

## Research Article

# Asymptotics for Nonlinear Evolution Equation with Module-Fractional Derivative on a Half-Line

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We consider the initial-boundary value problem for a nonlinear partial differential equation with module-fractional derivative on a half-line. We study the local and global in time existence of solutions to the initial-boundary value problem and the asymptotic behavior of solutions for large time.

## 1. Introduction

We study the local and global existence and asymptotic behavior for solutions to the initial-boundary value problem:

$$\begin{aligned}u_t + \mathcal{N}(u) + |\partial_x|^\alpha u &= 0, \quad t > 0, \quad x > 0, \\u(x, 0) &= u_0(x), \quad x > 0,\end{aligned}\tag{1.1}$$

where  $\mathcal{N}(u) = |u|^\sigma u$ ,  $\sigma > 1$ , and  $|\partial_x|^\alpha$  is the module-fractional derivative operator defined by

$$|\partial_x|^\alpha u(x, t) = \theta(x) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} |p|^\alpha \hat{u}(p, t) dp, \quad \alpha \in \left(\frac{1}{2}, 1\right),\tag{1.2}$$

where  $\hat{u}(p, t)$  is the Laplace transform for  $u(x, t)$  with respect to  $x$  and  $\theta(x)$  is the Heaviside function:

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (1.3)$$

The Cauchy problem for a wide class of nonlinear nonlocal dissipative equations has been studied extensively. In particular, the general approach for the study of the large time asymptotics to the Cauchy problem for different nonlinear equations was investigated in the book [1] and the references therein.

The boundary value problems are more natural for applications and play an important role in the contemporary mathematical physics. However, their mathematical investigations are more complicated even in the case of the differential equations, with more reason to the case of nonlocal equations. We need to answer such basic question as how many boundary values should be given in the problem for its solvability and the uniqueness of the solution? Also it is interesting to study the influence of the boundary data on the qualitative properties of the solution. For examples and details see [2–12] and references therein.

The general theory of nonlinear nonlocal equations on a half-line was developed in book [13], where the pseudodifferential operator  $\mathbb{K}$  on a half-line was introduced by virtue of the inverse Laplace transformation. In this definition it was important that the symbol  $K(p)$  must be analytic in the complex right half-plane. We emphasize that the pseudodifferential operator  $|\partial_x|^\alpha u$  in (1.1) has a nonanalytic nonhomogeneous symbol  $K(p) = |p|^\alpha$  and the general theory from book [13] cannot be applied to the problem (1.1) directly. As far as we know there are few results on the initial-boundary value problems with pseudodifferential equations having a nonanalytic symbol. The case of rational symbol  $K(p)$  which has some poles in the complex right half-plane was studied in [14, 15], where it was proposed a new method for constructing the Green operator based on the introduction of some necessary condition at the singular points of the symbol  $K(p)$ . In [16] there was considered the initial-boundary value problem for a pseudodifferential equation with a nonanalytic homogeneous symbol  $K(p) = |p|^{1/2}$ , where the theory of sectionally analytic functions was implemented for proving that the initial-boundary value problem is well posed. Since the symbol  $K(p) = |p|^{1/2}$  does not grow fast at infinity, so there were no boundary data in the corresponding problem.

In the present paper we consider the same problem as in [16] but with symbol  $K(p) = |p|^\alpha$ , where  $\alpha \in (1/2, 1)$ . The approach used in this paper is more general and simple than the one used in [16]; however to get the same result are necessary more accurate estimates than the ones obtained here for the Green operator.

To construct Green operator we proposed a new method based on the integral representation for a sectionally analytic function and the theory of singular integro-differential equations with Hilbert kernel (see [16, 17]). We arrive to a boundary condition of type  $\Omega^+(p) = W(p)\Omega^-(p) + g(p)$ , where  $-i\infty < p < +i\infty$ . The aim is to find two analytic functions,  $\Omega^+$  and  $\Omega^-$  (a sectionally analytic function  $\Omega$ ), in the left and right complex semi-planes, respectively, such that the boundary condition is satisfied. Two conditions are necessary to solve the problem: first, the function  $W$  must satisfy the Hölder condition both in the finite points and in the vicinity of the infinite point of the contour and, second, the index of function  $W$  must be zero. In our case both conditions do fail. To overcome this difficulty, we introduce an auxiliary function such that the Hölder and zero-index conditions are fulfilled.

To state precisely the results of the present paper we give some notations. We denote  $\langle t \rangle = 1 + t$ ,  $\{t\} = t / \langle t \rangle$ . Here and below  $p^\alpha$  is the main branch of the complex analytic function in the complex half-plane  $\operatorname{Re} p \geq 0$ , so that  $1^\alpha = 1$  (we make a cut along the negative real axis  $(-\infty, 0)$ ). Note that due to the analyticity of  $p^\alpha$  for all  $\operatorname{Re} p > 0$  the inverse Laplace transform gives us the function which is equal to 0 for all  $x < 0$ . Direct Laplace transformation  $\mathcal{L}_{x \rightarrow p}$  is

$$\widehat{u}(p) \equiv \mathcal{L}_{x \rightarrow p}\{u\} = \int_0^{+\infty} e^{-px} u(x) dx, \quad (1.4)$$

and the inverse Laplace transformation  $\mathcal{L}_{p \rightarrow x}^{-1}$  is defined by

$$u(x) \equiv \mathcal{L}_{p \rightarrow x}^{-1}\{\widehat{u}\} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} u(\xi) d\xi. \quad (1.5)$$

Weighted Lebesgue space is  $\mathbf{L}^{s,\mu}(\mathbf{R}^+) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^{s,\mu}} < \infty\}$ , where

$$\|\varphi\|_{\mathbf{L}^{s,\mu}} = \left( \int_0^{+\infty} x^{\mu s} |\varphi(x)|^s dx \right)^{1/s} \quad (1.6)$$

for  $\mu > 0$ ,  $1 < s < \infty$  and

$$\|\varphi\|_{\mathbf{L}^\infty} = \operatorname{ess\,sup}_{x \in \mathbf{R}^+} |\varphi(x)|. \quad (1.7)$$

Now, we define the metric spaces

$$\mathbf{Z} = \mathbf{L}^1(\mathbf{R}^+) \cap \mathbf{L}^{1,\mu}(\mathbf{R}^+) \cap \mathbf{L}^\infty(\mathbf{R}^+), \quad (1.8)$$

where  $\mu \in (0, 1)$ , with the norm

$$\begin{aligned} \|\phi\|_{\mathbf{Z}} &= \|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,\mu}} + \|\phi\|_{\mathbf{L}^\infty}, \\ \mathbf{X} &= \mathbf{C}([0, \infty); \mathbf{L}^2(\mathbf{R}^+)) \cap \mathbf{C}((0, \infty); \mathbf{L}^s(\mathbf{R}^+) \cap \mathbf{L}^{s,\mu}(\mathbf{R}^+) \cap \mathbf{L}^\infty(\mathbf{R}^+)), \end{aligned} \quad (1.9)$$

where  $s > 1$  and  $\mu \in (0, 1)$ , with the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t \geq 0} \|\phi\|_{\mathbf{L}^2} + \sup_{t > 0} \{t\}^{\gamma/\alpha} \left( \langle t \rangle^{\gamma/\alpha} \|\phi\|_{\mathbf{L}^s} + \langle t \rangle^{(1/\alpha)(\gamma-\mu)} \|\phi\|_{\mathbf{L}^{s,\mu}} + \langle t \rangle^{1/\alpha} \|\phi\|_{\mathbf{L}^\infty} \right), \quad (1.10)$$

$|\gamma - \mu| < \alpha$ ,  $\gamma = 1 - 1/s$ .

Now we state the main results. We introduce  $\Lambda(s) \in L^\infty(\mathbf{R}^+)$  by formula

$$\begin{aligned} \Lambda(s) = & \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{ps-|p|^\alpha} dp \\ & - \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\xi e^\xi \int_{-i\infty}^{i\infty} e^{ps} \frac{e^{\Gamma_0^+(p,\xi)} (p/(p-1))^{\alpha/2} - e^{-i(\pi/4)\alpha}}{|p|^\alpha + \xi} dp, \end{aligned} \quad (1.11)$$

where

$$\Gamma_0^+(p, \xi) = \lim_{\substack{z \rightarrow p \\ \operatorname{Re} z < 0}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \left\{ \frac{|q|^\alpha + \xi}{q^\alpha + \xi} \left( \frac{q-1}{q+1} \right)^{\alpha/2} \right\} dq. \quad (1.12)$$

We define the linear functional  $f$ :

$$f(\phi) = \int_0^{+\infty} \phi(y) dy. \quad (1.13)$$

**Theorem 1.1.** *Suppose that for small  $\mu > 0$  the initial data  $u_0 \in \mathbf{Z}$  are such that the norm  $\|u_0\|_{\mathbf{Z}} \leq \varepsilon$  is sufficiently small. Then, there exists a unique global solution  $u \in \mathbf{X}$  to the initial-boundary value problem (1.1). Moreover the following asymptotic is valid:*

$$u(x, t) = At^{-1/\alpha} \Lambda(xt^{-1/\alpha}) + O\left(t^{-(1/\alpha)(1+\kappa)}\right), \quad (1.14)$$

for  $t \rightarrow \infty$  in  $L^\infty(\mathbf{R}^+)$ , where  $\kappa \in (0, \mu)$  and

$$A = f(u_0) - \int_0^{+\infty} f(\mathcal{N}(u)) d\tau. \quad (1.15)$$

## 2. Preliminaries

In subsequent consideration we shall have frequently to use certain theorems of the theory of functions of complex variable, the statements of which we now quote. The proofs may be found in all text-book of the theory. Let  $L$  be smooth contour and  $\phi(q)$  a function of position on it.

*Definition 2.1.* The function  $\phi(q)$  is said to satisfy on the curve  $L$  the Hölder condition, if for two arbitrary points of this curve

$$|\phi(q_1) - \phi(q_2)| \leq C|q_1 - q_2|^\lambda, \quad (2.1)$$

where  $C$  and  $\lambda$  are positive numbers.

