Hindawi Publishing Corporation
Boundary Value Problems
Volume 2011, Article ID 845413, 18 pages
doi:10.1155/2011/845413

Research Article

Lagrangian Stability of a Class of Second-Order Periodic Systems

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Received 24 November 2010; Accepted 5 January 2011

Academic Editor: Claudianor O. Alves

We study the following second-order differential equation: $(\Phi_p(x'))' + F(x,t)x' + \omega^p \Phi_p(x) + \alpha|x|^l x + e(x,t) = 0$, where $\Phi_p(s) = |s|^{(p-2)} s$ $(p > 1)$, $\alpha > 0$ and $\omega > 0$ are positive constants, and l satisfies $-1 < l < p-2$. Under some assumptions on the parities of $F(x,t)$ and $e(x,t)$, by a small twist theorem of reversible mapping we obtain the existence of quasiperiodic solutions and boundedness of all the solutions.

1. Introduction and Main Result

In the early 1960s, Littlewood [1] asked whether or not the solutions of the Duffing-type equation

$$x'' + g(x,t) = 0 \tag{1.1}$$

are bounded for all time, that is, whether there are resonances that might cause the amplitude of the oscillations to increase without bound.

The first positive result of boundedness of solutions in the *superlinear case* (i.e., $g(x,t)/x \to \infty$ as $|x| \to \infty$) was due to Morris [2]. By means of KAM theorem, Morris proved that every solution of the differential equation (1.1) is bounded if $g(x,t) = 2x^3 - p(t)$, where $p(t)$ is piecewise continuous and periodic. This result relies on the fact that the nonlinearity $2x^3$ can guarantee the twist condition of KAM theorem. Later, several authors (see [3–5]) improved Morris's result and obtained similar result for a large class of superlinear function $g(x,t)$.

When $g(x)$ satisfies

$$0 \leq k \leq \frac{g(x)}{x} \leq K \leq +\infty, \quad \forall x \in R, \tag{1.2}$$

that is, the differential equation (1.1) is *semilinear*, similar results also hold, but the proof is more difficult since there may be resonant case. We refer to [6–8] and the references therein.

In [8] Liu considered the following equation:

$$x'' + \lambda^2 x + \varphi(x) = e(t), \tag{1.3}$$

where $\varphi(x) = o(x)$ as $|x| \to +\infty$ and $e(t)$ is a 2π-periodic function. After introducing new variables, the differential equation (1.3) can be changed into a Hamiltonian system. Under some suitable assumptions on $\varphi(x)$ and $e(t)$, by using a variant of Moser's small twist theorem [9] to the Pioncaré map, the author obtained the existence of quasi-periodic solutions and the boundedness of all solutions.

Then the result is generalized to a class of p-Laplacian differential equation.Yang [10] considered the following nonlinear differential equation

$$(\Phi_p(x'))' + \alpha\Phi_p(x^+) - \beta\Phi_p(x^-) + f(x) = e(t), \tag{1.4}$$

where $f(x) \in C^5(R \setminus 0) \cap C^0(R)$ is bounded, $e(t) \in C^6(R \setminus 2\pi Z)$ is periodic. The idea is also to change the original problem to Hamiltonian system and then use a twist theorem of area-preserving mapping to the Pioncaré map.

The above differential equation essentially possess Hamiltonian structure. It is well known that the Hamiltonian structure and reversible structure have many similar property. Especially, they have similar KAM theorem.

Recently, Liu [6] studied the following equation:

$$x'' + F_x(x,t)x' + a^2 x + \varphi(x) + e(x,t) = 0, \tag{1.5}$$

where a is a positive constant and $e(x,t)$ is 2π-periodic in t. Under some assumption of F, φ and e, the differential equation (1.5) has a reversible structure. Suppose that $\varphi(x)$ satisfies

$$\gamma x\varphi(x) \geq x^2\varphi'(x) > 0, \quad x\varphi(x) \geq \alpha\Phi(x), \quad \forall x \neq 0, \tag{1.6}$$

where $\Phi(x) = \int_0^x \varphi(t)dt$ and $0 < \gamma < 1 < \alpha < 2$. Moreover,

$$\left| x^k \frac{d^k\Phi(x)}{dx^k} \right| \leq c \cdot \Phi(x), \quad \text{for } 3 \leq k \leq 6, \tag{1.7}$$

where c is a constant. Note that here and below we always use c to indicate some constants. Assume that there exists $\sigma \in (0, \alpha - 1)$ such that

$$\left| x^k \frac{\partial^{k+l} F(x,t)}{\partial x^k \partial t^l} \right| \leq c \cdot |x|^{\sigma}, \quad \left| x^k \frac{\partial^{k+l} e(x,t)}{\partial x^k \partial t^l} \right| \leq c \cdot |x|^{\sigma} \quad \text{for } k, l \leq 6. \tag{1.8}$$

Then, the following conclusions hold true.

(i) There exist $\epsilon_0 > 0$ and a closed set $A \subset (a/2\pi, a/2\pi + \epsilon_0)$ having positive measure such that for any $\omega \in A$, there exists a quasi-periodic solution for (1.5) with the basic frequency $(\omega, 1)$.

(ii) Every solution of (1.5) is bounded.

Motivated by the papers [6, 8, 10], we consider the following p-Laplacian equation:

$$(\Phi_p(x'))' + F(x,t)x' + \omega^p \Phi_p(x) + \alpha |x|^l x + e(x,t) = 0. \tag{1.9}$$

where $\Phi_p(s) = |s|^{(p-2)} s (p > 1)$, $-1 < l < p-2$, and $\alpha, \omega > 0$ are constants. We want to generalize the result in [6] to a class of p-Laplacian-type differential equations of the form (1.9). The main idea is similar to that in [6]. We will assume that the functions F and e have some parities such that the differential system (1.9) still has a reversible structure. After some transformations, we change the systems (1.9) to a form of small perturbation of integrable reversible system. Then a KAM Theorem for reversible mapping can be applied to the Poincaré mapping of this nearly integrable reversible system and some desired result can be obtained.

Our main result is the following theorem.

Theorem 1.1. *Suppose that e and F are of class C^6 in their arguments and 2π-periodic with respect to t such that*

$$\begin{aligned} F(-x,-t) &= -F(x,t), \qquad e(-x,-t) = -e(x,t), \\ F(x,-t) &= -F(x,t), \qquad e(x,-t) = e(x,t). \end{aligned} \tag{1.10}$$

Moreover, suppose that there exists $\sigma < l$ such that

$$\left| x^k \frac{\partial^{k+m} F(x,t)}{\partial x^k \partial t^m} \right| \leq c \cdot |x|^{\sigma}, \qquad \left| x^k \frac{\partial^{k+m} e(x,t)}{\partial x^k \partial t^m} \right| \leq c \cdot |x|^{\sigma+1}, \tag{1.11}$$

for all $x \neq 0$, for all $0 \leq k \leq 6$, $0 \leq m \leq 6$. Then every solution of (1.9) is bounded.

