NONSMOOTH AND NONLOCAL DIFFERENTIAL EQUATIONS IN LATTICE-ORDERED BANACH SPACES

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We derive existence results for initial and boundary value problems in lattice-ordered Banach spaces. The considered problems can be singular, functional, discontinuous, and nonlocal. Concrete examples are also solved.

1. Introduction

In this paper, we apply fixed point results for mappings in partially ordered function spaces to derive existence results for initial and boundary value problems in an ordered Banach space E. Throughout this paper, we assume that E satisfies one of the following hypotheses.

(A) E is a Banach lattice whose every norm-bounded and increasing sequence is strongly convergent.

(B) E is a reflexive lattice-ordered Banach space whose lattice operation $E \ni x \mapsto x^+ = \sup\{0,x\}$ is continuous and $\|x^+\| \le \|x\|$ for all $x \in E$.

We note that condition (A) is equivalent with E being a weakly complete Banach lattice, see, for example, [11].

The problems that will be considered in this paper include many kinds of special types, such as, for example, the following:

(1) the differential equations may be singular;
(2) both the differential equations and the initial or boundary conditions may depend functionally on the unknown function;
(3) both the differential equations and the initial or boundary conditions may contain discontinuous nonlinearities;
(5) problems on unbounded intervals;
(6) finite and infinite systems of initial and boundary value problems;
(7) problems of random type.

The plan of the paper is as follows. In Section 2, we provide the basic abstract fixed point result which will be used in later sections. In Section 3, we deal with first-order initial value problems, and in Sections 4 and 5, second-order initial and boundary value problems are considered. Concrete examples are solved to demonstrate the applicability of the obtained results.

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2. Preliminaries

We will start with the following auxiliary result.

Lemma 2.1. *Let $J = (a,b) \subset \mathbb{R}$ be some interval. Given a function $w : J \to \mathbb{R}_+$, denote*

$$P = \{u \in C(J,E) \mid \|u(t)\| \le w(t) \ \text{for each } t \in J\}, \tag{2.1}$$

and assume that $C(J,E)$ is ordered pointwise. Then the following results hold.

(a) *The zero function 0 is an order center of P in the sense that* $\sup\{0,v\}$ *and* $\inf\{0,v\}$ *belong to P for each $v \in P$.*

(b) *If U is an equicontinuous subset of P, then U is relatively well-order complete in P in the sense that all well-ordered and inversely well-ordered chains of U have supremums and infimums in P.*

Proof. (a) In both cases (A) and (B), the mapping $x \mapsto x^+$ is continuous in E and $\|x^+\| \le \|x\|$ for each $x \in E$. Thus, for each $v \in C(J,E)$, the mapping $v^+ = \sup\{0,v\} = t \mapsto \sup\{0, v(t)\}$ belongs to $C(J,E)$, and $\|v^+(t)\| \le \|v(t)\|$ for all $t \in J$. These properties ensure that $v^+ = \sup\{0,v\}$, $v^- = \sup\{0,-v\}$, and $\inf\{0,v\} = -v^-$ belong to P for each $v \in P$.

(b) Assume next that U is an equicontinuous subset of P. If E is reflexive, then bounded and monotone sequences converge weakly in E. Consequently, if W is a well-ordered chain in U, then all its monotone sequences converge pointwise in E strongly in case (A) and weakly in case (B). Because W is also equicontinuous, it follows from [8, Proposition 4.3 and Remarks 4.1] that $u = \sup W$ exists in $C(J,E)$, and there is an increasing sequence (u_n) in W which converges pointwise strongly in case (A) and weakly in the case (B) to u. Moreover, in both cases,

$$\|u(t)\| \le \liminf_{n\to\infty} \|u_n(t)\| \le w(t), \quad t \in J, \tag{2.2}$$

so that $u = \sup W \in P$ by the definition (2.1) of P.

If W is an inversely well-ordered chain in U, then $-W$ is a well-ordered chain in $-U$. The above proof ensures that $v = \sup(-W)$ exists in $C(J,E)$ and belongs to P. Thus, $\inf W = -v$ exists and belongs to P. Noticing also that each well-ordered chain has a minimum and each inversely well-ordered chain has a maximum, the proof of (b) is complete. □

Let P be a nonempty subset of $C(J,E)$. We say that a mapping $G : P \to P$ is *increasing* if $Gu \le Gv$ whenever $u,v \in P$ and $u \le v$. Given a subset U of P, we say that $u \in U$ is the *least fixed point* of G in U if $u = Gu$, and if $u \le v$ whenever $v \in U$ and $v = Gv$. The greatest fixed point of G in U is defined similarly, by reversing the inequality. A fixed point u of G is called *minimal*, if $v \in P$, $v = Gv$, and $v \le u$ imply $v = u$, and *maximal*, if $v \in P$, $v = Gv$, and $u \le v$ imply $v = u$.

Our main existence results in later sections are based on the following fixed point lemma.