**Theorem 2.2.** Let  $\phi(q)$  be a complex function, which obeys the Hölder condition for all finite  $q$  and tends to a definite limit  $\phi(\infty)$  as  $q \rightarrow \infty$ , such that for large  $q$  the following inequality holds:

$$|\phi(q) - \phi(\infty)| \leq C|q|^{-\mu}, \quad \mu > 0. \quad (2.2)$$

Then Cauchy type integral

$$F(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z} dq \quad (2.3)$$

constitutes a function analytic in the left and right semiplanes. Here and below these functions will be denoted by  $F^+(z)$  and  $F^-(z)$ , respectively. These functions have the limiting values  $F^+(p)$  and  $F^-(p)$  at all points of imaginary axis  $\operatorname{Re} p = 0$ , on approaching the contour from the left and from the right, respectively. These limiting values are expressed by Sokhotzki-Plemelj formulae:

$$F^\pm(p) = \lim_{\substack{z \rightarrow p \\ \pm \operatorname{Re} z < 0}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-p} dq \pm \frac{1}{2} \phi(p). \quad (2.4)$$

Subtracting and adding the formula (2.4) we obtain the following two equivalent formulae:

$$F^+(p) - F^-(p) = \phi(p), \quad F^+(p) + F^-(p) = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-p} dq, \quad (2.5)$$

which will be frequently employed hereafter.

We consider the following linear initial-boundary value problem on half-line:

$$\begin{aligned} u_t + |\partial_x|^\alpha u &= 0, \quad t > 0, \quad x > 0, \\ u(x, 0) &= u_0(x), \quad x > 0. \end{aligned} \quad (2.6)$$

Setting  $K(q) = |q|^\alpha$ ,  $K_1(q) = q^\alpha$ , we define

$$G(t)\phi = \int_0^{+\infty} G(x, y, t)\phi(y) dy, \quad (2.7)$$

where the function  $G(x, y, t)$  is given by

$$G(x, y, t) = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} Z^-(p, \xi, y) dp, \quad (2.8)$$

$$Z^-(p, \xi, y) = \lim_{\substack{z \rightarrow p \\ \operatorname{Re} z > 0}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{Y^+(q, \xi)} e^{-qy} dq, \quad (2.9)$$

for  $x > 0$ ,  $y > 0$ ,  $t > 0$ . Here and below  $Y^\pm = e^{\pm} w^\pm$ , where  $\Gamma^+(p, \xi)$  and  $\Gamma^-(p, \xi)$  are a left and right limiting values of sectionally analytic function  $\Gamma(z, \xi)$  given by

$$\Gamma(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \left\{ \left( \frac{K(q) + \xi}{K_1(q) + \xi} \right) \frac{w^-(q)}{w^+(q)} \right\} dq, \quad (2.10)$$

where for some fixed real point  $z_0 > 0$ ,

$$w^-(z) = \frac{z^{\alpha/2}}{(z+z_0)^{\alpha/2}}, \quad w^+(z) = \frac{z^{\alpha/2}}{(z-z_0)^{\alpha/2}}. \quad (2.11)$$

All the integrals are understood in the sense of the principal values.

**Proposition 2.3.** *Let  $u_0(x) \in \mathbf{Z}$ . Then there exists a unique solution  $u(x, t)$  for the initial-boundary value problem (2.6), which has an integral representation:*

$$u = \mathcal{G}(t)u_0, \quad x > 0, \quad t > 0. \quad (2.12)$$

*Proof.* In order to obtain an integral representation for solutions of the problem (2.6) we suppose that there exist a solution  $u(x, t)$ , which is continued by zero outside of  $x > 0$ :

$$u(x, t) = 0, \quad \forall x < 0. \quad (2.13)$$

Let  $\phi(q)$  be a function of the complex variable  $q$ , which obeys the Hölder condition for all  $q$ , such that  $\operatorname{Re} q = 0$ . We define the operator  $\mathbb{P}$  by

$$\mathbb{P}_{q \rightarrow z} \{ \phi(q) \} = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \phi(q) dq, \quad \operatorname{Re} z \neq 0. \quad (2.14)$$

Using the Laplace transform we get

$$\mathbb{P}_{p \rightarrow q} \{ K(p) \hat{u}(p, t) \} = \mathcal{L}_{x \rightarrow q} \{ |\partial_x|^\alpha u(x, t) \}. \quad (2.15)$$

Since  $\hat{u}(q, t)$  is analytic for all  $\operatorname{Re} q > 0$ , we have

$$\mathbb{P}_{p \rightarrow q} \{ \hat{u}(p, t) \} = \hat{u}(q, t). \quad (2.16)$$

Therefore, applying the Laplace transform with respect to  $x$  to problem (2.6) and using (2.15) and (2.16), we obtain for  $t > 0$

$$\begin{aligned} \mathbb{P}_{p \rightarrow q} \{ \hat{u}_t(p, t) + K(p) \hat{u}(p, t) \} &= 0, \\ \hat{u}(p, 0) &= \hat{u}_0(p). \end{aligned} \quad (2.17)$$

We rewrite (2.17) in the form

$$\begin{aligned}\hat{u}_t(p, t) + K(p)\hat{u}(p, t) &= \Phi(p, t), \\ \hat{u}(p, 0) &= \hat{u}_0(p),\end{aligned}\tag{2.18}$$

with some function  $\Phi(p, t)$  such that

$$\mathbb{P}_{p \rightarrow q}\{\Phi(p, t)\} = 0, \quad \operatorname{Re} p > 0,\tag{2.19}$$

$$|\Phi(p, t)| \leq C|p|^{\alpha-1}, \quad |p| > 1.\tag{2.20}$$

Applying the Laplace transform with respect to time variable to (2.18), we find

$$\hat{\hat{u}}(p, \xi) = \frac{1}{K(p) + \xi} (\hat{u}_0(p) + \hat{\Phi}(p, \xi)),\tag{2.21}$$

where  $\operatorname{Re} p = 0$  and  $\operatorname{Re} \xi > 0$ . Here, the functions  $\hat{\hat{u}}(p, \xi)$  and  $\hat{\Phi}(p, \xi)$  are the Laplace transforms for  $\hat{u}(p, t)$  and  $\Phi(p, t)$  with respect to time, respectively. In order to obtain an integral formula for solutions to the problem (2.6) it is necessary to know the function  $\Phi(p, t)$ . We will find the function  $\hat{\Phi}(p, \xi)$  using the analytic properties of the function  $\hat{\hat{u}}$  in the right-half complex planes  $\operatorname{Re} p > 0$  and  $\operatorname{Re} \xi > 0$ . Equation (2.16) and the Sokhotzki-Plemelj formulae imply for  $\operatorname{Re} p = 0$

$$\hat{\hat{u}}(p, \xi) = -\frac{1}{\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-p} \hat{\hat{u}}(q, \xi) dq.\tag{2.22}$$

In view of Sokhotzki-Plemelj formulae via (2.21) the condition (2.22) can be written as

$$\Theta^+(p, \xi) = -\Lambda^+(p, \xi),\tag{2.23}$$

where the sectionally analytic functions  $\Theta(p, \xi)$  and  $\Lambda(p, \xi)$  are given by Cauchy type integrals:

$$\Theta(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{K(q) + \xi} \hat{\Phi}(q, \xi) dq,\tag{2.24}$$

$$\Lambda(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{K(q) + \xi} \hat{u}_0(q) dq.\tag{2.25}$$

To perform the condition (2.23) in the form of a nonhomogeneous Riemann-Hilbert problem we introduce the sectionally analytic function:

$$\Omega(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \Psi(q, \xi) dq,\tag{2.26}$$

where

$$\Psi(p, \xi) = \frac{K(p)}{K(p) + \xi} \widehat{\Phi}(p, \xi). \quad (2.27)$$

Taking into account the assumed condition (2.19), we get

$$\Theta^-(p, \xi) = -\frac{1}{\xi} \Omega^-(p, \xi). \quad (2.28)$$

Also observe that from (2.24) and (2.26) by Sokhotzki-Plemelj formulae,

$$K(p)(\Theta^+(p, \xi) - \Theta^-(p, \xi)) = \Psi(p, \xi) = \Omega^+(p, \xi) - \Omega^-(p, \xi). \quad (2.29)$$

Substituting (2.23) and (2.28) into this equation we obtain for  $\operatorname{Re} p = 0$

$$\Omega^+(p, \xi) = W(p, \xi)\Omega^-(p, \xi) + g(p, \xi), \quad (2.30)$$

where

$$W(p, \xi) = \frac{K(p) + \xi}{\xi}, \quad g(p, \xi) = -K(p)\Lambda^+(p, \xi). \quad (2.31)$$

Equation (2.30) is the boundary condition for a nonhomogeneous Riemann-Hilbert problem. It is required to find two functions for some fixed point  $\xi$ ,  $\operatorname{Re} \xi > 0$ :  $\Omega^+(z, \xi)$ , analytic in the left-half complex plane  $\operatorname{Re} z < 0$  and  $\Omega^-(z, \xi)$ , analytic in the right-half complex plane  $\operatorname{Re} z > 0$ , which satisfy on the contour  $\operatorname{Re} p = 0$  the relation (2.30).

Note that bearing in mind formula (2.27) we can find the unknown function  $\widehat{\Phi}(p, \xi)$ , which involved in the formula (2.21), by the relation

$$\widehat{\Phi}(p, \xi) = \frac{K(p) + \xi}{K(p)} (\Omega^+(p, \xi) - \Omega^-(p, \xi)). \quad (2.32)$$

The method for solving the Riemann problem  $F^+(p) = \phi(p)F^-(p) + \varphi(p)$  is based on the following results. The proofs may be found in [17].