Remark 1.2. Our main nonlinearity $\alpha |x|^l x$ in (1.9) corresponds to φ in (1.5). Although it is more special than φ, it makes no essential difference of proof and can simplify our proof greatly. It is easy to see from the proof that this main nonlinearity is used to guarantee the small twist condition.

2. The Proof of Theorem

The proof of Theorem 1.1 is based on Moser's small twist theorem for reversible mapping. It mainly consists of two steps. The first one is to find an equivalent system, which has a nearly integrable form of a reversible system. The second one is to show that Pincaré map of the equivalent system satisfies some twist theorem for reversible mapping.

2.1. Action-Angle Variables

We first recall the definitions of reversible system. Let $\Omega \subset \mathbb{R}^n$ be an open domain, and $Z = Z(z,t) : \Omega \times \mathbb{R} \to \mathbb{R}^n$ be continuous. Suppose $G : \mathbb{R}^n \to \mathbb{R}^n$ is an involution (i.e., G is a C^1-diffeomorphism such that $G^2 = Id$) satisfying $G(\Omega) = \Omega$. The differential equations system

$$z' = Z(z,t) \tag{2.1}$$

is called reversible with respect to G, if

$$G_* Z(z,-t) = DG(Gz)Z(Gz,-t) = -Z(z,t), \quad \forall z \in \Omega, \ \forall t \in R \tag{2.2}$$

with DG denoting the Jacobian matrix of G.

We are interested in the special involution $G(x,y) \to (x,-y)$ with $z = (x,y) \in R^2$. Let $Z = (Z_1, Z_2)$. Then $z' = Z(z,t)$ is reversible with respect to G if and only if

$$\begin{aligned} Z_1(x,-y,-t) &= -Z_1(x,y,t), \\ Z_2(x,-y,-t) &= Z_2(x,y,t). \end{aligned} \tag{2.3}$$

Below we will see that the symmetric properties (1.10) imply a reversible structure of the system (1.9).

Let $y = \Phi_p(x') = |x'|^{p-2}x'$. Then $x' = \Phi_q(y)$, where q satisfies $1/p + 1/q = 1$. Hence, the differential equation (1.9) is changed into the following planar system:

$$\begin{aligned} x' &= \Phi_q(y), \\ y' &= -\omega^p \Phi_p(x) - \alpha |x|^l x - e(x,t) - F(x,t)\Phi_q(y). \end{aligned} \tag{2.4}$$

By (1.10) it is easy to see that the system (2.4) is reversible with respect to the involution $G : (x,y) \to (x,-y)$.

Below we will write the reversible system (2.4) as a form of small perturbation. For this purpose we first introduce action-angle variables (r,θ).

Consider the homogeneous differential equation:

$$(\Phi_p(u'))' + \Phi_p(u) = 0. \tag{2.5}$$

This equation takes as an integrable part of (1.9). We will use its solutions to construct a pair of action-angle variables. One of solutions for (2.5) is the function $\sin_p$ as defined below. Let the number π_p defined by

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{\left[1 - s^p/(p-1)\right]^{1/p}}. \tag{2.6}$$

We define the function $w(t) : [0, \pi_p/2] \to [0, (p-1)^{1/p}]$, implicitly by

$$\int_0^{w(t)} \frac{ds}{\left[1 - s^p/(p-1)\right]^{1/p}} = t. \tag{2.7}$$

The function $w(t)$ will be extended to the whole real axis R as explained below, and the extension will be denoted by $\sin_p$. Define $\sin_p$ on $[\pi_p/2, \pi_p]$ by $\sin_p(t) = w(\pi_p - t)$. Then, we define $\sin_p$ on $[-\pi_p, 0]$ such that $\sin_p$ is an odd function. Finally, we extend $\sin_p$ to R by $2\pi_p$-periodicity. It is not difficult to verify that $\sin_p$ has the following properties:

(i) $\sin_p(0) = 0$, $\sin_p'(0) = 1$;

(ii) $(p-1)|\sin_p'(t)|^p + |\sin_p(t)|^p = p - 1$;

(iii) $\sin_p t$ is an odd periodic function with period $2\pi_p$.

It is easy to verify that $x = \sin_p(\omega t)$ satisfies

$$\left(\Phi_p(x')\right)' + \omega^p \Phi_p(x) = 0 \tag{2.8}$$

with initial condition $(x(0), x'(0)) = (0, \omega)$. Define a transformation $\Theta : (x, y) \mapsto (r, \theta)$ by

$$\begin{aligned} x &= r^{2/p} \sin_p \omega\theta, \\ y &= r^{2/q} \Phi_p\left(\omega \sin_p' \omega\theta\right). \end{aligned} \tag{2.9}$$

It is easy to see that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = -\frac{2}{q}\omega^p r. \tag{2.10}$$

Since the *Jacobian* matrix of Θ is nonsingular for $r > 0$, the transformation Θ is a local homeomorphism at each point (r, θ) of the set $R^+ \times [0, 2\pi_p/\omega)$, while $\Theta^{-1} : (r, \theta) \mapsto (x, y)$ is a global homeomorphism from $R^+ \times [0, 2\pi_p/\omega)$ to $R^2 \setminus \{0\}$. Under the transformation Θ the system (2.4) is changed to

$$\begin{aligned} r' &= f_1(t, \theta, r) = N_1(t, \theta, r) + P_1(t, \theta, r), \\ \theta' &= 1 + f_2(t, \theta, r) = 1 + N_2(t, \theta, r) + P_2(t, \theta, r), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
N_1(t,\theta,r) &= -\alpha\frac{q}{2}\frac{1}{\omega^{p-1}}r^{4/p-1+(2/p)l}\sin_p'\tilde{\theta}\left|\sin_p^l\tilde{\theta}\right|\sin_p\tilde{\theta},\\
P_1(t,\theta,r) &= -\frac{q}{2}\frac{1}{\omega^{p-1}}r^{1-2/q}\sin_p'\tilde{\theta}F\left(r^{2/p}\sin_p\tilde{\theta},t\right)\Phi_q\left(r^{2/q}\Phi_p\left(\omega\sin_p'\tilde{\theta}\right)\right)\\
&\quad-\frac{q}{2}\frac{1}{\omega^{p-1}}r^{1-2/q}\sin_p'\tilde{\theta}e\left(r^{2/p}\sin_p\tilde{\theta},t\right),\\
N_2(t,\theta,r) &= \alpha\frac{q}{p}\frac{1}{\omega^{p}}r^{4/p-2+(2/p)l}\left|\sin_p^l\tilde{\theta}\right|\sin_p^2\tilde{\theta},\\
P_2(t,\theta,r) &= \frac{q}{p}\frac{1}{\omega^{p}}r^{-2/q}\sin_p\tilde{\theta}F\left(r^{2/p}\sin_p\tilde{\theta},t\right)\Phi_q\left(r^{2/q}\Phi_p\left(\omega\sin_p'\tilde{\theta}\right)\right)\\
&\quad+\frac{q}{p}\frac{1}{\omega^{p}}r^{-2/q}\sin_p\tilde{\theta}e\left(r^{2/p}\sin_p\tilde{\theta},t\right),
\end{aligned} \tag{2.12}$$

with $\tilde{\theta} = \omega\theta$.