Lemma 2.2. *Let P be given by (2.1), and let $G : P \to P$ be an increasing mapping whose range $G[P]$ is equicontinuous. Then G has*

(a) *minimal and maximal fixed points;*

(b) *least and greatest fixed points u_* and u^* in $\{u \in P \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of the equation*

$$u(t) = -\big(Gu(t)\big)^-, \quad t \in J, \tag{2.3}$$

and $\overline{u}$ is the least solution of the equation

$$u(t) = \big(Gu(t)\big)^+, \quad t \in J. \tag{2.4}$$

Moreover, u_* and u^* are increasing with respect to G.

Proof. The hypotheses imply by Lemma 2.1 that P has an order center, and that $G[P]$ is relatively well-order complete in P. Thus the assertions follow from [9, Proposition 2.3], whose proof is based on a recursion method and generalized iteration methods introduced in [10]. For instance, $\overline{u}$ and u^* can be obtained as follows. The union C of those well-ordered subsets A of P whose elements satisfy $u = \sup\{(Gv)^+ \mid v \in A,\ v < u\}$ is well-ordered and $\overline{u} = \max C$. The union D of those inversely well-ordered subsets B of P whose elements are of the form $u = \inf\{\overline{u}, \{Gv \mid v \in B,\ u < v\}\}$ is inversely well-ordered, and $u^* = \min D$. By dual reasoning, one obtains $\underline{u}$ and u_*. □

Remark 2.3. In the case when the sets C and D in the above proof are finite, the fixed point u^* of G is the last member of the finite sequence $D \cup C$, which can be determined by the following.

Algorithm 2.4. $u_0 \equiv 0$*: For n from 0 while $u_n \ne Gu_n$ do: If $u_n < (Gu_n)^+$, then $u_{n+1} = (Gu_n)^+$ else $u_{n+1} = Gu_n$.*

3. Existence results for first-order initial value problems

In this section, we study initial value problems which can be represented in the form

$$\begin{gathered} \frac{d}{dt}\big(p(t)u(t)\big) = f(t,u) \quad \text{for almost every (a.e.) } t \in J := (a,b), \\ \lim_{t\to a+} p(t)u(t) = c(u), \end{gathered} \tag{3.1}$$

where $-\infty \le a < b \le \infty$, $p \in C(J)$, $f : J \times C(J,E) \to E$, and $c : C(J,E) \to E$.

We are looking for solutions of (3.1) from the set

$$X := \{u \in C(J,E) \mid pu \text{ is locally absolutely continuous and a.e. differentiable}\}. \tag{3.2}$$

We will first convert the IVP (3.1) to an integral equation.

Lemma 3.1. *Assume that $p(t) > 0$ on J, and that $u \in X$ and $f(\cdot,u) \in L^1(J,E)$. Then u is a solution of the IVP (3.1) if and only if u satisfies the integral equation*

$$u(t) = \frac{1}{p(t)}\left(c(u) + \int_a^t f(x,u)\,dx\right), \quad t \in J. \tag{3.3}$$

Proof. Assume that u is a solution of (3.1). The differential equation of (3.1) and the definition (3.2) of X imply that

$$\int_r^s \frac{d}{dt}(p(t)u(t))\,dt = p(s)u(s) - p(r)u(r) = \int_r^s f(t,u)\,dt, \quad a < r \le s < b. \tag{3.4}$$

In view of this result and the initial condition of (3.1), we obtain (3.3).

The converse part of the proof is trivial. □

Now we are ready to prove our main existence result for the IVP (3.1). Assuming that $L^1(J,E)$ is ordered a.e. pointwise, and that $C(J,E)$ is ordered pointwise, we impose the following hypotheses on the functions p, f, and c:

(p) $p(t) > 0$ for all $t \in J$,

(f0) $f(\cdot,u)$ is strongly measurable, and $\|f(\cdot,u)\| \le h_0 \in L^1(J)$ for all $u \in C(J,E)$,

(f1) $f(\cdot,u) \le f(\cdot,v)$ whenever $u,v \in C(J,E)$ and $u \le v$,

(c) c is bounded, and $c(u) \le c(v)$ whenever $u,v \in C(J,E)$ and $u \le v$.

Theorem 3.2. *Assume that the hypotheses (p), (f0), (f1), and (c) hold, and assume that the space X defined by (3.2) is ordered pointwise. Then the IVP (3.1) has*

(a) *minimal and maximal solutions in X;*

(b) *least and greatest solutions u_* and u^* in $\{u \in X \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of equation*

$$u(t) = -\frac{1}{p(t)}\left(c(u) + \int_a^t f(x,u)\,dx\right)^-, \quad t \in J, \tag{3.5}$$

and $\overline{u}$ is the least solution of equation

$$u(t) = \frac{1}{p(t)}\left(c(u) + \int_a^t f(x,u)\,dx\right)^+, \quad t \in J. \tag{3.6}$$

Moreover, u_* and u^* are increasing with respect to c and f.