**Lemma 2.4.** *An arbitrary function  $\varphi(p)$  given on the contour  $\operatorname{Re} p = 0$ , satisfying the Hölder condition, can be uniquely represented in the form*

$$\varphi(p) = U^+(p) - U^-(p), \quad (2.33)$$

where  $U^\pm(p)$  are the boundary values of the analytic functions  $U^\pm(z)$  and the condition  $U^\pm(\infty) = 0$  holds. These functions are determined by

$$U(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \varphi(q) dq. \quad (2.34)$$



**Lemma 2.5.** *An arbitrary function  $\phi(p)$  given on the contour  $\operatorname{Re} p = 0$ , satisfying the Hölder condition, and having zero index,*

$$\operatorname{ind} \phi(p) := \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d \ln \phi(p) = 0, \quad (2.35)$$

*is uniquely representable as the ratio of the functions  $X^+(p)$  and  $X^-(p)$ , constituting the boundary values of functions,  $X^+(z)$  and  $X^-(z)$ , analytic in the left and right complex semiplane and having in these domains no zero. These functions are determined to within an arbitrary constant factor and given by*

$$X^\pm(z) = e^{\Gamma^\pm(z)}, \quad \Gamma(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \phi(q) dq. \quad (2.36)$$

In the formulations of Lemmas 2.4 and 2.5 the coefficient  $\phi(p)$  and the free term  $\varphi(p)$  of the Riemann problem are required to satisfy the Hölder condition on the contour  $\operatorname{Re} p = 0$ . This restriction is essential. On the other hand, it is easy to observe that both functions  $W(p, \xi)$  and  $g(p, \xi)$  do not have limiting value as  $p \rightarrow \pm i\infty$ . So we cannot find the solution using  $\ln W(p, \xi)$ . The principal task now is to get an expression equivalent to the boundary value problem (2.30), such that the conditions of lemmas are satisfied. First, we introduce the function

$$\phi(p, \xi) = \left( \frac{K(p) + \xi}{K_1(p) + \xi} \right) \frac{w^-(p)}{w^+(p)}, \quad w^\pm(p) = \frac{p^{\alpha/2}}{(p \mp z_0)^{\alpha/2}}, \quad (2.37)$$

where  $K(p) = |p|^\alpha$ ,  $K_1(p) = p^\alpha$ , and  $z_0 > 0$ . We make a cut in the plane  $z$  from point  $z_0$  to point  $-\infty$  through 0. Owing to the manner of performing the cut the functions  $w^-(z)$  and  $K_1(z)$  are analytic for  $\operatorname{Re} z > 0$  and the function  $w^+(z)$  is analytic for  $\operatorname{Re} z < 0$ .

We observe that the function  $\phi(p, \xi)$ , given on the contour  $\operatorname{Re} p = 0$ , satisfies the Hölder condition and  $K_1(p) + \xi$  does not vanish for any  $\operatorname{Re} \xi > 0$ . Also we have

$$\operatorname{ind} \phi(p, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d \ln \phi(p, \xi) = 0. \quad (2.38)$$

Therefore in accordance with Lemma 2.5 the function  $\phi(p, \xi)$  can be represented in the form of the ratio

$$\phi(p, \xi) = \frac{X^+(p, \xi)}{X^-(p, \xi)}, \quad X^\pm(p, \xi) = \lim_{\substack{z \rightarrow p \\ \pm \operatorname{Re} z < 0}} e^{\Gamma(z, \xi)}, \quad (2.39)$$

where

$$\Gamma(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \{ \phi(q, \xi) \} dq. \quad (2.40)$$

From (2.37) and (2.39) we get

$$\frac{Y^+(p, \xi)}{Y^-(p, \xi)} = \frac{K(p) + \xi}{K_1(p) + \xi}, \quad (2.41)$$

where  $Y^\pm = e^{\Gamma^\pm} w^\pm$ . We note that (2.41) is equivalent to

$$\frac{K(p) + \xi}{\xi} = \frac{Y^+(p, \xi)}{Y^-(p, \xi)} \left( \frac{K_1(p) + \xi}{\xi} \right). \quad (2.42)$$

Now, we return to the nonhomogeneous Riemann-Hilbert problem defined by the boundary condition (2.30). We substitute the above equation in (2.30) and add  $-(\xi/Y^+)\Lambda^+$  in both sides to get

$$\frac{\Omega^+ - \xi\Lambda^+}{Y^+} = \left( \frac{K_1(p) + \xi}{\xi} \right) \frac{\Omega^-}{Y^-} - \left( \frac{K(p) + \xi}{Y^+} \right) \Lambda^+. \quad (2.43)$$

On the other hand, by Sokhotzki-Plemelj formulae and (2.25),  $\Lambda^+ - \Lambda^- = (1/(K(p) + \xi))\hat{u}_0(p)$ . Now, we substitute  $\Lambda^+$  from this equation in formula (2.43); then by (2.41) we arrive to

$$\frac{\Omega^+ - \xi\Lambda^+}{Y^+} = \left( \frac{K_1(p) + \xi}{\xi} \right) \left( \frac{\Omega^- - \xi\Lambda^-}{Y^-} \right) - \frac{1}{Y^+} \hat{u}_0(p). \quad (2.44)$$

In subsequent consideration we shall have to use the following property of the limiting values of a Cauchy type integral, the statement of which we now quote. The proofs may be found in [17].

**Lemma 2.6.** *If  $L$  is a smooth closed contour and  $\phi(q)$  a function that satisfies the Hölder condition on  $L$ , then the limiting values of the Cauchy type integral*

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{1}{q-z} \phi(q) dq \quad (2.45)$$

*also satisfy this condition.*

Since  $\hat{u}_0(p)$  satisfies on  $\text{Re } p = 0$  the Hölder condition, on basis of Lemma 2.6 the function  $(1/Y^+)\hat{u}_0(p)$  also satisfies this condition. Therefore, in accordance with Lemma 2.4 it can be uniquely represented in the form of the difference of the functions  $U^+(p, \xi)$  and  $U^-(p, \xi)$ , constituting the boundary values of the analytic function  $U(z, \xi)$ , given by

$$U(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{Y^+(q, \xi)} \hat{u}_0(q) dq. \quad (2.46)$$

Therefore, (2.44) takes the form

$$\frac{\Omega^+ - \xi\Lambda^+}{Y^+} + U^+ = \left( \frac{K_1(p) + \xi}{\xi} \right) \left( \frac{\Omega^- - \xi\Lambda^-}{Y^-} \right) + U^-. \quad (2.47)$$

The last relation indicates that the function  $(\Omega^+ - \xi\Lambda^+)/Y^+ + U^+$ , analytic in  $\operatorname{Re} z < 0$ , and the function  $((K_1(p) + \xi)/\xi)((\Omega^- - \xi\Lambda^-)/Y^-) + U^-$ , analytic in  $\operatorname{Re} z > 0$ , constitute the analytic continuation of each other through the contour  $\operatorname{Re} z = 0$ . Consequently, they are branches of a unique analytic function in the entire plane. According to Liouville theorem this function is some arbitrary constant  $A$ . Thus, we obtain the solution of the Riemann-Hilbert problem defined by the boundary condition (2.30):

$$\begin{aligned} \Omega^+(p, \xi) &= Y^+(p, \xi)(A - U^+(p, \xi)) + \xi\Lambda^+(p, \xi), \\ \Omega^-(p, \xi) &= \frac{\xi}{K_1(p) + \xi} Y^-(p, \xi)(A - U^-(p, \xi)) + \xi\Lambda^-(p, \xi). \end{aligned} \quad (2.48)$$

Since  $\Omega$  is defined by a Cauchy type integral, with density  $\Psi$ , we have  $\Omega^\pm(z, \xi) \rightarrow \pm(1/2)\Psi(\infty, \xi) = 0$ , as  $z \rightarrow \infty$  for  $\mp \operatorname{Re} z > 0$ . Using this property in (2.48) we get  $A = 0$  and the limiting values for  $\Omega$  are given by

$$\begin{aligned} \Omega^+(p, \xi) &= -Y^+(p, \xi)U^+(p, \xi) + \xi\Lambda^+(p, \xi), \\ \Omega^-(p, \xi) &= -\frac{\xi}{K_1(p) + \xi} Y^-(p, \xi)U^-(p, \xi) + \xi\Lambda^-(p, \xi). \end{aligned} \quad (2.49)$$

Now, we proceed to find the unknown function  $\widehat{\Phi}(p, \xi)$  involved in the formula (2.21) for the solution  $\widehat{u}(p, \xi)$  of the problem (2.6). First, we represent  $\Omega^-(p, \xi)$  as the limiting value of analytic functions on the left-hand side complex semiplane. From (2.41) and Sokhotzki-Plemelj formulae we obtain

$$\Omega^-(p, \xi) = -\frac{\xi}{K(p) + \xi} Y^+(p, \xi)U^+(p, \xi) + \xi\Lambda^+(p, \xi). \quad (2.50)$$

Now, making use of (2.49) and the above equation, we get

$$\Omega^+(p, \xi) - \Omega^-(p, \xi) = -\frac{K(p)}{K(p) + \xi} Y^+(p, \xi)U^+(p, \xi). \quad (2.51)$$

Thus, by formula (2.32),

$$\widehat{\Phi}(p, \xi) = -Y^+(p, \xi)U^+(p, \xi). \quad (2.52)$$

We observe that  $\widehat{\Phi}(p, \xi)$  is boundary value of a function analytic in the left-hand side complex semi-plane and therefore satisfies our basic assumption (2.19). Having determined the function  $\widehat{\Phi}$ , bearing in mind formula (2.21) we determine the required function  $\widehat{u}$ :

$$\widehat{u}(p, \xi) = \frac{1}{K(p) + \xi} (\widehat{u}_0(p) - Y^+(p, \xi)U^+(p, \xi)). \tag{2.53}$$

Now we prove that, in accordance with last relation, the function  $\widehat{u}(p, \xi)$  constitutes the limiting value of an analytic function in  $\text{Re } z > 0$ . In fact, making use of Sokhotzki-Plemelj formulae and using (2.41), we obtain

$$\widehat{u}(p, \xi) = -\frac{1}{K_1(p) + \xi} Y^-(p, \xi)U^-(p, \xi). \tag{2.54}$$

Thus, the function  $\widehat{u}(p, \xi)$  is the limiting value of an analytic function in  $\text{Re } z > 0$ . We note the fundamental importance of the proven fact, the solution  $\widehat{u}(p, \xi)$  constitutes an analytic function in  $\text{Re } z > 0$ , and, as a consequence, its inverse Laplace transform vanishes for all  $x < 0$ . We now return to solution  $u(x, t)$  of the problem (2.6). Taking inverse Laplace transform with respect to time and space variables, we obtain  $u(x, t) = \mathcal{G}(t)u_0 = \int_0^\infty G(x, y, t)u_0(y)dy$ , where the function  $G(x, y, t)$  is defined by formula (2.8). Thus, Proposition 2.3 has been proved.  $\square$

Now we collect some preliminary estimates of the Green operator  $\mathcal{G}(t)$ .