It is easily verified that $f_1(-t,-\theta,r) = -f_1(t,\theta,r)$ and $f_2(-t,-\theta,r) = f_2(t,\theta,r)$ and so the system (2.11) is reversible with respect to the involution $G : (r,\theta) \rightarrow (r,-\theta)$.

2.2. Some Lemmas

To estimate $f_1(t,\theta,r)$ and $f_2(t,\theta,r)$, we need some definitions and lemmas.

Lemma 2.1. *Let $F(t,\theta,r) = F(r^{2/p}\sin_p\theta,t)$, $e(t,\theta,r) = e(r^{2/p}\sin_p\theta,t)$. If $F(x,t)$ and $e(x,t)$ satisfy* (1.11)*, then*

$$\left|r^k\frac{\partial^{k+s}F(t,\theta,r)}{\partial r^k\partial t^s}\right| \leq c\cdot r^{(2/p)\sigma},\quad \left|r^k\frac{\partial^{k+s}e(t,\theta,r)}{\partial r^k\partial t^s}\right| \leq c\cdot r^{(2/p)(\sigma+1)}, \tag{2.13}$$

for $\forall\theta \in R$, $k+s \leq m$.

Proof. We only prove the second inequality since the first one can be proved similarly.

$$\begin{aligned}
\left|r^k\frac{\partial^{k+s}e(t,\theta,r)}{\partial r^k\partial t^s}\right| &= \left|r^k\frac{\partial^{k+s}e(x,t)}{\partial x^k\partial t^s}\left(\frac{\partial x}{\partial r}\right)^k+\cdots+r^k\frac{\partial^{1+s}e(x,t)}{\partial x\partial t^s}\frac{\partial^k x}{\partial r^k}\right|\\
&= \left|c_1(p)r^k\frac{\partial^{k+s}e(x,t)}{\partial x^k\partial t^s}\left(r^{2/p-1}\right)^k\sin_p^k\theta+\cdots+c_k(p)r^k\frac{\partial^{1+s}e(x,t)}{\partial x\partial t^s}r^{2/p-k}\sin_p\theta\right|\\
&= \left|cx^k\frac{\partial^{k+s}e(x,t)}{\partial x^k\partial t^s}+\cdots+cx\frac{\partial^{1+s}e(x,t)}{\partial x\partial t^s}\right|\\
&\leq c\cdot|x|^{\sigma+1} \leq c\cdot r^{(2/p)(\sigma+1)}.
\end{aligned} \tag{2.14}$$

□

To describe the estimates in Lemma 2.1, we introduce function space $M_n(\Psi)$, where Ψ is a function of r.

Definition 2.2. Let $n = (n_1, n_2) \in N^2$. We say $f \in M_n(\Psi)$, if for $0 < j \leq n_1, 0 < s \leq n_2$, there exist $r_0 > 0$ and $c > 0$ such that

$$r^j \left| D_r^j D_t^s f(t,\theta,r) \right| \leq c \cdot \Psi(r), \quad \forall r \geq r_0,\ \forall (t,\theta) \in S^1 \times S^1. \tag{2.15}$$

Lemma 2.3 (see [6]). *The following conclusions hold true:*

(i) *if $f \in M_n(\Psi)$, then $D_r^j f \in M_{n-(0,j)}(r^{-j}\Psi)$ and $D_t^s f \in M_{n-(s,0)}(\Psi)$;*

(ii) *if $f_1 \in M_n(\Psi_1)$ and $f_2 \in M_n(\Psi_2)$, then $f_1 f_2 \in M_n(\Psi_1 \Psi_2)$;*

(iii) *Suppose Ψ, Ψ_1, Ψ_2 satisfy that, there exists $c > 0$ such that for $\forall 0 \leq \xi \leq 2 \cdot r$,*

$$\begin{gathered} \Psi(\xi) \leq c\Psi(r), \\ \lim_{r \to +\infty} r^{-1}\Psi_1 = \lim_{r \to +\infty} \Psi_2 = 0. \end{gathered} \tag{2.16}$$

If $f \in M_n(\Psi)$, $g_1 \in M_n(\Psi_1)$, $g_2 \in M_n(\Psi_2)$, then, we have

$$f(t + g_1, \theta, r + g_2) \in M_{n'}(\Psi), \quad n' = (n_1', n_2') \text{ with } n_1' = n_2' = \min\{n_1, n_2\}. \tag{2.17}$$

Moreover,

$$\begin{gathered} f(t + g_1, \theta, r) - f(t,\theta,r) \in M_{(n_1 - 1, \min\{n_1, n_2\})}(\Psi \cdot \Psi_1), \\ f(t,\theta, r + g_2) - f(t,\theta,r) \in M_{(\min\{n_1,n_2\}, n_2 - 1)}\left(r^{-1}\Psi \cdot \Psi_2\right). \end{gathered} \tag{2.18}$$

Proof. This lemma was proved in [6], but we give the proof here for reader's convenience. Since (i) and (ii) are easily verified by definition, so we only prove (iii). Let

$$v(t,\theta,r) = t + g_1(t,\theta,r), \qquad u(t,\theta,r) = r + g_2(t,\theta,r). \tag{2.19}$$

Since $g_2 \in M_n(\Psi_2)$, we have $|r \cdot \partial g_2/\partial r| \leq c\Psi_2$. So $|\partial g_2/\partial r| \leq cr^{-1}\Psi_2 \to 0$ $(r \to \infty)$. Thus $|\partial g_2/\partial r|$ is bounded and so $|\partial u/\partial r| \leq 1 + |\partial g_2/\partial r| \leq c$. Similarly, we have

$$\left|\frac{\partial u}{\partial t}\right| \leq c \cdot \Psi_2, \qquad \left|\frac{\partial v}{\partial t}\right| \leq c, \qquad \left|\frac{\partial v}{\partial r}\right| \leq c \cdot r^{-1}\Psi_1. \tag{2.20}$$

For $j + s \geq 2$, we have

$$\frac{\partial^{j+s} u}{\partial r^j \partial t^s} = \frac{\partial^{j+s} g_2}{\partial r^j \partial t^s}, \qquad \frac{\partial^{j+s} v}{\partial r^j \partial t^s} = \frac{\partial^{j+s} g_1}{\partial r^j \partial t^s}. \tag{2.21}$$

Since $g_1 \in M_n(\Psi_1), g_2 \in M_n(\Psi_2)$, it follows that

$$\frac{\partial^{j+s}u}{\partial r^j \partial t^s} \in M_n\left(r^{-j}\Psi_2\right), \qquad \frac{\partial^{j+s}v}{\partial r^j \partial t^s} \in M_n\left(r^{-j}\Psi_1\right). \tag{2.22}$$

Let $g(t,\theta,r) = f(v(t,\theta,r),\theta,u(t,\theta,r))$. Since $g_2 \in M_n(\Psi_2)$, we know that for r sufficiently large

$$r_0 \ll r + g_2 \leq 2r. \tag{2.23}$$

By the property of Ψ, we have

$$\left|g(t,\theta,r)\right| \leq c \cdot \Psi(u) = c \cdot \Psi\left(r + g_2\right) \leq c \cdot \Psi(r), \tag{2.24}$$

for r_0 sufficiently large.

