac{b(0)}{c(0)} - \lambda\right)\right] y(0), \tag{4.4}$$

which may be rewritten as

$$y(1) := \frac{P_0^1 + P_1^1\lambda + \cdots + P_{s+1}^1\lambda^{s+1}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)} y(0), \tag{4.5}$$

where P_i^1, $i = 0, \ldots, s+1$ are real constants.

Now (1.1), for $n = 1$, together with (4.5) results in

$$\begin{aligned} y(2) &= \left[-\frac{c(0)}{c(1)} + \left(\frac{b(1)}{c(1)} - \lambda\right)\left\{\frac{P_0^1 + P_1^1\lambda + \cdots + P_{s+1}^1\lambda^{s+1}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)}\right\}\right] y(0) \\ &:= \left[\frac{P_0^2 + P_1^2\lambda + \cdots + P_{s+2}^2\lambda^{s+2}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)}\right] y(0), \end{aligned} \tag{4.6}$$

where P_i^2, $i = 0, \ldots, s+2$ are real constants.

Thus, by induction,

$$y(r-1) = \left[\frac{P_0^{r-1} + P_1^{r-1}\lambda + \cdots + P_{s+r-1}^{r-1}\lambda^{s+r-1}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)}\right] y(0), \tag{4.7}$$

for real constants P_i^{r-1}, $i = 0, \ldots, s+r-1$. Similarly

$$y(r) = \left[\frac{P_0^{r} + P_1^{r}\lambda + \cdots + P_{s+r}^{r}\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)}\right] y(0), \tag{4.8}$$

for real constants P_i^r, $i = 0, \ldots, s+r$.

Since $y(0) \not\equiv 0$, using boundary condition (4.2) we obtain the following eigencondition:

$$\begin{aligned} &\frac{P_0^{r-1} + P_1^{r-1}\lambda + \cdots + P_{s+r-1}^{r-1}\lambda^{s+r-1}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)} \\ &\quad = \left(\alpha\lambda + \beta - \sum_{j=1}^{p}\frac{\gamma_j}{\lambda - \sigma_j}\right)\left(\frac{P_0^{r} + P_1^{r}\lambda + \cdots + P_{s+r}^{r}\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)}\right) \\ &\quad := \left(\frac{Q_0 + Q_1\lambda + \cdots + Q_{p+1}\lambda^{p+1}}{(\lambda - \sigma_1)(\lambda - \sigma_2)\cdots(\lambda - \sigma_p)}\right)\left(\frac{P_0^{r} + P_1^{r}\lambda + \cdots + P_{s+r}^{r}\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_s)}\right), \end{aligned} \tag{4.9}$$

where Q_i, $i = 0, \ldots, p+1$, are real constants.

Thus, the numerator is a polynomial, in λ, of order $p+1+s+r$. Note that, none of the roots of this polynomial are given by d_k, $k = 1, \ldots, s$ or σ_j, $j = 1, \ldots, p$ since, from Figures 1 to 3, it is easy to see that none of the eigenvalues of the boundary value problem are equal to the poles of the boundary conditions. Also $\lambda = \pm\infty$ is not a problem as the curve of the Nevanlinna function never intersects with the horizontal or oblique asymptote. This means that there are no common factors to cancel out. Hence the eigencondition has $p+1+s+r$ roots giving that the boundary value problem has $p+1+s+r$ eigenvalues. □

As a direct consequence of Theorems 2.1, 3.2, 3.3, 3.4, and Lemma 4.1 we have the following theorem.

Theorem 4.2. *For the original boundary value problem we consider twelve cases, (see Table 1 in the Appendix), each of which has s+l+m+1 eigenvalues. The corresponding transformed boundary value problem for each of the twelve cases, together with the number of eigenvalues for that transformed boundary value problem, is given in Table 1 (see the appendix).*

Remark 4.3. To summarise we have the following.

(a) If z obeys the boundary conditions at both ends the transformed boundary value problem will have one less eigenvalue than the original boundary value problem, namely, λ_0.

(b) If z obeys the boundary condition at one end only the transformed boundary value problem will have the same eigenvalues as the original boundary value problem.

(c) If z does not obey any of the boundary conditions the transformed boundary value problem will have one more eigenvalue than the original boundary value problem, namely, λ_0.

Corollary 4.4. *If $\lambda_1, \ldots, \lambda_{s+l+m+1}$ are the eigenvalues of any one of the original boundary value problems (1)–(9), in Theorem 4.2, with corresponding eigenfunctions $u_1, \ldots, u_{s+l+m+1}$ then*

(i) *$\lambda_0, \ldots, \lambda_{s+l+m+1}$ are the eigenvalues of the corresponding transformed boundary value problems (1)–(3), in Theorem 4.2, with corresponding eigenfunctions $z, u_1, \ldots, u_{s+l+m+1}$;*

(ii) *$\lambda_1, \ldots, \lambda_{s+l+m+1}$ are the eigenvalues of the corresponding transformed boundary value problems (4)–(9), in Theorem 4.2, with corresponding eigenfunctions $u_1, \ldots, u_{s+l+m+1}$.*

Also, if $\lambda_0, \ldots, \lambda_{s+l+m}$ are the eigenvalues of any one of the original boundary value problems (10)–(12), in Theorem 4.2, with corresponding eigenfunctions $z, u_1, \ldots, u_{s+l+m}$ then $\lambda_1, \ldots, \lambda_{s+l+m}$ are the eigenvalues of the corresponding transformed boundary value problems (10)–(12), in Theorem 4.2, with corresponding eigenfunctions $u_1, \ldots, u_{s+l+m}$.