**Lemma 2.7.** *The following estimates are true, provided that the right-hand sides are finite:*

$$\begin{aligned} \|\mathcal{G}(t)\phi - t^{-1/\alpha}\Lambda(\cdot)t^{-1/\alpha}f(\phi)\|_{L^\infty} &\leq Ct^{-(1/\alpha)(1+\mu)}\|\phi\|_{L^{1,\mu}}, \\ \|\mathcal{G}(t)\phi\|_{L^\infty} &\leq C\{t\}^{-\gamma/\alpha}\langle t \rangle^{-1/\alpha}(\|\phi\|_{L^1} + \|\phi\|_{L^\infty}), \\ \|\mathcal{G}(t)\phi\|_{L^2} &\leq C(\|\phi\|_{L^1} + \|\phi\|_{L^\infty}), \\ \|\mathcal{G}(t)\phi\|_{L^{s,\mu}} &\leq Ct^{-1/\alpha(1/r-1/s-\mu)}\|\phi\|_{L^r} + Ct^{-(1/\alpha)(1/r-1/s)}\|\phi\|_{L^{r,\mu}} \end{aligned} \tag{2.55}$$

where  $0 < \gamma < \alpha$ ,  $|1/r - 1/s - \mu| < \alpha$ ,  $0 \leq \mu < 2\alpha - 1$ ,  $1 \leq r < s \leq \infty$ , and  $f(\phi)$  and  $\Lambda(s)$  are given by (1.13) and (1.11), respectively.

*Proof.* First, we estimate the function

$$\Gamma(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln\{\phi(q, \xi)\} dq, \quad \phi(q, \xi) = \frac{K(q) + \xi}{K_1(q) + \xi} \left(\frac{q-z_0}{q+z_0}\right)^{\alpha/2}. \tag{2.56}$$

We note that  $\arg\{\phi(q, \xi)\} \rightarrow -(\pi/2)\alpha$ , as  $q \rightarrow \pm i\infty$ , and write  $\Gamma$  in the form

$$\Gamma(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \left(\ln\left\{e^{i(\pi/2)\alpha}\phi(q, \xi)\right\} + \ln\left\{e^{-i(\pi/2)\alpha}\right\}\right) dq. \tag{2.57}$$

For first integral in (2.57), we obtain the estimate

$$\left| \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \left\{ e^{i(\pi/2)\alpha} \phi(q, \xi) \right\} dq \right| \leq C \left( \frac{|\xi|}{|z|^\alpha} + \frac{z_0^{1-\epsilon}}{|z|^{1-\epsilon}} \right), \quad (2.58)$$

where  $0 < \epsilon < 1$ , and for second integral we have

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \left\{ e^{-i(\pi/2)\alpha} \right\} dq = i \frac{\pi}{4} \alpha \operatorname{sgn}(\operatorname{Re} z). \quad (2.59)$$

Therefore, substituting (2.58) and (2.59) in (2.57), we get for  $\Gamma$

$$\left| \Gamma(z, \xi) - i \frac{\pi}{4} \alpha \operatorname{sgn}(\operatorname{Re} z) \right| \leq C \left( \frac{|\xi|}{|z|^\alpha} + \frac{z_0^{1-\epsilon}}{|z|^{1-\epsilon}} \right). \quad (2.60)$$

Now, we estimate function  $Z$  defined by

$$Z(z, \xi, y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \frac{1}{Y^+(q, \xi)} e^{-qy} dq. \quad (2.61)$$

Using (2.60), we get for  $Y^\pm = e^{\Gamma^\pm} w^\pm$  the estimate

$$\left| Y(z, \xi)^{\pm 1} - e^{\pm i(\pi/4)\alpha \operatorname{sgn}(\operatorname{Re} z)} \right| \leq C \left( \frac{|\xi|}{|z|^\alpha} + \frac{z_0^{1-\epsilon}}{|z|^{1-\epsilon}} \right), \quad (2.62)$$

where  $\operatorname{Re} z \neq 0$ . Then, by (2.62) and Cauchy Theorem,

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \left( \frac{1}{Y^+(q, \xi)} - e^{i(\pi/4)\alpha} \right) dq &= 0, \quad \operatorname{Re} z > 0, \\ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} e^{-qy} dq &= -e^{-zy}, \quad \operatorname{Re} z > 0. \end{aligned} \quad (2.63)$$

Equations (2.63) imply that we can write  $Z$  in the form

$$Z(z, \xi, y) = e^{i(\pi/4)\alpha} e^{-zy} + Z_0(z, \xi, y), \quad \operatorname{Re} z > 0, \quad (2.64)$$

where

$$Z_0(z, \xi, y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \left( \frac{1}{Y^+(q, \xi)} - e^{i(\pi/4)\alpha} \right) (e^{-qy} - e^{-zy}) dq. \quad (2.65)$$

Thus, for  $\operatorname{Re} p = 0$ ,

$$Z^-(p, \xi, y) = e^{i(\pi/4)\alpha} e^{-py} + Z_0^-(p, \xi, y), \quad (2.66)$$

where  $Z_0^-$  satisfies

$$|Z_0^-(p, \xi, y)| \leq C y^\mu \left( \frac{|\xi|}{|p|^{\alpha-\mu}} + \frac{z_0^{1-\epsilon}}{|p|^{1-\epsilon-\mu}} \right), \quad 0 \leq \mu < \alpha. \quad (2.67)$$

In fact, we use (2.62) and inequality  $|e^{-qy} - e^{-py}| \leq C|q - p|^\mu y^\mu$ , where  $\operatorname{Re} q = \operatorname{Re} p = 0$  and  $0 \leq \mu \leq 1$ , to obtain

$$|Z_0^-(p, \xi, y)| \leq C y^\mu \int_{-i\infty}^{i\infty} \frac{1}{|q - p|^{1-\mu}} \left( \frac{|\xi|}{|q|^\alpha} + \frac{z_0^{1-\epsilon}}{|q|^{1-\epsilon}} \right) |dq|. \quad (2.68)$$

Making the change of variable  $q = pz$ , (2.67) follows. Now, substituting (2.66) in (2.8), for Green function  $G$ , we obtain

$$G(x, y, t) = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} e^{i(\pi/4)\alpha} e^{-py} dp + R_0, \quad (2.69)$$

where

$$R_0(x, y, t) = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} Z_0^-(p, \xi, y) dp. \quad (2.70)$$

The function  $R_0$  defined in (2.70) satisfies the estimate

$$|R_0(x, y, t)| \leq C t^{-(1/\alpha)(\mu+1)} y^\mu, \quad 0 \leq \mu < 2\alpha - 1. \quad (2.71)$$

In fact, using (2.67) we get

$$|R_0| \leq C y^\mu \int_{C_1} |d\xi| e^{-\lambda|\xi|t} \int_{-i\infty}^{i\infty} \frac{1}{|K(p) + \xi|} \left( \frac{|\xi|}{|p|^{\alpha-\mu}} + \frac{z_0^{1-\epsilon}}{|p|^{1-\epsilon-\mu}} \right) |dp|. \quad (2.72)$$

Here,

$$C_1 = \left\{ \xi \in \left( \infty e^{-i(\pi/2+\epsilon_1)}, 0 \right) \cup \left( \infty e^{i(\pi/2+\epsilon_1)}, 0 \right) \right\}, \quad \epsilon_1 > 0. \quad (2.73)$$

We have used inequality  $|e^{\xi t}| \leq e^{-\lambda|\xi|t}$ , where  $\xi \in C_1$  and  $\lambda > 0$  is some positive constant. Taking  $z_0 = t^{-1/\alpha}$  and making the change of variables:  $p = qt^{-1/\alpha}$  and  $\xi = q_1 t^{-1}$ , we obtain (2.71). Now, let us split (2.69):

$$\begin{aligned} G(x, y, t) = & -\frac{1}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{1}{K(p) + \xi} dp \\ & - \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi) - e^{-i(\pi/4)\alpha}}{K(p) + \xi} dp \\ & - \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi) - e^{-i(\pi/4)\alpha}}{K(p) + \xi} (e^{-py} - 1) dp \\ & - \frac{1}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{1}{K(p) + \xi} (e^{-py} - 1) dp + R_0(x, y, t). \end{aligned} \quad (2.74)$$

By Fubini's theorem and Cauchy's theorem, from the first and fourth summands we obtain

$$\begin{aligned} G(x, y, t) = & \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{px-K(p)t} dp \\ & - \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi) - e^{-i(\pi/4)\alpha}}{K(p) + \xi} dp \\ & - \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi) - e^{-i(\pi/4)\alpha}}{K(p) + \xi} (e^{-py} - 1) dp \\ & + \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{px-K(p)t} (e^{-py} - 1) dp + R_0(x, y, t). \end{aligned} \quad (2.75)$$

Then,

$$G(x, y, t) = G_0(x, t) + R_0(x, y, t) + R_1(x, y, t) + R_2(x, y, t), \quad (2.76)$$

where

$$G_0 = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{px-K(p)t} dp - \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi) - e^{-i(\pi/4)\alpha}}{K(p) + \xi} dp, \quad (2.77)$$

$$R_0 = -\frac{1}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} Z_0^-(p, \xi, y) dp,$$

$$R_1 = -\frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\xi e^{\xi t} \int_{-\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi) - e^{-i(\pi/4)\alpha}}{K(p) + \xi} (e^{-py} - 1) dp, \quad (2.78)$$

$$R_2 = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{px-K(p)t} (e^{-py} - 1) dp.$$

Now, we show that function  $R_1$  defined by (2.78) satisfies

$$|R_1(x, y, t)| \leq Ct^{-(1/\alpha)(\mu+1)}y^\mu, \quad 0 \leq \mu < 2\alpha - 1. \quad (2.79)$$

In fact, using (2.62) and the inequality  $|e^{-py} - 1| \leq C|p|^\mu y^\mu$ , where  $\operatorname{Re} p = 0$  and  $0 \leq \mu \leq 1$ , we get

$$|R_1| \leq Cy^\mu \int_{C_1} |d\xi| e^{-\lambda|\xi|t} \int_{-i\infty}^{i\infty} \frac{|dp|}{|K(p) + \xi|} \left( \frac{|\xi|}{|p|^{\alpha-\mu}} + \frac{|z_0|^{1-\epsilon}}{|p|^{1-\epsilon-\mu}} \right) |dp|. \quad (2.80)$$

Then, taking  $z_0 = t^{-1/\alpha}$ ,  $\epsilon < 1 - \alpha$  and making the change of variable  $p = qt^{-1/\alpha}$  and  $\xi = q_1 t^{-1}$ , we obtain (2.79). In the same way, we show that function  $R_2$  defined in (2.78) satisfies the inequality:

$$|R_2(x, y, t)| \leq Ct^{-(1/\alpha)(\mu+1)}y^\mu, \quad 0 \leq \mu < 2\alpha - 1. \quad (2.81)$$

In fact, using the inequality  $|e^{-py} - 1| \leq C|p|^\mu y^\mu$ , where  $\operatorname{Re} p = 0$  and  $0 \leq \mu \leq 1$ , we get

$$|R_2(x, y, t)| \leq C \int_{-i\infty}^{i\infty} e^{-K(p)t} |e^{-py} - 1| |dp| \leq Cy^\mu \int_{-i\infty}^{i\infty} e^{-K(p)t} |p|^\mu |dp|. \quad (2.82)$$