If $k + s \geq 1$, then by a direct computation, we have

$$\frac{\partial^{k+s}g}{\partial r^k \partial t^s} = \sum \frac{\partial^{b+m} f(v,\theta,u)}{\partial r^b \partial t^m} \cdot \frac{\partial^{j_1+j_1'}u}{\partial r^{j_1}\partial t^{j_1'}} \cdots \frac{\partial^{j_b+j_b'}u}{\partial r^{j_b}\partial t^{j_b'}} \cdot \frac{\partial^{i_1+i_1'}v}{\partial r^{i_1}\partial t^{i_1'}} \cdots \frac{\partial^{i_m+i_m'}v}{\partial r^{i_m}\partial t^{i_m'}}, \tag{2.25}$$

where the sum is found for the indices satisfying

$$j_1 + \cdots + j_b + i_1 + \cdots + i_m = k, \qquad j_1' + \cdots + j_b' + i_1' + \cdots + i_m' = s. \tag{2.26}$$

Without loss of generality, we assume that

$$\begin{gathered} j_1 + j_1' = 1, \ldots, j_{b_1} + j_{b_1}' = 1, \\ i_1 + i_1' = 1, \ldots, i_{m_1} + i_{m_1}' = 1. \end{gathered} \tag{2.27}$$

Furthermore, we suppose that among $j_1, \ldots, j_{b_1}$, there are b_2 numbers which equal to 0, and among $i_1, \ldots, i_{m_1}$, there are m_2 numbers which equal to 0.

Since

$$\begin{aligned} \frac{\partial^{k+s}g}{\partial r^k \partial t^s} = \sum & \frac{\partial^{b+m} f(v,\theta,u)}{\partial r^b \partial t^m} \cdot \frac{\partial^{j_1+j_1'}u}{\partial r^{j_1}\partial t^{j_1'}} \cdots \frac{\partial^{j_{b_2}+j_{b_2}'}u}{\partial r^{j_{b_2}}\partial t^{j_{b_2}'}} \\ & \cdot \frac{\partial^{j_{b_2+1}+j_{b_2+1}'}u}{\partial r^{j_{b_2+1}}\partial t^{j_{b_2+1}'}} \cdots \frac{\partial^{j_{b_1}+j_{b_1}'}u}{\partial r^{j_{b_1}}\partial t^{j_{b_1}'}} \cdot \frac{\partial^{j_{b_1+1}+j_{b_1+1}'}u}{\partial r^{j_{b_1+1}}\partial t^{j_{b_1+1}'}} \cdots \frac{\partial^{j_b+j_b'}u}{\partial r^{j_b}\partial t^{j_b'}} \\ & \cdot \frac{\partial^{i_1+i_1'}v}{\partial r^{i_1}\partial t^{i_1'}} \cdots \frac{\partial^{i_{m_2}+i_{m_2}'}v}{\partial r^{i_{m_2}}\partial t^{i_{m_2}'}} \cdot \frac{\partial^{i_{m_2+1}+i_{m_2+1}'}v}{\partial r^{i_{m_2+1}}\partial t^{i_{m_2+1}'}} \cdots \frac{\partial^{i_{m_1}+i_{m_1}'}v}{\partial r^{i_{m_1}}\partial t^{i_{m_1}'}} \\ & \cdot \frac{\partial^{i_{m_1+1}+i_{m_1+1}'}v}{\partial r^{i_{m_1+1}}\partial t^{i_{m_1+1}'}} \cdots \frac{\partial^{i_m+i_m'}v}{\partial r^{i_m}\partial t^{i_m'}}, \end{aligned} \tag{2.28}$$

we have

$$\begin{aligned}\frac{\partial^{k+s} g}{\partial r^k \partial t^s} &\leq \sum c \cdot r^{-b} \Psi r^{-(j_{b_1+1}+\cdots+j_b)} r^{m_2-m_1} \Psi_1^{b-b_1+b_2} r^{-(i_{m_1+1}+\cdots+i_m)} \Psi_2^{(m-m_2+(m_2-m_1))} \\ &\leq c \cdot r^{(b_2-b_1)-(j_{b_1+1}+\cdots+j_b)+(m_2-m_1)-(i_{m_1+1}+\cdots+i_m)} \left(r^{-(b+b_2-b_1)} \Psi_1^{b+b_2-b_1}\right) \Psi_2^{m-m_1} \\ &\leq c \cdot r^{-k} \Psi,\end{aligned} \tag{2.29}$$

and then,

$$f(t+g_1, \theta, r+g_2) \in M_{n'}(\Psi). \tag{2.30}$$

Obviously

$$f(t+g_1, \theta, r) - f(t, \theta, r) = \int_0^1 \frac{\partial f}{\partial t}(t+\eta g_1, \theta, r) g_1 d\eta. \tag{2.31}$$

Since

$$\frac{\partial f}{\partial t} \in M_{n-(1,0)}(\Psi), \quad \lim_{r \to +\infty} (\eta g_1) = 0, \quad \eta \in [0,1]. \tag{2.32}$$

By the condition of (iii) we obtain

$$f(t+g_1, \theta, r) - f(t, \theta, r) \in M_{(n_1-1, \min\{n_1, n_2\})}(\Psi \cdot \Psi_1), \tag{2.33}$$

In the same way we can consider $f(t, \theta, r+g_2) - f(t, \theta, r)$ and we omit the details. □

2.3. Some Estimates

The following lemma gives the estimate for $f_1(t, \theta, r)$ and $f_2(t, \theta, r)$.