Proof. By Theorems 2.1, 3.2, 3.3, and 3.4, we have that (2.1) transforms eigenfunctions of the original boundary value problems (1)–(9) to eigenfunctions of the corresponding transformed boundary value problems. In particular, if $\lambda_1, \ldots, \lambda_{s+l+m+1}$ are the eigenvalues of one of the original boundary value problems, (1)–(9), with eigenfunctions $u_1, \ldots, u_{s+l+m+1}$ then

(i) $z, u_1, \ldots, u_{s+l+m+1}$ are the eigenfunctions of the corresponding transformed boundary value problem, (1)–(3), with eigenvalues $\lambda_0, \ldots, \lambda_{s+l+m+1}$. Since the transformed boundary value problems, (1)–(3), have $s + l + m + 2$ eigenvalues it follows that $\lambda_0, \ldots, \lambda_{s+l+m+1}$ constitute all the eigenvalues of the transformed boundary value problem;

(ii) $u_1, \ldots, u_{s+l+m+1}$ are the eigenfunctions of the corresponding transformed boundary value problem, (4)–(9), with eigenvalues $\lambda_1, \ldots, \lambda_{s+l+m+1}$. Since the transformed boundary value problems, (4)–(9), have $s + l + m + 1$ eigenvalues it follows that $\lambda_1, \ldots, \lambda_{s+l+m+1}$ constitute all the eigenvalues of the transformed boundary value problem.

Also, again by Theorems 2.1, 3.2, 3.3, and 3.4, we have that (2.1) transforms eigenfunctions of the original boundary value problems (10)–(12) to eigenfunctions of the corresponding transformed boundary value problems. In particular, if $\lambda_0, \lambda_1, \ldots, \lambda_{s+l+m}$ are the eigenvalues of one of the original boundary value problems, (10)–(12), with eigenfunctions $z, u_1, \ldots, u_{s+l+m}$ then $u_1, \ldots, u_{s+l+m}$ are the eigenfunctions of the corresponding transformed boundary value problem, (10)–(12), with eigenvalues $\lambda_1, \ldots, \lambda_{s+l+m}$. Since the transformed boundary value problems, (10)–(12), have $s + l + m$ eigenvalues it follows that $\lambda_1, \ldots, \lambda_{s+l+m}$ constitute all the eigenvalues of the transformed boundary value problem. □

Appendix

Twelve Cases for Theorem 4.2

See Table 1.

Table 1

	Original BVP: (1.1) with bc's	Trans. BVP: (2.2) with bc's	No. of evals of Trans. BVP
1	(3.1) with $b = 0$ and (3.76) z does not obey (3.1) or (3.76)	(3.14) and (3.77)	$s + l + 1 + m + 1 = s + l + m + 2$ That is, one extra eval $\lambda = \lambda_0$
2	(3.1) with $b > 0$ and (3.76) z does not obey (3.1) or (3.76)	(3.15) and (3.77)	$s + l + 1 + m + 1 = s + l + m + 2$ That is, one extra eval $\lambda = \lambda_0$
3	(3.50) and (3.76) z does not obey (3.50) or (3.76)	(3.51) and (3.77)	$s + l + 1 + m + 1 = s + l + m + 2$ That is, one extra eval $\lambda = \lambda_0$
4	(3.1) with $b = 0$ and (3.76) z obeys (3.1) but not (3.76)	(3.16) and (3.77)	$s - 1 + l + 1 + m + 1 = s + l + m + 1$ That is, same number of evals
5	(3.1) with $b > 0$ and (3.76) z obeys (3.1) but not (3.76)	(3.17) and (3.77)	$s - 1 + l + 1 + m + 1 = s + l + m + 1$ That is, same number of evals
6	(3.50) and (3.76) z obeys (3.50) but not (3.76)	(3.52) and (3.77)	$s + l + 1 + m = s + l + m + 1$ That is, same number of evals
7	(3.1) with $b = 0$ and (3.76) z obeys (3.76) but not (3.1)	(3.14) and (3.78)	$s + l + m + 1 = s + l + m + 1$ That is, same number of evals
8	(3.1) with $b > 0$ and (3.76) z obeys (3.76) but not (3.1)	(3.15) and (3.78)	$s + l + m + 1 = s + l + m + 1$ That is, same number of evals
9	(3.50) and (3.76) z obeys (3.76) but not (3.1)	(3.51) and (3.78)	$s + 1 + l + m = s + l + m + 1$ That is, same number of evals
10	(3.1) with $b = 0$ and (3.76) z obeys both (3.1) and (3.76)	(3.16) and (3.78)	$s - 1 + l + m + 1 = s + l + m$ That is, one less eval $\lambda = \lambda_0$
11	(3.1) with $b > 0$ and (3.76) z obeys both (3.1) and (3.76)	(3.17) and (3.78)	$s - 1 + l + m + 1 = s + l + m$ That is, one less eval $\lambda = \lambda_0$
12	(3.50) and (3.76) z obeys both (3.50) and (3.76)	(3.52) and (3.78)	$s - 1 + l + m + 1 = s + l + m$ That is, one less eval $\lambda = \lambda_0$

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