Making the change of variable  $p = qt^{-1/\alpha}$ , we arrive to

$$|R_2(x, y, t)| \leq Ct^{-(1/\alpha)(\mu+1)}y^\mu \int_{-i\infty}^{i\infty} e^{-K(q)t} |q|^\mu |dq|. \quad (2.83)$$

Thus, (2.81) follows. Finally, we show that

$$G_0(x, t) = t^{-1/\alpha} \Lambda(xt^{-1/\alpha}), \quad (2.84)$$

where  $\Lambda$  is given by (1.11). Making the change of variable  $p = t^{-1/\alpha}q_0$ ,  $\xi = t^{-1}q_1$ , and choosing  $z_0 = t^{-1/\alpha}$ , we get

$$Y^+(p, \xi) = e^{\Gamma_0^+(q_0, q_1)} \left( \frac{q_0}{q_0 - 1} \right)^{\alpha/2}, \quad (2.85)$$

where

$$\Gamma_0^+(q_0, q_1) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q - q_0} \ln \left\{ \left( \frac{K(q) + q_1}{K_1(q) + q_1} \right) \left( \frac{q - 1}{q + 1} \right)^{\alpha/2} \right\} dq. \quad (2.86)$$



Now, making the change of variable  $p = t^{-1/\alpha}q_0$  and  $\xi = t^{-1}q_1$  in equation for  $G_0$  we obtain

$$G_0 = t^{-1/\alpha} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{q_0 x t^{-(1/\alpha)} - K(q_0)} dq_0 - t^{-1/\alpha} \frac{e^{i(\pi/4)\alpha}}{(2\pi i)^2} \int_{-i\infty}^{i\infty} dq_1 e^{q_1} \int_{-i\infty}^{i\infty} e^{q_0 x t^{-1/\alpha}} \frac{e^{\Gamma_0^+(q_0, q_1)} (q_0/q_0 - 1)^{\alpha/2} - e^{-i(\pi/4)\alpha}}{K(q_0) + q_1} dq_0. \quad (2.87)$$

Therefore, (2.84) follows. Finally, using estimates (2.71), (2.79), (2.81), and (2.84), we get the asymptotic for the Green function  $G$ :

$$\left| G(x, y, t) - t^{-1/\alpha} \Lambda(xt^{-1/\alpha}) \right| \leq Ct^{-(1/\alpha)(\mu+1)} y^\mu, \quad (2.88)$$

where  $\Lambda$  is given by (1.11) and  $0 \leq \mu < 2\alpha - 1$ . By last inequality

$$\left| \int_0^{+\infty} (G(x, y, t) - t^{-1/\alpha} \Lambda(xt^{-1/\alpha})) \phi(y) dy \right| \leq Ct^{-(1/\alpha)(\mu+1)} \|\phi\|_{L^{1,\mu}}. \quad (2.89)$$

Therefore,

$$\left\| \mathcal{G}(t)\phi - t^{-1/\alpha} \Lambda(\cdot t^{-1/\alpha}) f(\phi) \right\|_{L^\infty} \leq Ct^{-(1/\alpha)(\mu+1)} \|\phi\|_{L^{1,\mu}}, \quad (2.90)$$

where  $f$  and  $\Lambda$  are given by (1.13) and (1.11), respectively, and  $0 \leq \mu < 2\alpha - 1$ . Thus, the first estimate in Lemma 2.7 has been proved.

Now, we are going to prove the second estimate in Lemma 2.7. First, for large  $t$ , using Sokhotzki-Plemelj formulae, we have for function  $Z$ , defined in (2.61),

$$Z^+(p, \xi, y) - Z^-(p, \xi, y) = \frac{1}{Y^+(p, \xi)} e^{-py}. \quad (2.91)$$

Substituting last equation in (2.8), we get

$$G(x, y, t) = J_1(x - y, t) + J_2(x, y, t), \quad (2.92)$$

where

$$J_1(x - y, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{p(x-y) - K(p)t} dp, \quad (2.93)$$

$$J_2(x, y, t) = -\frac{1}{(2\pi i)^2} \int_{i\infty}^{-i\infty} d\xi e^{\xi t} \int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} Z^+(p, \xi, y) dp.$$

Making the change of variable  $p = zt^{-1/\alpha}$  we get for  $J_1$

$$|J_1(x - y, t)| \leq Ct^{-1/\alpha}. \quad (2.94)$$

To estimate  $J_2$ , we consider an extension to the function  $K(p)$ :

$$K(p) = \begin{cases} (-ip)^\alpha, & \operatorname{Im} p > 0, \\ (ip)^\alpha, & \operatorname{Im} p \leq 0, \end{cases} \quad (2.95)$$

and we use the contours

$$\begin{aligned} C_1 &= \{\xi \in (\infty e^{-i(\pi/2+\varepsilon_1)}, 0) \cup (0, \infty e^{i(\pi/2+\varepsilon_1)})\}, & \varepsilon_1 > 0, \\ C_2 &= \{p \in (\infty e^{-i(\pi/2+\varepsilon_2)}, 0) \cup (0, \infty e^{i(\pi/2+\varepsilon_2)})\}, & \varepsilon_2 > 0, \end{aligned} \quad (2.96)$$

to obtain for  $J_2$

$$J_2(x, y, t) = -\frac{1}{(2\pi i)^2} \int_{C_1} d\xi e^{\xi t} \int_{C_2} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} Z^+(p, \xi, y) dp. \quad (2.97)$$

Let us write the function  $Z$ , defined in (2.61), in the form

$$Z = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \left( \frac{1}{Y^+(q, \xi)} - e^{i(\pi/4)\alpha} \right) e^{-qy} dq + \frac{e^{i(\pi/4)\alpha}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} e^{-qy} dq, \quad (2.98)$$

where  $\operatorname{Re} z \neq 0$ . Then, by Cauchy Theorem, for  $z < 0$  the second summand in last equation is zero. Thus, using (2.62) we obtain for  $\operatorname{Re} p = 0$

$$|Z^+(p, \xi, y)| \leq C \left( \frac{|\xi|}{|p|^\alpha} + \frac{z_0^{1-\varepsilon}}{|p|^{1-\varepsilon}} \right). \quad (2.99)$$

From the last inequality and (2.97) we get

$$|J_2| \leq C \int_{C_1} |d\xi| e^{-C|\xi|t} \int_{C_2} \frac{1}{|K(p) + \xi|} \left( \frac{|\xi|}{|p|^\alpha} + \frac{z_0^{1-\varepsilon}}{|p|^{1-\varepsilon}} \right) |dp|. \quad (2.100)$$

Taking  $z_0 = t^{-1/\alpha}$  and making the change of variables  $p = zt^{-1/\alpha}$  and  $\xi = \xi_1 t^{-1}$ , in the last inequality, we obtain

$$|J_2(x, y, t)| \leq Ct^{-1/\alpha}. \quad (2.101)$$

From (2.92) and the estimates (2.94) and (2.101) we get the estimate  $|G(x, y, t)| \leq Ct^{-1/\alpha}$ . Thus,

$$\|G(t)\phi\|_{L^\infty} = \sup_{x \in \mathbb{R}^+} \left| \int_0^{+\infty} G(x, y, t) \phi(y) dy \right| \leq t^{-1/\alpha} \|\phi\|_{L^1}. \quad (2.102)$$

Now, for small  $t$ , we are going to prove the estimate

$$\|G(t)\phi\|_{L^\infty} \leq Ct^{-(1/\alpha)(1-\gamma)} (\|\phi\|_{L^1} + \|\phi\|_{L^\infty}), \quad (2.103)$$

where  $1 - \alpha < \gamma < 1$ . First, we rewrite the Green function  $G$  in the form

$$G(x, y, t) = J_1(x - y, t) + J_2(x, y, t), \quad (2.104)$$

where

$$\begin{aligned} J_1(r, t) &= \frac{1}{2\pi i} \int_{C_\pm} e^{pr-K(p)t} dp, \quad \pm r > 0, \\ J_2 &= -\frac{1}{(2\pi i)^3} \int_{C_1} d\xi e^{\xi t} \int_{C_2} dp e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} \int_{C_3} \frac{1}{q-p} \frac{1}{Y^-(q, \xi)} \left( \frac{K_1(q) + \xi}{K(q) + \xi} \right) e^{-qy} dq. \end{aligned} \quad (2.105)$$

The contours  $C_1$  and  $C_2$  are defined in (2.96) and

$$\begin{aligned} C_\pm &= \{p \in (\infty e^{-i(\pi/2 \pm \varepsilon)}, 0) \cup (0, \infty e^{i(\pi/2 \pm \varepsilon)})\}, \quad \varepsilon > 0, \\ C_3 &= \{q \in (\infty e^{-i(\pi/2 - \varepsilon_3)}, 0) \cup (0, \infty e^{i(\pi/2 - \varepsilon_3)})\}, \quad \varepsilon_3 > 0. \end{aligned} \quad (2.106)$$

Moreover, we have extended the function  $K(p)$  as in (2.95). Making the change of variable  $p = zt^{-1/\alpha}$  and using the inequality  $|e^z| \leq |z|^{-\gamma}$ ,  $\gamma > 0$ , we obtain the estimate  $|J_1(r, t)| \leq Ct^{-(1/\alpha)(1-\gamma)} |r|^{-\gamma}$  or

$$|J_1(x - y, t)| \leq Ct^{-(1/\alpha)(1-\gamma)} |x - y|^{-\gamma}, \quad (2.107)$$

for  $x, y > 0$ . Now, we estimate  $J_2$ . Using  $|e^{-qy}| \leq |q|^{-\gamma} y^{-\gamma}$ , for  $\operatorname{Re} q > 0$ ,  $y > 0$ , and  $\gamma > 0$ , we get

$$|J_2(x, y, t)| \leq Cy^{-\gamma} \int_{C_1} |d\xi| e^{-C|\xi|t} \int_{C_2} \frac{|dp|}{|K(p) + \xi|} \int_{C_3} \frac{1}{|q-p|} \frac{|dq|}{|q|^\gamma}. \quad (2.108)$$

Making the change of variables  $p = zt^{-1/\alpha}$ ,  $q = z_1 t^{-1/\alpha}$  and  $\xi = \xi_1 t^{-1}$ , into the last inequality, we obtain

$$|J_2(x, y, t)| \leq Ct^{-(1/\alpha)(1-\gamma)} y^{-\gamma}, \quad 1 - \alpha < \gamma < 1. \quad (2.109)$$

By (2.104) and the estimates (2.107) and (2.109) we get

$$|G(x, y, t)| \leq Ct^{-(1/\alpha)(1-\gamma)} (|x - y|^{-\gamma} + y^{-\gamma}), \quad 1 - \alpha < \gamma < 1. \quad (2.110)$$

Thus,

$$\begin{aligned} \|\mathcal{G}(t)\phi\|_{L^\infty} &\leq \sup_{x \in \mathbb{R}^+} \int_0^{+\infty} |G(x, y, t)| |\phi(y)| dy \\ &\leq Ct^{-(1/\alpha)(1-\gamma)} \int_0^{+\infty} (|x-y|^{-\gamma} + y^{-\gamma}) |\phi(y)| dy \\ &\leq Ct^{-(1/\alpha)(1-\gamma)} (\|\phi\|_{L^1} + \|\phi\|_{L^\infty}). \end{aligned} \quad (2.111)$$

Thus, we get (2.103) and the second estimate in Lemma 2.7 has been proved.