Lemma 2.4. *$f_1(t, \theta, r) \in M_{(5,5)}(r^{\beta+1})$, $f_2(t, \theta, r) \in M_{(5,5)}(r^{\beta})$, where $\beta = 2(2-p+l)/p$.*

Proof. Since $f_1(t, \theta, r) = P_1(t, \theta, r) + N_1(t, \theta, r)$, we first consider $P_1(t, \theta, r)$ and $N_1(t, \theta, r)$. By Lemma 2.1, $F(t, \theta, r) \in M_{(5,5)}(r^{(2/p)\sigma})$. Again $\Phi_q(r^{2/q}\Phi_p(\omega \sin'_p \widetilde{\theta})) = r^{2/p}\Phi_q(\Phi_p(\omega \sin'_p \widetilde{\theta})) \in M_{(5,5)}(r^{2/p})$, using the conclusion (iii) of Lemma 2.3, we have $P_1(t, \theta, r) \in M_{(5,5)}(r^{\beta'+1})$, where $\beta' = 2(2-p+\sigma)/p$. Note that $N_1(t, \theta, r) \in M_{(5,5)}(r^{\beta+1})$ and $\beta' < \beta$, we have $f_1(t, \theta, r) \in M_{(5,5)}(r^{\beta+1})$. In the same way we can prove $f_2(t, \theta, r) \in M_{(5,5)}(r^{\beta})$. Thus Lemma 2.4 is proved. □

Since $-1 < l < p-2$, we get $\beta < 0$. So $|f_2| \le r^{\beta} \ll 1$ for sufficiently large r. When $r \gg 1$ the system (2.11) is equivalent to the following system:

$$\begin{aligned} \frac{dr}{d\theta} &= f_1(t,\theta,r)\left(1+f_2(t,\theta,r)\right)^{-1}, \\ \frac{dt}{d\theta} &= \left(1+f_2(t,\theta,r)\right)^{-1}. \end{aligned} \tag{2.34}$$

It is easy to see that $f_1(-t,-\theta,r) = -f_1(t,\theta,r)$ and $f_2(-t,-\theta,r) = f_2(t,\theta,r)$. Hence, system (2.34) is reversible with respect to the involution $G : (r,t) \to (r,-t)$.

We will prove that the Poincaré mapping can be a small perturbation of integrable reversible mapping. For this purpose, we write (2.34) as a small perturbation of an integrable reversible system. Write the system (2.34) in the form

$$\begin{aligned} \frac{dr}{d\theta} &= f_1(t,\theta,r) + h_1(t,\theta,r) = N_1(t,\theta,r) + (P_1(t,\theta,r) + h_1(t,\theta,r)), \\ \frac{dt}{d\theta} &= 1 - f_2(t,\theta,r) + h_2(t,\theta,r) = 1 - N_2(t,\theta,r) + (-P_2(t,\theta,r) + h_2(t,\theta,r)), \end{aligned} \tag{2.35}$$

where $h_1(t,\theta,r) = -f_1 f_2/(1+f_2)$, $h_2(t,\theta,r) = f_2^2/(1+f_2)$, with $f_1(t,\theta,r)$ and $f_2(t,\theta,r)$ defined in (2.11). It follows $h_1(-t,-\theta,r) = -h_1(t,\theta,r), h_2(-t,-\theta,r) = h_2(t,\theta,r)$, and so (2.35) is also reversible with respect to the involution $G : (r,t) \to (r,-t)$. Below we prove that $h_1(t,\theta,r)$ and $h_2(t,\theta,r)$ are smaller perturbations.

Lemma 2.5. $h_1(t,\theta,r) \in M_{(5,5)}(r^{2\beta+1})$, $h_2(t,\theta,r) \in M_{(5,5)}(r^{2\beta})$.

Proof. If r_0 is sufficiently large, then $|f_2(t,\theta,r)| < 1/2$ and so $1/(1+f_2(t,\theta,r)) = \sum_{s=0}^{+\infty}(-1)^s f_2^s(t,\theta,r)$. Hence

$$h_1(t,\theta,r) = \sum_{s=0}^{\infty}(-1)^s f_2^{s+1}(t,\theta,r) f_1(t,\theta,r). \tag{2.36}$$

It is easy to verify that

$$\frac{\partial^{k+m}}{\partial r^k \partial t^m} f_2^{s+1} f_1 = \sum_{|i|=k,|j|=m,} c_{i,j} \frac{\partial^{i_1+j_1}}{\partial r^{i_1}\partial t^{j_2}} f_1 \frac{\partial^{i_2+j_2}}{\partial r^{i_2}\partial t^{j_2}} f_2 \cdots \frac{\partial^{i_{s+2}+j_{s+2}}}{\partial r^{i_{s+2}}\partial t^{j_{s+2}}} f_2, \tag{2.37}$$

where $i = (i_1, \ldots, i_{l+2})$, $|i| = i_1 + \cdots + i_{s+2}$, and j and $|j|$ are defined in the same way as i and $|i|$.

So, we have

$$\frac{\partial^{k+m}}{\partial r^k \partial t^m} h_1 = \sum_{|i|=k,|j|=m,n\ge 2} c_{i,j} \frac{\partial^{i_1+j_1}}{\partial r^{i_1}\partial t^{j_1}} f_1 \frac{\partial^{i_2+j_2}}{\partial r^{i_2}\partial t^{j_2}} f_2 \cdots \frac{\partial^{i_n+j_n}}{\partial r^{i_n}\partial t^{j_n}} f_2, \tag{2.38}$$

where

$$\frac{\partial^{i_\tau+j_\tau}}{\partial r^{i_\tau}\partial t^{j_\tau}} f_2 \le c, \quad \tau = 2,\ldots,n \quad \text{for } f_2 \in M_{(5,5)}\left(r^{\beta}\right). \tag{2.39}$$

So

$$\begin{aligned}\left|\frac{\partial^{k+m}}{\partial r^k \partial t^m} h_1\right| &\le c_{i,j} r^{\beta+1-i_1} r^{\beta-i_2} \cdots r^{\beta-i_n} \\ &\le c_1 r^{\beta+1} r^{\beta}\left(r^{\beta}\right)^{n-2} r^{-(i_1+\cdots+i_n)} \\ &\le c r^{-k} r^{2\beta+1}.\end{aligned} \tag{2.40}$$

Thus, $h_1 \in M_{(5,5)}(r^{2\beta+1})$. In the same way, we have $h_2 \in M_{(5,5)}(r^{2\beta})$. □

Now we change system (2.35) to

$$\begin{aligned}\frac{dr}{d\theta} &= N_1(t,\theta,r) + g_1(t,\theta,r), \\ \frac{dt}{d\theta} &= 1 - N_2(t,\theta,r) + g_2(t,\theta,r),\end{aligned} \tag{2.41}$$

where $g_1(t,\theta,r) = P_1(t,\theta,r) + h_1(t,\theta,r)$ and $g_2(t,\theta,r) = -P_2(t,\theta,r) + h_2(t,\theta,r)$. By the proof of Lemma 2.4, we know $P_1 \in M_{(5,5)}(r^{\beta'+1})$ and $P_2 \in M_{(5,5)}(r^{\beta'})$. Thus, $g_1(t,\theta,r) \in M_{(5,5)}(r^{\beta+1-\tilde{\sigma}})$ and $g_2(t,\theta,r) \in M_{(5,5)}(r^{\beta-\tilde{\sigma}})$ where

$$\tilde{\sigma} = \min\left\{-\beta, -\frac{2}{p}(\sigma - l)\right\} > 0, \tag{2.42}$$

with $\sigma < l < p-2$, $-1 < l$.