Proof. Let P be defined by (2.1) with w given by

$$w(t) := \frac{1}{p(t)}\left(c_0 + \int_a^t h_0(x)\,dx\right), \quad t \in J, \tag{3.7}$$

where $c_0 = \sup\{\|c(u)\| \mid u \in C(J,E)\}$, and the function $h_0 \in L^1(J)$ is as in the hypothesis (f0). The given hypotheses imply that the relation

$$Gu(t) = \frac{1}{p(t)}\left(c(u) + \int_a^t f(x,u)\,dx\right), \quad t \in J, \tag{3.8}$$

defines an increasing mapping $G : P \to P$, and that $G[P]$ is an equicontinuous subset of P. Thus G satisfies the hypotheses of Lemma 2.2. Moreover, it is easy to verify that each solution of (3.1) in X belongs to P, and that Gu increases if $c(u)$ or $f(\cdot,u)$ increases. Thus the assertions follow from Lemmas 3.1 and 2.2. □

As a special case, we obtain an existence result for the IVP

$$\begin{gathered}\frac{d}{dt}(p(t)u(t)) = g(t,u(t)) \quad \text{for a.e. } t \in J, \\ \lim_{t\to a+} p(t)u(t) = c.\end{gathered} \tag{3.9}$$

Corollary 3.3. *Let the hypothesis (p) hold, and let $g : J \times E \to E$ satisfy the following hypotheses:*

(g0) *$g(\cdot,u(\cdot))$ is strongly measurable and $\|g(\cdot,u(\cdot))\| \le h_0 \in L^1(J)$ for all $u \in C(J,E)$,*
(g1) *$g(t,x) \le g(t,y)$ for a.e. $t \in J$ and whenever $x \le y$ in E.*

Then the IVP (3.9) has, for each choice of $c \in E$,

(a) *minimal and maximal solutions in X;*
(b) *least and greatest solutions u_* and u^* in $\{u \in X \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of equation*

$$u(t) = -\frac{1}{p(t)}\left(c + \int_a^t g(x,u(x))\,dx\right)^-, \quad t \in J, \tag{3.10}$$

and $\overline{u}$ is the least solution of equation

$$u(t) = \frac{1}{p(t)}\left(c + \int_a^t g(x,u(x))\,dx\right)^+, \quad t \in J. \tag{3.11}$$

Moreover, u_* and u^* are increasing with respect to c and g.

Proof. If $c \in E$, the IVP (3.9) is reduced to (3.1) when we define

$$\begin{gathered}f(t,u) = g(t,u(t)), \quad t \in J,\ u \in C(J,E), \\ c(u) \equiv c, \quad u \in C(J,E).\end{gathered} \tag{3.12}$$

The hypotheses (g0) and (g1) imply that f satisfies the hypotheses (f0) and (f1). The hypothesis (c) is also valid, whence the asserted results follow from Theorem 3.2. □

Example 3.4. Consider the following system of IVPs:

$$\begin{aligned}\frac{d}{dt}(\sqrt{t}u(t)) &= \frac{t}{4} + \frac{[\int_1^2 v(s)ds]}{(1+|[\int_1^2 v(s)ds]|)} \quad \text{a.e. in } (0,\infty), \\ \frac{d}{dt}(\sqrt{t}v(t)) &= \sqrt{t} + 2\frac{[\int_1^2 u(s)ds]}{(1+|[\int_1^2 u(s)ds]|)}, \quad \text{a.e. in } (0,\infty), \\ \lim_{t\to 0+}\sqrt{t}u(t) &= \frac{[v(1)]}{1+|[v(1)]|}, \qquad \lim_{t\to 0+}\sqrt{t}v(t) = 2\frac{[u(1)]}{1+|[u(1)]|},\end{aligned} \tag{3.13}$$

where $[z]$ denotes the greatest integer $\le z$.

Solution 3.5. The system (3.13) is a special case of (3.1) when $E = \mathbb{R}^2$, ordered coordinatewise, $a = 0$, $b = \infty$, $p(t) = \sqrt{t}$,

$$f(t,(u,v)) = \left(\frac{t}{4} + \frac{[\int_1^2 v(s)ds]}{1 + |[\int_1^2 v(s)ds]|}, \sqrt{t} + 2\frac{[\int_1^2 u(s)ds]}{1 + |[\int_1^2 u(s)ds]|}\right),$$
$$c((u,v)) = \left(\frac{[v(1)]}{1 + |[v(1)]|}, 2\frac{[u(1)]}{1 + |[u(1)]|}\right). \tag{3.14}$$