Let us introduce the operators

$$\mathcal{J}_1(t)\phi = \theta(x) \int_0^{+\infty} J_1(x-y, t)\phi(y) dy, \quad (2.112)$$

$$\mathcal{J}_2(t)\phi = \theta(x) \int_0^{+\infty} J_2(x, y, t)\phi(y) dy, \quad (2.113)$$

where  $J_1$  and  $J_2$  are defined in (2.104). Then, the operator  $\mathcal{G}(t)$  can be written in the form

$$\mathcal{G}(t) = \mathcal{J}_1(t) + \mathcal{J}_2(t). \quad (2.114)$$

Now, we are going to prove the third estimate in Lemma 2.7,

$$\|\mathcal{G}(t)\phi\|_{L^2} \leq C(\|\phi\|_{L^1} + \|\phi\|_{L^\infty}). \quad (2.115)$$

First, we estimate the operator  $\mathcal{J}_1$ . Making the change of variable  $p = qt^{-1/\alpha}$ , we get for the function  $J_1$

$$|J(r, t)| \leq Ct^{-1/\alpha}. \quad (2.116)$$

Now, we make the change of variable  $z = t^{-1/\alpha}r$ :

$$J_1(r, t) = \frac{t^{-1/\alpha}}{2\pi i} \int_{-i\infty}^{i\infty} e^{qz-K(q)} dq. \quad (2.117)$$

Integrating by parts the last equation we obtain

$$J_1(r, t) = \frac{t^{-1/\alpha}}{2\pi i} \left(\frac{1}{z}\right) \int_{-i\infty}^{i\infty} e^{-K(q)} de^{qz} = \frac{t^{-1/\alpha}}{2\pi i} \left(\frac{\alpha}{z}\right) \int_{-i\infty}^{i\infty} \frac{K(q)}{q} e^{qz-K(q)} dq. \quad (2.118)$$

Then,

$$|J_1(r, t)| \leq Ct^{-1/\alpha} \frac{1}{|z|^{1+\gamma}} \int_{C_\pm} |q|^{\alpha-1-\gamma} e^{-C|q|^\alpha} |dq|, \quad (2.119)$$

for  $\pm r > 0$ , where  $C_{\pm}$  are defined as above. Thus, for  $\gamma < \alpha$ ,

$$|J_1(r, t)| \leq Ct^{-1/\alpha} \frac{1}{|z|^{1+\gamma}}. \quad (2.120)$$

Therefore, from the inequalities (2.116) and (2.120) we have

$$|J_1(r, t)| \leq C \frac{t^{-1/\alpha}}{1 + (t^{-1/\alpha}|r|)^{1+\gamma}}, \quad \gamma < \alpha. \quad (2.121)$$

We remember some well-known inequalities.

- (i) *Young's Inequality.* Let  $f \in \mathbf{L}^p(\mathbb{R})$  and  $g \in \mathbf{L}^q(\mathbb{R})$ , where  $1 \leq p, q \leq \infty$ ,  $1/p + 1/q \geq 1$ . Then, the convolution  $h(x) \equiv \int_{\mathbb{R}} f(x-y)g(y)dy$  belongs to  $\mathbf{L}^r(\mathbb{R})$ , where  $1/r = 1/p + 1/q - 1$  and Young's inequality

$$\|h\|_{\mathbf{L}^r} \leq \|f\|_{\mathbf{L}^p} \|g\|_{\mathbf{L}^q} \quad (2.122)$$

holds.

- (ii) *Minkowski's Inequality.* Let  $f, g \in \mathbf{L}^p$  and  $1 \leq p \leq \infty$ ; then

$$\|f + g\|_{\mathbf{L}^p} \leq \|f\|_{\mathbf{L}^p} + \|g\|_{\mathbf{L}^p}. \quad (2.123)$$

- (iii) *Interpolation Inequality.* Let  $f \in \mathbf{L}^p(\mathbb{R}) \cap \mathbf{L}^q(\mathbb{R})$  with  $1 \leq p \leq q \leq \infty$ ; then  $f \in \mathbf{L}^r(\mathbb{R})$  for any  $p \leq r \leq q$ , and the interpolation inequality holds:

$$\|f\|_{\mathbf{L}^r} \leq \|f\|_{\mathbf{L}^p}^{\alpha} \|f\|_{\mathbf{L}^q}^{1-\alpha}, \quad (2.124)$$

where  $1/r = \alpha/p + (1-\alpha)/q$  and  $0 \leq \alpha \leq 1$ .

- (iv) *Arithmetic-Geometric Mean Inequality.* If  $a$  and  $b$  are nonnegative, then

$$\sqrt{ab} \leq \frac{a+b}{2}. \quad (2.125)$$

Then, by (2.121) and Young's inequality (2.122), we obtain

$$\|\mathcal{J}_1(t)\phi\|_{\mathbf{L}^2} \leq \|J_1(\cdot, t)\|_{\mathbf{L}^1} \|\phi\|_{\mathbf{L}^2} \leq C \|\phi\|_{\mathbf{L}^2}, \quad (2.126)$$

since

$$\|J_1(\cdot, t)\|_{\mathbf{L}^1} \leq C \int_{-\infty}^{+\infty} \frac{t^{-1/\alpha}}{1 + (t^{-1/\alpha}|r|)^{1+\gamma}} dr = C \int_{-\infty}^{+\infty} \frac{1}{1 + |r|^{1+\gamma}} dr \leq C. \quad (2.127)$$

Finally, using the Interpolation Inequality (2.124) and the arithmetic-geometric mean inequality (2.125), we obtain

$$\|\phi\|_{L^2} \leq \|\phi\|_{L^1}^{1/2} \|\phi\|_{L^\infty}^{1/2} \leq \frac{1}{2} (\|\phi\|_{L^1} + \|\phi\|_{L^\infty}). \quad (2.128)$$

Therefore,

$$\|\mathcal{J}_1(t)\phi\|_{L^2} \leq C(\|\phi\|_{L^1} + \|\phi\|_{L^\infty}). \quad (2.129)$$

Now, we estimate the operator  $\mathcal{J}_2$ . First, by Cauchy Theorem we get for  $\operatorname{Re} z < 0$

$$Z(z, \xi, y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-qy}}{q-z} \left( \frac{1}{Y^+(q, \xi)} - \frac{1}{Y^-(q, \xi)} \right) dq. \quad (2.130)$$

By (2.41) we get

$$\frac{1}{Y^+} - \frac{1}{Y^-} = \frac{1}{Y^-} \left( \frac{K_1(p) + \xi}{K(p) + \xi} - 1 \right) = \frac{1}{Y^-} \left( \frac{K_1(p) - K(p)}{K(p) + \xi} \right). \quad (2.131)$$

Then, using (2.131) and the inequalities  $e^{-C|q|y} \leq Cy^{-\gamma}|q|^{-\gamma}$ , where  $\operatorname{Re} q > 0$  and  $\gamma > 0$ , and

$$\frac{1}{|K(q) + \xi|} \leq C \frac{1}{|q|^{\alpha(1-\gamma_1)} |\xi|^{\gamma_1}}, \quad 0 < \gamma_1 < 1, \quad (2.132)$$

we obtain

$$|Z(z, \xi, y)| \leq C \frac{y^{-\gamma}}{|\xi|^{\gamma_1}} \int_{C_3} \frac{1}{|q-z|} \frac{1}{|q|^{\gamma-\alpha\gamma_1}} |dq| \leq C \frac{y^{-\gamma}}{|\xi|^{\gamma_1} |z|^{\gamma-\alpha\gamma_1}}. \quad (2.133)$$

Then, using the inequalities (2.132) and  $\|e^{-C|p|x}\|_{L^2} \leq C|p|^{-1/2}$ , we obtain for  $J_2$

$$\|J_2(\cdot, y, t)\|_{L^2} \leq Cy^{-\gamma} \int_{-i\infty}^{i\infty} \frac{|d\xi|}{|\xi|^{2\gamma_1}} \int_{C_2} \frac{|dp|}{|p|^{1/2+\gamma+\alpha(1-2\gamma_1)}} \leq Cy^{-\gamma}. \quad (2.134)$$

Therefore,

$$\|\mathcal{J}_2(t)\phi\|_{L^2} \leq \int_0^{+\infty} \|J_2(\cdot, y, t)\|_{L^2} |\phi(y)| dy \leq C(\|\phi\|_{L^1} + \|\phi\|_{L^\infty}). \quad (2.135)$$

Thus, the last estimate and (2.129) imply the third estimate in Lemma 2.7.