2.4. Coordination Transformation

Lemma 2.6. *There exists a transformation of the form*

$$t = t, \qquad \lambda = r + S(r,\theta), \tag{2.43}$$

such that the system (2.41) *is changed into the form*

$$\begin{aligned}\frac{d\lambda}{d\theta} &= \tilde{g}_1(t,\theta,\lambda), \\ \frac{dt}{d\theta} &= 1 - N_2(t,\theta,\lambda) + \tilde{g}_2(t,\theta,\lambda),\end{aligned} \tag{2.44}$$

where $\widetilde{g}_1, \widetilde{g}_2$ *satisfy:*

$$\widetilde{g}_1 \in M_{(5,5)}\left(\lambda^{\beta+1-\widetilde{\sigma}}\right), \qquad \widetilde{g}_2 \in M_{(5,5)}\left(\lambda^{\beta-\widetilde{\sigma}}\right). \tag{2.45}$$

Moreover, the system (2.44) *is reversible with respect to the involution* G*:* $(\lambda, -t) \mapsto (\lambda, t)$.

Proof. Let

$$S(r,\theta) = \int_0^{\theta} N_1(t,\theta,r)d\theta = \frac{q}{2}\frac{\alpha}{\omega^{p-1}}\frac{1}{l+2}\left|\sin_p^{l+2}\widetilde{\theta}\right| r^{\beta+1}, \tag{2.46}$$

then

$$S(r,\theta) = S\left(r, \theta + 2\pi_p\right), \qquad S(r,-\theta) = S(r,\theta). \tag{2.47}$$

It is easy to see that

$$S(r,\theta) \in M_{(5,5)}\left(r^{\beta+1}\right). \tag{2.48}$$

Hence the map $(r,\theta) \to (\lambda, t)$ with $\lambda = r + S(r,\theta)$ is diffeomorphism for $r \gg 1$. Thus, there is a function $L = L(\lambda,\theta)$ such that

$$r = \lambda + L(\lambda,\theta) \tag{2.49}$$

where

$$L\left(\lambda,\theta + 2\pi_p\right) = L(\lambda,\theta), \qquad L(\lambda,-\theta) = L(\lambda,\theta), \qquad L(\lambda,\theta) \in M_{(5,5)}\left(\lambda^{\beta+1}\right). \tag{2.50}$$

Under this transformation, the system (2.41) is changed to (2.44) with

$$\widetilde{g}_1(t,\theta,\lambda) = g_1(t,\theta,\lambda+L), \qquad \widetilde{g}_2(t,\theta,\lambda) = N_2(t,\theta,\lambda) - N_2(t,\theta,\lambda+L) + g_2(t,\theta,\lambda+L). \tag{2.51}$$

Below we estimate g_1 and g_2. We only consider g_2 since g_1 can be considered similarly or even simpler.

Obviously,

$$\lim_{\lambda\to\infty}\left(\lambda^{-1}\lambda^{4/p-1+(2/p)l}\right) = \lim_{\lambda\to\infty}\left(\lambda^{2\beta}\right) = 0. \tag{2.52}$$

Note that

$$g_2(t,\theta,r) \in M_{(5,5)}\left(r^{\beta-\widetilde{\sigma}}\right). \tag{2.53}$$

By the third conclusion of Lemma 2.3, we have that

$$g_2(t,\theta,\lambda+L) \in M_{(5,5)}\left(\lambda^{\beta-\tilde{\sigma}}\right). \tag{2.54}$$

In the same way as the above, we have

$$N_2(t,\theta,r) = N_2(t,\theta,\lambda+L) \in M_{(5,5)}\left(\lambda^{\beta}\right) \tag{2.55}$$

and so

$$\begin{aligned} N_2(t,\theta,r) - N_2(t,\theta,\lambda) &= N_2(t,\theta,\lambda+L) - N_2(t,\theta,\lambda) \in M_{(5,5)}\left(\lambda^{-1}\lambda^{\beta}\lambda^{4/p-1+(2/p)\sigma}\right) \\ &= M_{(5,5)}\left(\lambda^{\beta+\beta'}\right). \end{aligned} \tag{2.56}$$

By (2.54) and (2.56), noting that $\beta' < \beta$, it follows that

$$\tilde{g}_2(t,\theta,\lambda) \in M_{(5,5)}\left(\lambda^{\beta-\tilde{\sigma}}\right). \tag{2.57}$$

Since $L(\lambda,-\theta) = L(\lambda,\theta)$, the system (2.44) is reversible in θ with respect to the involution $(\lambda,t) \to (\lambda,-t)$. Thus Lemma 2.6 is proved. □

Now we make average on the nonlinear term $N_2(t,\theta,\lambda)$ in the second equation of (2.44).

Lemma 2.7. *There exists a transformation of the form*

$$\tau = t + \tilde{S}(\lambda,\theta), \quad \lambda = \lambda \tag{2.58}$$

which changes (2.44) *to the form*

$$\frac{d\lambda}{d\theta} = H_1(\lambda,\tau,\theta), \qquad \frac{d\tau}{d\theta} = 1 - [N_2] + H_2(\lambda,\tau,\theta), \tag{2.59}$$

where $[N_2] = \tilde{\alpha} \cdot \lambda^{\beta}$ *with* $\tilde{\alpha} = (1/2\pi_p)(q/p)(\alpha/\omega^p)(2/p)\int_0^{2\pi_p/\omega} |\sin_p^l \tilde{\theta}|^{l+2} d\tilde{\theta}$ *and the new perturbations* $H_1(\lambda,\tau,\theta), H_2(\lambda,\tau,\theta)$ *satisfy:*

$$\left|\lambda^k \frac{\partial^{k+s}}{\partial\lambda^k \partial t^s} H_1(\lambda,\tau,\theta)\right|, \qquad \left|\lambda^{k+1} \frac{\partial^{k+s}}{\partial\lambda^k \partial t^s} H_2(\lambda,\tau,\theta)\right| \leq C \cdot \lambda^{\beta+1-\tilde{\sigma}}. \tag{2.60}$$

Moreover, the system (2.59) *is reversible with respect to the involution G:* $(\lambda,\tau) \mapsto (\lambda,-\tau)$.