The hypotheses (f0), (f1), and (c) are satisfied, with respect to 1-norm of $\mathbb{R}^2$, when $h_0(t) = t/4 + \sqrt{t} + 3$ and $c_0 = 3$. Thus the results of Theorem 3.2 can be applied. In this case, the chains needed in the proof of Theorem 3.2 (cf. the proof of Lemma 2.2) are reduced to finite ordinary iteration sequences. Thus one can apply algorithms of the form (2.4) presented in Remark 2.3 to calculate solutions to the system (3.13). Calculations, which are carried out by the use of a simple Maple program, show that the least and the greatest solutions of (3.13) between $\underline{u}$, which is the zero function, and $\overline{u}$ are equal to $\overline{u}$, and this solution (u^*, v^*) is the only solution of (3.13) between $\underline{u}$ and $\overline{u}$. Moreover, (3.13) has only one minimal solution, (u_-, v_-) and only one maximal solution (u^+, v^+), and thus they are the least and the greatest of all the solutions of (3.13). The exact expressions of these solutions are

$$(u^*(t), v^*(t)) = \left(\frac{1}{8}t\sqrt{t} + \frac{1}{2}\sqrt{t}, \frac{2}{3}t\right),$$
$$(u^+(t), v^+(t)) = \left(\frac{1}{8}t\sqrt{t} + \frac{3}{4}\sqrt{t} + \frac{2}{3\sqrt{t}}, \frac{2}{3}t + \sqrt{t} + \frac{1}{\sqrt{t}}\right), \tag{3.15}$$
$$(u_-(t), v_-(t)) = \left(\frac{1}{8}t\sqrt{t} - \frac{2}{3}\sqrt{t} - \frac{2}{3\sqrt{t}}, \frac{2}{3}t - \frac{4}{3}\sqrt{t} - \frac{4}{3\sqrt{t}}\right).$$

4. Existence results for second-order initial value problems

Next we will study initial value problems which can be represented in the form

$$\frac{d}{dt}(p(t)u'(t)) = f(t,u) \quad \text{for a.e. } t \in J := (a,b),$$
$$\lim_{t\to a+} p(t)u'(t) = c(u), \qquad \lim_{t\to a+} u(t) = d(u), \tag{4.1}$$

where $-\infty \le a < b \le \infty$, $p \in C(J)$, $f : J \times C(J,E) \to E$, and $c, d : C(J,E) \to E$.

Now we are looking for solutions from the set

$$Y := \{u \in C^1(J,E) \mid p \cdot u' \text{ is locally absolutely continuous and a.e. differentiable}\}. \tag{4.2}$$

The method is similar to that applied in Section 3, that is, we will first convert the IVP (4.1) to an integral equation, and then apply Lemma 2.2.

Lemma 4.1. *Assume that $p(t) > 0$ on J, and that $f(\cdot,u) \in L^1(J,E)$ for all $u \in C(J,E)$. Then $u \in Y$ is a solution of the IVP (4.1) if and only if u satisfies the integral equation*

$$u(t) = d(u) + \int_a^t \frac{1}{p(s)}\left(c(u) + \int_a^s f(x,u)dx\right)ds, \quad t \in J. \tag{4.3}$$

Proof. Assume that $u \in Y$ is a solution of (4.1). The differential equation of (4.1) and the definition (4.2) of Y ensure that

$$\int_r^s \frac{d}{dt}\big(p(t)u'(t)\big)dt = p(s)u'(s) - p(r)u'(r) = \int_r^s f(t,u)dt, \quad a < r \le s < b. \tag{4.4}$$

In view of this result and the first initial condition of (4.1), we obtain

$$u'(s) = \frac{1}{p(s)}\left(c(u) + \int_a^s f(x,u)dx\right), \quad s \in J. \tag{4.5}$$

Because the right-hand side of (4.5) is continuous in s, we can integrate it to obtain

$$u(t) - u(r) = \int_r^t \frac{1}{p(s)}\left(c(u) + \int_a^s f(x,u)dx\right)ds, \quad a < r \le t < b. \tag{4.6}$$

Applying the second initial condition of (4.1) to the above equation, we see that u satisfies the integral equation (4.3).

The converse part of the proof is trivial. □

To prove our main existence result for the IVP (4.1), we assume the following hypotheses for the functions p, f, c, and d:

(p0) $p(t) > 0$ and $\int_a^t ds/p(s) < \infty$ for all $t \in J$,

(f0) $f(\cdot,u)$ is strongly measurable, and $\|f(\cdot,u)\| \le h_0 \in L^1(J)$ for all $u \in C(J,E)$,

(f1) $f(\cdot,u) \le f(\cdot,v)$ whenever $u,v \in C(J,E)$ and $u \le v$,

(c) c is bounded, and $c(u) \le c(v)$ whenever $u,v \in C(J,E)$ and $u \le v$,

(d) d is bounded, and $d(u) \le d(v)$ whenever $u,v \in C(J,E)$ and $u \le v$.