Now, we are going to prove the fourth estimate in Lemma 2.7. We use (2.114). First, we estimate the operator  $\mathcal{Q}_1$ , defined in (2.112). Using the inequality  $x^\mu \leq |x - y|^\mu + y^\mu$ , where  $0 \leq \mu \leq 1$ , and Minkowski's inequality (2.123), we obtain

$$\begin{aligned} \|\mathcal{Q}_1(t)\phi\|_{\mathbf{L}^{s,\mu}} &\leq \left( \int_0^{+\infty} \left( \int_0^{+\infty} |x-y|^\mu |J_1(x-y,t)| |\phi(y)| dy \right)^s dx \right)^{1/s} \\ &\quad + \left( \int_0^{+\infty} \left( \int_0^{+\infty} y^\mu |J_1(x-y,t)| |\phi(y)| dy \right)^s dx \right)^{1/s}. \end{aligned} \quad (2.136)$$

Then, Young's inequality (2.122) implies

$$\|\mathcal{Q}_1(t)\phi\|_{\mathbf{L}^{s,\mu}} \leq \|J_1(\cdot,t)\|_{\mathbf{L}^{p,\mu}} \|\phi\|_{\mathbf{L}^r} + \|J_1(\cdot,t)\|_{\mathbf{L}^p} \|\phi\|_{\mathbf{L}^{r,\mu}}, \quad (2.137)$$

where  $1/s = 1/p + 1/r - 1$ ,  $1 \leq p, r \leq \infty$ ,  $1 \leq 1/p + 1/r \leq 2$ , and  $0 \leq \mu \leq 1$ . Then, by the inequality (2.121) and the change of variables  $x = t^{-1/\alpha}|r|$ , we get

$$\|J_1(\cdot,t)\|_{\mathbf{L}^{p,\mu}} \leq Ct^{-(1/\alpha)(1-1/p-\mu)} \left( \int_{-\infty}^{+\infty} \left( \frac{|x|^\mu}{(1+|x|)^{1+\gamma}} \right)^p dx \right)^{1/p}. \quad (2.138)$$

Thus,  $\|J_1(\cdot,t)\|_{\mathbf{L}^{p,\mu}} \leq Ct^{-(1/\alpha)(1-1/p-\mu)}$ , provided  $1 + \gamma - \mu > 1/p$ . Using  $1/s = 1/p + 1/r - 1$ , it follows that

$$\|J_1(\cdot,t)\|_{\mathbf{L}^{p,\mu}} \leq Ct^{-(1/\alpha)(1/r-1/s-\mu)}, \quad (2.139)$$

where  $1/r - 1/s - \mu + \gamma > 0$ . We note that  $-(1/\alpha)((1/r) - (1/s) - \mu) < 1$ , since  $\gamma < \alpha$ . Substituting (2.139) in (2.137), we get

$$\|\mathcal{Q}_1(t)\phi\|_{\mathbf{L}^{s,\mu}} \leq Ct^{-(1/\alpha)(1/r-1/s-\mu)} \|\phi\|_{\mathbf{L}^r} + Ct^{-(1/\alpha)(1/r-1/s)} \|\phi\|_{\mathbf{L}^{r,\mu}}, \quad (2.140)$$

where  $-(1/\alpha)((1/r) - (1/s) - \mu) < 1$ ,  $1 \leq s, r \leq \infty$ , and  $0 \leq \mu \leq 1$ . Now, we estimate the operator  $\mathcal{Q}_2$ , defined in (2.113). First, we use that function  $J_2$  satisfies the following inequality:

$$|J_2(x,y,t)| \leq C \int_{c_1} |d\xi| e^{-C|\xi|t} \int_{c_2} |dp| e^{-C|p|x} \frac{1}{|K(p) + \xi|} \int_{c_3} \frac{1}{|q-p|} e^{-C|q|y} |dq|. \quad (2.141)$$

Then, by the inequality  $|\mathcal{Q}_2(t)\phi| \leq \int_0^{+\infty} |J_2(x,y,t)| |\phi(y)| dy$ , we obtain

$$|\mathcal{Q}_2(t)\phi| \leq C \int_{c_1} |d\xi| e^{-C|\xi|t} \int_{c_2} |dp| \frac{e^{-C|p|x}}{|K(p) + \xi|} \int_{c_3} \frac{|dq|}{|q-p|} \int_0^{+\infty} e^{-C|q|y} |\phi(y)| dy. \quad (2.142)$$

Substituting in the last inequality the estimate

$$\int_0^{+\infty} e^{-C|q|y} |\phi(y)| dy \leq \|e^{-C|q|y}\|_{L^l} \|\phi\|_{L^r} = C|q|^{-1/l} \|\phi\|_{L^r}, \quad (2.143)$$

where  $1/l + 1/r = 1$ , we obtain

$$|\mathcal{J}_2(t)\phi| \leq C\|\phi\|_{L^r} \int_{c_1} |d\xi| e^{-C|\xi|t} \int_{c_2} e^{-C|p|x} \frac{1}{|K(p) + \xi|} \frac{1}{|p|^{1/l}} |dp|. \quad (2.144)$$

Then, using  $\|e^{-C|p|x}\|_{L^{s,\mu}} = C|p|^{-1/s-\mu}$  and  $1/l + 1/r = 1$ , we get

$$\|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} \leq C\|\phi\|_{L^r} \int_{c_1} |d\xi| e^{-C|\xi|t} \int_{c_2} \frac{1}{|K(p) + \xi|} \frac{1}{|p|^{1-1/r+1/s+\mu}} |dp|. \quad (2.145)$$

Therefore,

$$\|\mathcal{J}_2(t)\phi\|_{L^{s,\mu}} \leq Ct^{-(1/\alpha)(1/r-1/s-\mu)} \|\phi\|_{L^r}, \quad (2.146)$$

where  $0 < 1/r - 1/s - \mu < \alpha$ ,  $1 \leq r < s \leq \infty$ , and  $0 \leq \mu < 1$ . Finally, from estimates (2.140) and (2.146) we obtain the fourth estimate in Lemma 2.7. Then, we have proved Lemma 2.7.  $\square$

**Theorem 2.8.** *Let the initial data be  $u_0 \in \mathbf{Z}$ , with  $\mu \in (0, 1)$ . Then, for some  $T > 0$  there exists a unique solution*

$$u \in \mathbf{C}([0, T]; L^2(\mathbf{R}^+)) \cap \mathbf{C}((0, T]; L^s(\mathbf{R}^+) \cap L^{s,\mu}(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)), \quad s > 1, \quad (2.147)$$

to the initial boundary-value problem (1.1). Moreover, the existence time  $T$  can be chosen as follows:  $T = C\|u_0\|_{\mathbf{Z}}^{-\sigma/\kappa}$ , where  $\kappa \in (0, 1)$ .

### 3. Proof of Theorem 1.1

By the Local Existence Theorem 2.8, it follows that the global solution (if it exist) is unique. Indeed, on the contrary, we suppose that there exist two global solutions with the same initial data. And these solutions are different at some time  $t > 0$ . By virtue of the continuity of solutions with respect to time, we can find a maximal time segment  $[0, T]$ , where the solutions are equal, but for  $t > T$  they are different. Now, we apply the local existence theorem taking the initial time  $T$  and obtain that these solutions coincide on some interval  $[T, T_1]$ , which give us a contradiction with the fact that  $T$  is the maximal time of coincidence. So our main purpose in the proof of Theorem 1.1 is to show the global in time existence of solutions.



First, we note that Lemma 2.7 implies for the Green operator  $\mathcal{G} : \mathbf{Z} \rightarrow \mathbf{X}$  the inequality  $\|\mathcal{G}(t)u_0\|_{\mathbf{X}} \leq C\|u_0\|_{\mathbf{Z}}$ . Now, we show the estimate

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)))d\tau \right\|_{\mathbf{X}} \leq C\|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma, \quad (3.1)$$

for all  $u, v \in \mathbf{X}$ , where  $\mathcal{N}(u) = |u|^\sigma u$ ,  $\sigma > 1$ . In fact, using the inequality

$$\left| |u|^\sigma u - |v|^\sigma v \right| \leq C|u-v|(|u|^\sigma + |v|^\sigma), \quad (3.2)$$

we get

$$\begin{aligned} 1\|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathbf{L}^{1,\mu}} &\leq C\|u-v\|_{\mathbf{L}^\infty}(\|u\|_{\mathbf{L}^{\sigma,\mu/\sigma}}^\sigma + \|v\|_{\mathbf{L}^{\sigma,\mu/\sigma}}^\sigma) \\ &\leq C\{\tau\}^{-(\gamma/\alpha)(\sigma+1)}\langle\tau\rangle^{-(1/\alpha)(\gamma\sigma+1-\mu)}\|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma, \end{aligned} \quad (3.3)$$

where  $0 \leq \mu < 1$ , and

$$\begin{aligned} \|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathbf{L}^\infty} &\leq C\|u-v\|_{\mathbf{L}^\infty}(\|u\|_{\mathbf{L}^\infty}^\sigma + \|v\|_{\mathbf{L}^\infty}^\sigma) \\ &\leq C\{\tau\}^{-(\gamma/\alpha)(\sigma+1)}\langle\tau\rangle^{-(1/\alpha)(\sigma+1)}\|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma \end{aligned} \quad (3.4)$$

Then, the estimates (3.3), (3.4), and Lemma 2.7 imply

$$\begin{aligned} &\|\mathcal{G}(t-\tau)(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^2} \\ &\leq C\left(\{\tau\}^{-(\gamma/\alpha)(\sigma+1)}\langle\tau\rangle^{-(1/\alpha)(\gamma\sigma+1)} + \{\tau\}^{-(\gamma/\alpha)(\sigma+1)}\langle\tau\rangle^{-(1/\alpha)(\sigma+1)}\right) \times \|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma, \\ &\|\mathcal{G}(t-\tau)(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^{s,\mu}} \\ &\leq C\left((t-\tau)^{-(1/\alpha)(\gamma-\mu)}\{\tau\}^{-(\gamma/\alpha)(\sigma+1)}\langle\tau\rangle^{-(1/\alpha)(\gamma\sigma+1)}\right. \\ &\quad \left.+ (t-\tau)^{-\gamma/\alpha}\{\tau\}^{-(\gamma/\alpha)(\sigma+1)}\langle\tau\rangle^{-(1/\alpha)(\gamma\sigma+1-\mu)}\right) \|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma, \end{aligned} \quad (3.5)$$

where  $0 \leq \mu < 1$ , and

$$\begin{aligned} &\|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau) - \mathcal{N}(v)(\tau))\|_{\mathbf{L}^\infty} \\ &\leq C\left(\{t-\tau\}^{-\gamma/\alpha}\langle t-\tau\rangle^{-1/\alpha}\{\tau\}^{-\gamma/\alpha(\sigma+1)}\langle\tau\rangle^{-1/\alpha(\gamma\sigma+1)}\right. \\ &\quad \left.+ \{t-\tau\}^{-\gamma/\alpha}\langle t-\tau\rangle^{-1/\alpha}\{\tau\}^{-\gamma/\alpha(\sigma+1)}\langle\tau\rangle^{-1/\alpha(\sigma+1)}\right) \\ &\quad \times \|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma. \end{aligned} \quad (3.6)$$