Proof. We choose

$$\widetilde{S}(\lambda,\theta) = \int_0^{\theta} (N_2(\lambda) - [N_2])d\overline{\theta}. \tag{2.61}$$

Then

$$\widetilde{S}(\lambda,-\theta) = \widetilde{S}(\lambda,\theta), \qquad \widetilde{S}(\lambda, 2\pi_p + \theta) = \widetilde{S}(\lambda,\theta), \qquad \widetilde{S}(\lambda,\theta) \in M_{(5,5)}\left(\lambda^{\beta}\right). \tag{2.62}$$

Defined a transformation by

$$\tau = t + \widetilde{S}(\lambda,\theta), \quad \lambda = \lambda. \tag{2.63}$$

Then the system of (2.44) becomes

$$\frac{d\lambda}{d\theta} = H_1(\lambda,\tau,\theta), \qquad \frac{d\tau}{d\theta} = 1 - [N_2] + H_2(\lambda,\tau,\theta), \tag{2.64}$$

where

$$\begin{gathered} H_1(\lambda,\tau,\theta) = \widetilde{g}_1\left(\lambda, \tau - \widetilde{S}(\lambda,\theta), \theta\right), \\ H_2(\lambda,\tau,\theta) = \widetilde{g}_2\left(\lambda, \tau - \widetilde{S}(\lambda,\theta), \theta\right) + \frac{\partial \widetilde{S}}{\partial \lambda}\widetilde{g}_1\left(\lambda, \tau - \widetilde{S}(\lambda,\theta), \theta\right). \end{gathered} \tag{2.65}$$

It is easy to very that

$$H_1(\lambda,-\tau,-\theta) = -H_1(\lambda,-\tau,-\theta), \qquad H_2(\lambda,-\tau,-\theta) = H_2(\lambda,\tau,\theta), \tag{2.66}$$

which implies that the system (2.59) is reversible with respect to the involution G: $(\lambda,\tau) \mapsto (\lambda,-\tau)$. In the same way as the proof of $g_1(\lambda,t,\theta)$ and $g_2(\lambda,t,\theta)$, we have

$$\left|\lambda^k \frac{\partial^{k+s}}{\partial\lambda^k \partial t^s} H_1(\lambda,\tau,\theta)\right|, \qquad \left|\lambda^{k+1} \frac{\partial^{k+s}}{\partial\lambda^k \partial t^s} H_2(\lambda,\tau,\theta)\right| \le C \cdot \lambda^{\beta+1-\widetilde{\sigma}}. \tag{2.67}$$

Thus Lemma 2.7 is proved. □

Below we introduce a small parameter such that the system (2.4) is written as a form of small perturbation of an integrable.

Let

$$[N_2] = \epsilon\rho. \tag{2.68}$$

Since

$$[N_2] = \widetilde{\alpha} \cdot \lambda^{\beta} \longrightarrow 0 \quad \text{as } \lambda \longrightarrow +\infty, \tag{2.69}$$

then

$$\lambda \longrightarrow +\infty \Longleftrightarrow \epsilon \longrightarrow 0^{+}. \tag{2.70}$$

Now, we define a transformation by

$$\lambda = \left(\frac{\epsilon\rho}{\widetilde{\alpha}}\right)^{1/\beta}, \quad \tau = \tau. \tag{2.71}$$

Then the system (2.59) has the form

$$\frac{d\rho}{d\theta} = g_1(\rho,\tau,\theta,\epsilon), \qquad \frac{d\tau}{d\theta} = 1 - \epsilon\rho + g_2(\rho,\tau,\theta,\epsilon), \tag{2.72}$$

where

$$g_1(\rho,\tau,\theta,\epsilon) = \varepsilon^{-1}\frac{d[N_2]}{d\lambda}H_1(\lambda(\epsilon,\rho),\tau,\theta), \qquad g_2(\rho,\tau,\theta,\epsilon) = H_2(\lambda(\epsilon,\rho),\tau,\theta). \tag{2.73}$$

Lemma 2.8. *The perturbations g_1 and g_2 satisfy the following estimates:*

$$\left|\frac{\partial^{k+s}}{\partial\rho^k\partial\tau^s}g_1\right| \le c\cdot\epsilon^{1+\sigma_0}, \quad \left|\frac{\partial^{k+s}}{\partial\rho^k\partial\tau^s}g_2\right| \le c\cdot\epsilon^{1+\sigma_0}, \quad \sigma_0 = -\frac{\widetilde{\sigma}}{\beta} > 0. \tag{2.74}$$

Proof. By (2.73), (2.60) and noting that $\lambda = (\epsilon\rho/\widetilde{\alpha})^{1/\beta}$, it follows that

$$\begin{aligned} |g_1| &= \left|\frac{[N]'}{\epsilon}\widetilde{H}_1\right| \le c\cdot\left|\epsilon^{-1}\lambda^{\beta+1}\widetilde{H}_1\right| \\ &\le c\cdot\epsilon^{-1}\lambda^{\beta-1}\lambda^{\beta+1-\widetilde{\sigma}} \le c\cdot\epsilon^{-1}\lambda^{2\beta-\widetilde{\sigma}} \le c\cdot\epsilon^{1+\sigma_0}. \end{aligned} \tag{2.75}$$

In the same way, $|g_2| = |\widetilde{H}_2| \le c\cdot\lambda^{\beta-\widetilde{\sigma}} \le c\cdot\epsilon^{1+\sigma_0}$. The estimates (2.74) for $k+s \ge 1$ follow easily from (2.60). □

2.5. Poincaré Map and Twist Theorems for Reversible Mapping

We can use a small twist theorem for reversible mapping to prove that the Pioncaré map P has an invariant closed curve, if ϵ is sufficiently small. The earlier result was due to Moser [11, 12], and Sevryuk [13]. Later, Liu [14] improved the previous results. Let us first recall the theorem in [14].

Let $A = [a, b] \times S^1$ be a finite part of cylinder $C = S^1 \times R$, where $S^1 = R/2\pi Z$, we denote by Γ the class of Jordan curves in C that are homotopic to the circle $r =$ constant. The subclass of Γ composed of those curves lying in A will be denoted by Γ_A, that is,

$$\Gamma_A = \{L \in \Gamma : L \subset A\}. \tag{2.76}$$

Consider a mapping $f_\epsilon : A \subset C \to C$, which is reversible with respect to $G : (\rho, \tau) \mapsto (\rho, -\tau)$. Moreover, a lift of f_ϵ can be expressed in the form:

$$\begin{aligned} \tau_1 &= \tau + \omega + \epsilon l_1(\rho, \tau) + \epsilon \tilde{g}_1(\rho, \tau, \epsilon), \\ \rho_1 &= \rho + \epsilon l_2(\rho, \tau) + \epsilon \tilde{g}_2(\rho, \tau, \epsilon), \end{aligned} \tag{2.77}$$

where ω is a real number, $\epsilon \in [0, 1]$ is a small parameter, the functions l_1, l_2, $\tilde{g}_1$, and $\tilde{g}_2$ are 2π periodic.