Theorem 4.2. *Assume that the hypotheses (p0), (f0), (f1), (c), and (d) hold, and assume that the space Y defined by (4.2) is ordered pointwise. Then the IVP (4.1) has*

(a) *minimal and maximal solutions in Y;*

(b) *least and greatest solutions u_* and u^* in $\{u \in Y \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of equation*

$$u(t) = -\left(d(u) + \int_a^t \frac{1}{p(s)}\left(c(u) + \int_a^s f(x,u)dx\right)ds\right)^-, \quad t \in J, \tag{4.7}$$

and $\overline{u}$ is the least solution of equation

$$u(t) = \left(d(u) + \int_a^t \frac{1}{p(s)}\left(c(u) + \int_a^s f(x,u)dx\right)ds\right)^+, \quad t \in J. \tag{4.8}$$

Moreover, u_* and u^* are increasing with respect to c, d, and f.

Proof. Let P be defined by (2.1) with

$$w(t) := d_0 + \int_a^t \frac{1}{p(s)}\left(c_0 + \int_a^s h_0(x)dx\right)ds, \quad t \in J, \tag{4.9}$$

where $c_0 = \sup\{\|c(u)\| \mid u \in C(J,E)\}$, $d_0 = \sup\{\|d(u)\| \mid u \in C(J,E)\}$, and the function $h_0 \in L^1(J)$ is as in the hypothesis (f0). The given hypotheses imply that the relation

$$Gu(t) = d(u) + \int_a^t \frac{1}{p(s)}\left(c(u) + \int_a^s f(x,u)dx\right)ds, \quad t \in J, \tag{4.10}$$

defines an increasing mapping $G : P \to P$, and that

$$\|Gu(t) - Gu(\bar{t})\| \le \left(c_0 + \|h_0\|_1\right)\left|\int_t^{\bar{t}} \frac{ds}{p(s)}\right| \quad \forall u \in P,\ t,\bar{t} \in J. \tag{4.11}$$

The above inequality implies that $G[P]$ is an equicontinuous subset of P, whence G satisfies the hypotheses of Lemma 2.2. Moreover, it is easy to verify that each solution of (4.1) in Y belongs to P, and that Gu increases if $c(u)$, $d(u)$, or $f(\cdot,u)$ increases. Thus the assertions follow from Lemmas 4.1 and 2.2. □

As a special case, we obtain an existence result for the IVP

$$\begin{gathered}\frac{d}{dt}(p(t)u'(t)) = g(t,u(t)) \quad \text{for a.e. } t \in J,\\ \lim_{t\to a+} p(t)u'(t) = c, \qquad \lim_{t\to a+} u(t) = d.\end{gathered} \tag{4.12}$$

Corollary 4.3. *Let the hypothesis (p0) hold, and let $g : J \times E \times E \to E$ satisfy the following hypotheses:*

(g0) *$g(\cdot,u(\cdot))$ is strongly measurable and $\|g(\cdot,u(\cdot))\| \le h_0 \in L^1(J)$ for all $u \in C(J,E)$,*
(g1) *$g(t,x) \le g(t,y)$ for a.e. $t \in J$ and whenever $x \le y$ in E.*

Then the IVP (4.12) has, for each choice of $c,d \in E$,

(a) *minimal and maximal solutions in Y;*
(b) *least and greatest solutions u_* and u^* in $\{u \in Y \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of equation*

$$u(t) = -\left(d + \int_a^t \frac{1}{p(s)}\left(c + \int_a^s g(x,u(x))dx\right)ds\right)^-, \quad t \in J, \tag{4.13}$$

and $\overline{u}$ is the least solution of equation

$$u(t) = \left(d + \int_a^t \frac{1}{p(s)}\left(c + \int_a^s g(x,u(x))dx\right)ds\right)^+, \quad t \in J. \tag{4.14}$$

Moreover, u_* and u^* are increasing with respect to c, d, and f.

Proof. If $c,d \in E$, the IVP (4.12) is reduced to (4.1) when we define

$$\begin{gathered} f(t,u) = g(t,u(t)), \quad t \in J,\ u \in C(J,E), \\ c(v) \equiv c, \quad v \in C(J,E), \quad d(v) \equiv d, \quad v \in C(J,E). \end{gathered} \tag{4.15}$$

The hypotheses (g0) and (g1) imply that f satisfies the hypotheses (f0) and (f1). The hypotheses (c) and (d) are also valid, whence the asserted results follow from Theorem 4.2. □

Example 4.4. Consider the following system of IVPs:

$$\begin{aligned} \frac{d}{dt}(\sqrt{t}u'(t)) &= t + \frac{[\int_1^2 v(s)ds]}{1+|[\int_1^2 v(s)ds]|} \quad \text{a.e. in } (0,\infty), \\ \frac{d}{dt}(\sqrt{t}v'(t)) &= \sqrt{t} + 2\frac{[\int_1^2 u(s)ds]}{1+|[\int_1^2 u(s)ds]|} \quad \text{a.e. in } (0,\infty), \\ \lim_{t\to 0+}\sqrt{t}u'(t) &= 2\frac{[\int_1^2 v(s)ds]}{1+|[\int_1^2 v(s)ds]|}, \qquad u(0) = \frac{[v(1)]}{1+|[v(1)]|}, \\ \lim_{t\to 0+}\sqrt{t}v'(t) &= \frac{\left[\int_1^2 u(s)ds\right]}{1+\left|\left[\int_1^2 u(s)ds\right]\right|}, \qquad v(0) = 2\frac{[u(1)]}{1+|[u(1)]|}, \end{aligned} \tag{4.16}$$

where $[z]$ denotes the greatest integer $\leq z$.