Now, we integrate with respect to  $\tau$ , on the interval  $[0, t]$ , the inequalities (3.5) and (3.6). Then, we get for  $\gamma < \alpha/(\sigma + 1)$ ,

$$\begin{aligned} & \int_0^t \|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau) - \mathcal{N}(v)(\tau))\|_{L^2} \\ & \leq C \left( \{t\}^{1-(\gamma/\alpha)(\sigma+1)} \langle t \rangle^{1-1/\alpha-(\gamma/\alpha)(2\sigma+1)} + \{t\}^{1-(\gamma/\alpha)(\sigma+1)} \langle t \rangle^{1-(1/\alpha)(\sigma+1)(\gamma+1)} \right) \\ & \quad \times \|u - v\|_{\mathbf{X}} (\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma \leq C \|u - v\|_{\mathbf{X}} (\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma, \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \int_0^t \|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau) - \mathcal{N}(v)(\tau))\|_{L^{s,\mu}} \\ & \leq C \{t\}^{-\gamma/\alpha} \langle t \rangle^{-(1/\alpha)(\gamma-\mu)} \left( \{t\}^{1-(\gamma/\alpha)(\sigma+1)} \langle t \rangle^{1-(1/\alpha)(2\gamma\sigma+1)} \right. \\ & \quad \left. + \{t\}^{1-(\gamma/\alpha)(\sigma+1)+\mu/\alpha} \langle t \rangle^{1-(1/\alpha)(2\gamma\sigma+1)-\mu/\alpha} \right) \|u - v\|_{\mathbf{X}} (\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma \\ & \leq C \{t\}^{-\gamma/\alpha} \langle t \rangle^{-(1/\alpha)(\gamma-\mu)} \|u - v\|_{\mathbf{X}} (\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma, \end{aligned} \quad (3.8)$$

where  $0 \leq \mu < 1$ , and

$$\begin{aligned} & \int_0^t \|\mathcal{G}(t-\tau)(\mathcal{N}(u)(\tau) - \mathcal{N}(v)(\tau))\|_{L^\infty} \\ & \leq C \{t\}^{-\gamma/\alpha} \langle t \rangle^{-1/\alpha} \left( \{t\}^{1-\gamma/\alpha(\sigma+1)} \langle t \rangle^{1-1/\alpha(\gamma\sigma+1)} + \{t\}^{1-\gamma/\alpha(\sigma+1)} \langle t \rangle^{1-1/\alpha(\sigma+1)} \right) \\ & \quad \times \|u - v\|_{\mathbf{X}} (\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma \\ & \leq C \{t\}^{-\gamma/\alpha} \langle t \rangle^{-1/\alpha} \|u - v\|_{\mathbf{X}} (\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma. \end{aligned} \quad (3.9)$$

Then, the definition of the norm on the space  $\mathbf{X}$  and the estimates (3.7), (3.8), and (3.9) imply (3.1). Now, we apply the Contraction Mapping Principle on a ball with ratio  $\rho > 0$  in the space  $\mathbf{X}$ ,  $\mathbf{X}_\rho = \{\phi \in \mathbf{X} : \|\phi\|_{\mathbf{X}} \leq \rho\}$ , where  $\rho = 2C\|u_0\|_{\mathbf{Z}}$ . Here, the constant  $C$  coincides with the one that appears in estimate (3.1). First, we show that

$$\|\mathcal{M}(u)\|_{\mathbf{X}} \leq \rho, \quad (3.10)$$

where  $u \in \mathbf{X}_\rho$ . Indeed, from the integral formula

$$\mathcal{M}(u) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau \quad (3.11)$$

and the estimate (3.1) (with  $v \equiv 0$ ), we obtain

$$\begin{aligned} \|\mathcal{M}(u)\|_{\mathbf{X}} &\leq \|G(t)u_0\|_{\mathbf{X}} + \left\| \int_0^t G(t-\tau)\mathcal{N}(u)(\tau)d\tau \right\|_{\mathbf{X}} \\ &\leq C\|u_0\|_{\mathbf{Z}} + C\|u\|_{\mathbf{X}}^{\sigma+1} \leq \frac{\rho}{2} + C\rho^{\sigma+1} \leq \rho, \end{aligned} \quad (3.12)$$

since  $\rho > 0$  is sufficient small. Therefore, the operator  $\mathcal{M}$  transforms a ball of ratio  $\rho > 0$  into itself, in the space  $\mathbf{X}$ . In the same way we estimate the difference of two functions  $u, v \in \mathbf{X}_\rho$ :

$$\begin{aligned} \|\mathcal{M}(u) - \mathcal{M}(v)\|_{\mathbf{X}} &\leq \left\| \int_0^t G(t-\tau)(\mathcal{N}(u)(\tau) - \mathcal{N}(v)(\tau))d\tau \right\|_{\mathbf{X}} \\ &\leq C\|u - v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma \leq C(2\rho)^\sigma \leq \frac{1}{2}\|u - v\|_{\mathbf{X}}, \end{aligned} \quad (3.13)$$

since  $\rho > 0$  is sufficient small. Thus,  $\mathcal{M}$  is a contraction mapping in  $\mathbf{X}_\rho$ . Therefore, there exists a unique solution  $u \in \mathbf{X}$  to the Cauchy problem (1.1). Now we can prove asymptotic formula:

$$u(x, t) = At^{-1/\alpha} \Lambda(xt^{-1/\alpha}) + O(t^{-(1/\alpha)(1+\kappa)}), \quad \kappa \in (0, \mu), \quad (3.14)$$

where  $A = f(u_0) - \int_0^{+\infty} f(\mathcal{N}(u(\tau)))d\tau$ . We denote  $G_0(t) = t^{-1/\alpha} \Lambda(xt^{-1/\alpha})$ . From Lemma 2.7 we have

$$\|G(t)\phi - G_0(t)f(\phi)\|_{\mathbf{L}^\infty} \leq C\langle t \rangle^{-(1/\alpha)(1+\mu)} \|\phi\|_{\mathbf{Z}} \quad (3.15)$$

for all  $t > 1$ . Also in view of the definition of the norm  $\mathbf{X}$  we have

$$\begin{aligned} |f(\mathcal{N}(u(\tau)))| &\leq \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1} \leq \|u(\tau)\|_{\mathbf{L}^\infty}^{\sigma-1} \|u(\tau)\|_{\mathbf{L}^2}^2 \\ &\leq C\{\tau\}^{-(\gamma/\alpha)(\sigma-1)} \langle \tau \rangle^{-(1/\alpha)(\sigma-1)} \|u\|_{\mathbf{X}}^{\sigma+1}. \end{aligned} \quad (3.16)$$

By a direct calculation we have for  $t > 1$

$$\begin{aligned} &\left\| \int_0^{t/2} (G_0(t-\tau) - G_0(t))f(\mathcal{N}(u(\tau)))d\tau \right\|_{\mathbf{L}^\infty} \\ &\leq C\|u\|_{\mathbf{X}}^{\sigma+1} \int_0^{t/2} \|G_0(t-\tau) + G_0(t)\|_{\mathbf{L}^\infty} \{\tau\}^{-(\gamma/\alpha)(\sigma-1)} \langle \tau \rangle^{-(1/\alpha)(\sigma-1)} d\tau \\ &\leq C\langle t \rangle^{-1/\alpha} \|u\|_{\mathbf{X}}^{\sigma+1} \int_0^{t/2} \{\tau\}^{-(\gamma/\alpha)(\sigma-1)} \langle \tau \rangle^{-(1/\alpha)(\sigma-1)} d\tau \leq C\langle t \rangle^{-(1/\alpha)(1+\kappa)} \|u\|_{\mathbf{X}}^{\sigma+1}, \end{aligned} \quad (3.17)$$

where  $\kappa = \sigma - 1 - \alpha$ , provided that  $\sigma < 1 + \alpha/\gamma$ , and in the same way

$$\left\| G_0(t) \int_{t/2}^{\infty} f(\mathcal{N}(u(\tau))) d\tau \right\|_{L^\infty} \leq C \langle t \rangle^{-(1/\alpha)(1+\kappa)} \|u\|_X^{\sigma+1}, \quad (3.18)$$

provided that  $\sigma > 1 + \alpha$ . Also we have

$$\begin{aligned} & \left\| \int_0^{t/2} (\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G_0(t-\tau)f(\mathcal{N}(u(\tau)))) d\tau \right\|_{L^\infty} + \left\| \int_{t/2}^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) d\tau \right\|_{L^\infty} \\ & \leq C \int_0^{t/2} (t-\tau)^{-1/\alpha} \|\mathcal{N}(u(\tau))\|_{L^1} d\tau \\ & \quad + C \int_{t/2}^t \{t-\tau\}^{-\gamma/\alpha} \langle t-\tau \rangle^{-1/\alpha} (\|\mathcal{N}(u(\tau))\|_{L^1} + \|\mathcal{N}(u(\tau))\|_{L^\infty}) d\tau \\ & \leq C \langle t \rangle^{-(1/\alpha)(1+\kappa)} \|u\|_X^{\sigma+1} \end{aligned} \quad (3.19)$$

for all  $t > 1$ . By virtue of the integral equation (3.11) we get

$$\begin{aligned} & \langle t \rangle^{(1/\alpha)(1+\kappa)} \|u(t) - AG_0(t)\|_{L^\infty} \\ & \leq \langle t \rangle^{(1/\alpha)(1+\kappa)} \|\mathcal{G}(t)u_0 - G_0(t)f(u_0)\|_{L^\infty} \\ & \quad + \langle t \rangle^{(1/\alpha)(1+\kappa)} \left\| \int_0^{t/2} (\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G_0(t-\tau)f(\mathcal{N}(u(\tau)))) d\tau \right\|_{L^\infty} \\ & \quad + \langle t \rangle^{(1/\alpha)(1+\kappa)} \left\| \int_{t/2}^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) d\tau \right\|_{L^\infty} \\ & \quad + \langle t \rangle^{(1/\alpha)(1+\kappa)} \left\| G_0(t) \int_{t/2}^{\infty} f(\mathcal{N}(u(\tau))) d\tau \right\|_{L^\infty} \\ & \quad + \langle t \rangle^{(1/\alpha)(1+\kappa)} \left\| \int_0^{t/2} (G_0(t-\tau) - G_0(t))f(\mathcal{N}(u(\tau))) d\tau \right\|_{L^\infty}. \end{aligned} \quad (3.20)$$

All summands in the right-hand side of (3.20) are estimated by  $C\|u_0\|_Z + C\|u\|_X^{\sigma+1}$  via estimates (3.17)–(3.19). Thus by (3.20) the asymptotic (3.14) is valid. Theorem 1.1 is proved.

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