Lemma 2.9 (see [14, Theorem 2]). *Let $\omega = 2n\pi$ with an integer n and the functions l_1, l_2, $\tilde{g}_1$, and $\tilde{g}_2$ satisfy*

$$\begin{gathered} l_1 \in C^6(A), \quad l_1 > 0, \quad \frac{\partial l_1}{\partial \rho} > 0, \quad \forall (\rho, \tau) \in A, \\ l_2(\cdot, \cdot), \quad \tilde{g}_1(\cdot, \cdot, \epsilon), \quad \tilde{g}_2(\cdot, \cdot, \epsilon) \in C^5(A). \end{gathered} \tag{2.78}$$

In addition, we assume that there is a function $I : A \to R$ satisfying

$$\begin{gathered} I \in C^6(A), \quad \frac{\partial I}{\partial \rho} > 0, \quad \forall (\rho, \tau) \in A, \\ l_1(\rho, \tau) \cdot \frac{\partial I}{\partial \tau}(\rho, \tau) + l_2(\rho, \tau) \cdot \frac{\partial I}{\partial \rho}(\rho, \tau) = 0, \quad \forall (\rho, \tau) \in A. \end{gathered} \tag{2.79}$$

Moreover, suppose that there are two numbers $\tilde{a}$, and $\tilde{b}$ such that $a < \tilde{a} < \tilde{b} < b$ and

$$I_M(a) < I_m(\tilde{a}) \le I_M(\tilde{a}) < I_m\left(\tilde{b}\right) \le I_M\left(\tilde{b}\right) < I_m(b), \tag{2.80}$$

where

$$I_M(r) = \max_{\rho \in S^1} I(\rho, \tau), \qquad I_m(r) = \min_{\rho \in S^1} I(\rho, \tau). \tag{2.81}$$

Then there exist $\varsigma > 0$ and $\Delta > 0$ such that, if $\epsilon < \Delta$ and

$$\left\|\tilde{g}_1(\cdot, \cdot, \epsilon)\right\|_{C^5(A)} + \left\|\tilde{g}_2(\cdot, \cdot, \epsilon)\right\|_{C^5(A)} < \varsigma \tag{2.82}$$

the mapping f_ϵ has an invariant curve in Γ_A, the constant ς and Δ depend on $a, \tilde{a}, \tilde{b}, b, l_1, l_2$, and I. In particular, ς is independent of ϵ.

Remark 2.10. If $-l_1, l_2, \tilde{g}_1, \tilde{g}_2$ satisfy all the conditions of Lemma 2.9, then Lemma 2.9 still holds.

Lemma 2.11 (see [14, Theorem 1]). *Assume that* $\omega \notin 2\pi Q$ *and* $l_1(\cdot,\cdot)$, $l_2(\cdot,\cdot)\tilde{g}_1(\cdot,\cdot,\epsilon)$ *and* $\tilde{g}_2(\cdot,\cdot,\epsilon) \in C^4(A)$. *If*

$$\int_0^{2\pi} \frac{\partial l_1}{\partial \rho}(\tau,\rho)d\tau > 0, \quad \forall \rho \in [a,b]. \tag{2.83}$$

then there exist $\Delta > 0$ *and* $\varsigma > 0$ *such that* f_ϵ *has an invariant curve in* Γ_A *if* $0 < \epsilon < \Delta$ *and*

$$\left\|\tilde{g}_1(\cdot,\cdot,\epsilon)\right\|_{C^4(A)} + \left\|\tilde{g}_2(\cdot,\cdot,\epsilon)\right\|_{C^4(A)} < \varsigma. \tag{2.84}$$

The constants ς *and* Δ *depend on* ω, l_1, l_2 *only.*

We use Lemmas 2.9 and 2.11 to prove our Theorem 1.1. For the reversible mapping (2.86), $l_1 = -2\pi_p\epsilon\rho$, $l_2 = 0$.

2.6. Invariant Curves

From (2.73) and (2.66), we have

$$g_1(\rho,-\tau,-\theta,\epsilon) = -g_1(\rho,\tau,\theta,\epsilon), \qquad g_2(\rho,-\tau,-\theta,\epsilon) = g_2(\rho,\tau,\theta,\epsilon) \tag{2.85}$$

which yields that system (2.72) is reversible in θ with respect to the involution $G : (\rho,\tau) \mapsto (\rho,-\tau)$. Denote by P the Poincare map of (2.72), then P is also reversible with the same involution $G : (\rho,\tau) \mapsto (\rho,-\tau)$ and has the form

$$P : \begin{cases} \tau_1 = \tau + 2\pi_p - 2\epsilon\pi_p\rho + \tilde{g}_1(\rho,\tau,\epsilon), \\ \rho_1 = \rho + \tilde{g}_2(\rho,\tau,\epsilon), \end{cases} \tag{2.86}$$

where $\tau \in S^1$ and $\rho \in [1,2]$. Moreover, $\tilde{g}_1$ and $\tilde{g}_2$ satisfy

$$\left|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}\tilde{g}_1\right|, \qquad \left|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}\tilde{g}_2\right| \le c\cdot\epsilon^{1+\sigma_0}. \tag{2.87}$$

Case 1 ($2\pi_p$ is rational). Let $I = -l_1 = 2\pi_p\rho$, it is easy to see that

$$\begin{gathered} l_1(\rho,\tau) \in C^6(A), \quad l_1(\rho,\tau) = -2\pi_p\rho < 0, \frac{\partial l_1(\rho,\tau)}{\partial\rho} < 0, \\ I(\rho,\tau) \in C^6(A), \quad \frac{\partial I}{\partial\rho}(\rho,\tau) > 0, \quad l_2(\rho,\tau) = 0, \\ l_1(\rho,\tau)\frac{\partial I}{\partial\tau}(\rho,\tau) + l_2(\rho,\tau)\frac{\partial I}{\partial\rho}(\rho,\tau) = 0. \end{gathered} \tag{2.88}$$

Since I only depends on ρ, and $(\partial I/\partial\rho)(\rho,\tau) > 0$, all conditions in Lemma 2.9 hold.

Case 2 ($2\pi_p$ is irrational). Since

$$\int_0^{2\pi_p} \frac{\partial l_1}{\partial \rho}(\tau, \rho) d\tau = -(2\pi_p)^2 < 0, \tag{2.89}$$

all the assumptions in Lemma 2.11 hold.

Thus, in the both cases, the Poincare mapping P always have invariant curves for ϵ being sufficient small. Since $\epsilon \ll 1 \Leftrightarrow \lambda \gg 1$, we know that for any $\lambda \gg 1$, there is an invariant curve of the Poincare mapping, which guarantees the boundedness of solutions of the system (2.11). Hence, all the solutions of (1.9) are bounded.

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