Solution 4.5. The system (4.16) is a special case of (4.1) when $E = \mathbb{R}^2$, ordered coordinatewise, $a = 0$, $b = \infty$, $p(t) = \sqrt{t}$,

$$\begin{aligned} f(t,(u,v)) &= \left(t + \frac{[\int_1^2 v(s)ds]}{1+|[\int_1^2 v(s)ds]|}, \sqrt{t} + 2\frac{[\int_1^2 u(s)ds]}{1+|[\int_1^2 u(s)ds]|}\right), \\ c((u,v)) &= \left(2\frac{[\int_1^2 v(s)ds]}{1+|[\int_1^2 v(s)ds]|}, \frac{[\int_1^2 u(s)ds]}{1+|[\int_1^2 u(s)ds]|}\right), \\ d((u,v)) &= \left(\frac{[v(1)]}{1+|[v(1)]|}, 2\frac{[u(1)]}{1+|[u(1)]|}\right). \end{aligned} \tag{4.17}$$

The hypotheses (f0), (f1), (c), and (d) hold, with respect to 1-norm of $\mathbb{R}^2$, when $h_0(t) = 3t + 2\sqrt{t} + 4$ and $c_0 = d_0 = 3$. Thus the results of Theorem 4.2 can be applied. It turns out that the chains needed in the proof of Theorem 4.2 (cf. the proof of Lemma 2.2) are reduced to finite ordinary iteration sequences. Thus algorithms of the form (2.4) presented in Remark 2.3 can be used to calculate solutions to the system (4.16). Calculations, carried out by the use of a simple Maple program, show that the least and the greatest solutions of (4.16) between $\underline{u}$, which is the zero function, and $\overline{u}$ are equal to $\overline{u}$, and this solution

(u^*,v^*) is the only solution of (4.16) between $\underline{u}$ and $\overline{u}$. Moreover, (4.16) has only one minimal solution (u_-,v_-) and only one maximal solution (u^+,v^+), and thus they are the least and the greatest of all the solutions of (4.16). The exact expressions of these solutions are

$$\begin{aligned}
(u^*(t),v^*(t)) &= \left(\frac{1}{5}t^2\sqrt{t},\frac{1}{3}t^2\right),\\
(u^+,v^+) &= \left(\frac{4}{5}+\frac{24}{7}\sqrt{t}+\frac{4}{7}t\sqrt{t}+\frac{1}{5}t^2\sqrt{t},\frac{5}{3}+\frac{12}{7}\sqrt{t}+\frac{8}{7}t\sqrt{t}+\frac{1}{2}t^2\right),\\
(u_-(t),v_-(t)) &= \left(-\frac{5}{6}-\frac{24}{7}\sqrt{t}-\frac{4}{7}t\sqrt{t}+\frac{1}{5}t^2\sqrt{t},-\frac{5}{3}-\frac{12}{7}\sqrt{t}-\frac{8}{7}t\sqrt{t}+\frac{1}{3}t^2\right).
\end{aligned} \tag{4.18}$$

5. Existence results for second-order boundary value problems

This section is devoted to the study of boundary value problems which can be represented in the form

$$\begin{aligned}
&-\frac{d}{dt}(p(t)u'(t)) = f(t,u) \quad \text{for a.e. } t\in J := (a,b),\\
&\lim_{t\to a+} p(t)u'(t) = c(u), \qquad \lim_{t\to b-} u(t) = d(u),
\end{aligned} \tag{5.1}$$

where $-\infty \le a < b \le \infty$, $p\in C(J)$, $f : J\times C(J,E)\to E$, and $c,d : C(J,E)\to E$.

We are looking for solutions of the set Y, defined in (4.2). In our main existence result for the BVP (5.1), we assume that the functions p, f, c, and d satisfy the following hypotheses:

(p1) $p(t) > 0$ and $\int_t^b ds/p(s) < \infty$ for all $t\in J$,
(f0) $f(\cdot,u)$ is strongly measurable, and $\|f(\cdot,u)\| \le h_0 \in L^1(J)$ for all $u\in C(J,E)$,
(f1) $f(\cdot,u) \le f(\cdot,v)$ whenever $u,v\in C(J,E)$ and $u\le v$,
(c0) c is bounded, and $c(u)\ge c(v)$ whenever $u,v\in C(J,E)$ and $u\le v$,
(d) d is bounded, and $d(u)\le d(v)$ whenever $u,v\in C(J,E)$ and $u\le v$.

To apply Lemma 2.2, we will first convert the BVP (5.1) to an integral equation.

Lemma 5.1. *Assume that $p(t) > 0$ on J, and that $f(\cdot,u)\in L^1(J,E)$ for all $u\in C(J,E)$. Then $u\in Y$ is a solution of the BVP (5.1) if and only if u satisfies the integral equation*

$$u(t) = d(u) - \int_t^b \frac{1}{p(s)}\left(c(u) - \int_a^s f(x,u)dx\right)ds, \quad t\in J. \tag{5.2}$$

Proof. Assume that $u\in Y$ is a solution of (5.1). The differential equation and the definition (4.2) of Y ensure that

$$-\int_r^s \frac{d}{dt}(p(t)u'(t))dt = p(r)u'(r) - p(s)u'(s) = \int_r^s f(t,u)dt, \quad a<r\le s<b. \tag{5.3}$$

In view of this result and the first boundary condition of (5.1), we obtain

$$u'(s) = \frac{1}{p(s)}\left(c(u) - \int_a^s f(x,u)dx\right), \quad s \in J. \tag{5.4}$$

Because the right-hand side of (5.4) is continuous in s, we can integrate it to obtain

$$u(r) - u(t) = \int_t^r \frac{1}{p(s)}\left(c(u) - \int_a^s f(x,u)dx\right)ds, \quad a < t \le r < b. \tag{5.5}$$

Applying the second boundary condition of (5.1) to the above equation, we see that u satisfies the integral equation (5.2).

The converse part of the proof is trivial. □

The main result of this section is the following existence theorem.

Theorem 5.2. *Assume that the hypotheses (p1), (f0), (f1), (c0), and (d) hold, and assume that the space Y defined by (4.2) is ordered pointwise. Then the BVP (5.1) has*

(a) *minimal and maximal solutions in Y;*

(b) *least and greatest solutions u_* and u^* in $\{u \in Y \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of equation*

$$u(t) = -\left(d(u) - \int_t^b \frac{1}{p(s)}\left(c(u) - \int_a^s f(x,u)dx\right)ds\right)^-, \quad t \in J, \tag{5.6}$$

and $\overline{u}$ is the least solution of equation

$$u(t) = \left(d(u) - \int_t^b \frac{1}{p(s)}\left(c(u) - \int_a^s f(x,u)dx\right)ds\right)^+, \quad t \in J. \tag{5.7}$$

Moreover, u_* and u^* are increasing with respect to d and f, and decreasing with respect to c.

Proof. Let P be defined by (2.1) with

$$w(t) := d_0 + \int_t^b \frac{1}{p(s)}\left(c_0 + \int_a^s h_0(x)dx\right)ds, \quad t \in J, \tag{5.8}$$

where $c_0 = \sup\{\|c(u)\| \mid u \in C(J,E)\}$, $d_0 = \sup\{\|d(u)\| \mid u \in C(J,E)\}$, and the function $h_0 \in L^1(J)$ is as in the hypothesis (f0). The given hypotheses imply that the relation

$$Gu(t) = d(u) - \int_t^b \frac{1}{p(s)}\left(c(u) - \int_a^s f(x,u)dx\right)ds, \quad t \in J, \tag{5.9}$$

defines an increasing mapping $G : P \to P$, and that

$$\|Gu(t) - Gu(\bar{t})\| \le (c_0 + \|h_0\|_1)\left|\int_t^{\bar{t}} \frac{ds}{p(s)}\right| \quad \forall u \in P,\ t,\bar{t} \in J. \tag{5.10}$$

The above inequality implies that $G[P]$ is an equicontinuous subset of P, whence G satisfies the hypotheses of Lemma 2.2. Moreover, it is easy to verify that each solution in Y belongs to P, and that Gu increases if $c(u)$ decreases, or if $d(u)$ or $f(\cdot,u)$ increases. Thus the assertions follow from Lemmas 5.1 and 2.2. □

As a special case, we obtain an existence result for the BVP

$$\begin{aligned} -\frac{d}{dt}\big(p(t)u'(t)\big) &= g\big(t,u(t)\big) \quad \text{for a.e. } t \in J, \\ \lim_{t\to a+} p(t)u'(t) = c&, \qquad \lim_{t\to b-} u(t) = d. \end{aligned} \tag{5.11}$$

Corollary 5.3. *Let the hypothesis (p1) hold, and let $g : J \times E \times E \to E$ satisfy the following hypotheses:*
(g0) *$g(\cdot,u(\cdot))$ is strongly measurable and $\|g(\cdot,u(\cdot))\| \le h_0 \in L^1(J)$ for all $u \in C(J,E)$,*
(g1) *$g(t,x) \le g(t,y)$ for a.e. $t \in J$ and whenever $x \le y$ in E.*
Then the BVP (5.11) has, for each choice of $c,d \in E$,
(a) *minimal and maximal solutions in Y;*
(b) *least and greatest solutions u_* and u^* in $\{u \in Y \mid \underline{u} \le u \le \overline{u}\}$, where $\underline{u}$ is the greatest solution of equation*

$$u(t) = -\left(d - \int_t^b \frac{1}{p(s)}\left(c - \int_a^s g(x,u(x))\,dx\right)ds\right)^-, \quad t \in J, \tag{5.12}$$

and $\overline{u}$ is the least solution of equation

$$u(t) = \left(d - \int_t^b \frac{1}{p(s)}\left(c - \int_a^s g(x,u(x))\,dx\right)ds\right)^+, \quad t \in J. \tag{5.13}$$

Moreover, u_* and u^* are increasing with respect to d and f, and decreasing with respect to c.

Proof. If $c,d \in E$, the BVP (5.11) is reduced to (5.1) when we define

$$\begin{aligned} f(t,u) = g\big(t,u(t)\big), \quad & t \in J,\ u \in C(J,E), \\ c(v) \equiv c, \quad v \in C(J,E), \quad & d(v) \equiv d, \quad v \in C(J,E). \end{aligned} \tag{5.14}$$

The hypotheses (g0) and (g1) imply that f satisfies the hypotheses (f0) and (f1). The hypotheses (c0) and (d) are also valid, whence the asserted results follow from Theorem 5.2. □

Example 5.4. Consider the following system of BVPs:

$$
\begin{aligned}
-\frac{d}{dt}\left(t\sqrt{t}u'(t)\right) &= 2t+1+\left[10\arctan\left(\int_1^2 v(s)\,ds\right)\right] \quad \text{a.e. in } (0,3),\\
-\frac{d}{dt}\left(t\sqrt{t}v'(t)\right) &= \sqrt{t}+1+\left[5\arctan\left(\int_1^2 u(s)\,ds\right)\right] \quad \text{a.e. in } (0,3),\\
\lim_{t\to 0+} t\sqrt{t}u'(t) &= -2\frac{[\int_1^2 v(s)ds]}{1+|[\int_1^2 v(s)ds]|}, \qquad u(3)=\frac{[5v(1)]}{1+|[5v(1)]|},\\
\lim_{t\to 0+} t\sqrt{t}v'(t) &= -\frac{[\int_1^2 u(s)ds]}{1+|[\int_1^2 u(s)ds]|}, \qquad v(3)=\frac{[10u(1)]}{1+|[10u(1)]|},
\end{aligned}
\tag{5.15}
$$

where $[z]$ denotes the greatest integer $\le z$.

Solution 5.5. The system (5.15) is a special case of (5.1) when $E=\mathbb{R}^2$, ordered coordinatewise, $a=0$, $b=3$, $p(t)=t\sqrt{t}$,

$$
\begin{aligned}
f(t,(u,v)) &= \left(2t+1+\left[10\arctan\left(\int_1^2 v(s)\,ds\right)\right], \sqrt{t}+1+\left[5\arctan\left(\int_1^2 u(s)\,ds\right)\right]\right),\\
c((u,v)) &= \left(-2\frac{[\int_1^2 v(s)ds]}{1+|[\int_1^2 v(s)ds]|}, -\frac{[\int_1^2 u(s)ds]}{1+|[\int_1^2 u(s)ds]|}\right),\\
d((u,v)) &= \left(\frac{[5v(1)]}{1+|[5v(1)]|}, \frac{[10u(1)]}{1+|[10u(1)]|}\right).
\end{aligned}
\tag{5.16}
$$

The hypotheses (f0), (f1), (c0), and (d) hold, with respect to 1-norm of $\mathbb{R}^2$, when $h_0(t)=2t+\sqrt{t}+26$, $c_0=3$ and $d_0=2$. Thus the results of Theorem 5.2 hold. Also, in this case, the chains needed in the proof of Theorem 5.2 (cf. the proof of Lemma 2.2) are reduced to finite ordinary iteration sequences. Applying a simple Maple program, one can show that the least and the greatest solutions of (5.15) between $\underline{u}$, which is the zero function, and $\overline{u}$ are equal to $\overline{u}$, and this solution (u^*,v^*) is also the greatest of all the solutions of (5.15). Moreover, (5.15) has only one minimal solution (u_-,v_-) and thus is the least of all the solutions of (5.15). The exact expressions of these solutions are

$$
\begin{aligned}
&(u^*(t),v^*(t)) =\\
&\quad\left(\frac{74}{75}+\frac{1016}{33}\sqrt{3}-\frac{2}{3}t\sqrt{t}-30\sqrt{t}+\frac{40}{11\sqrt{t}}, \frac{818}{273}+\frac{461}{30}\sqrt{3}-\frac{2}{3}t-16\sqrt{t}+\frac{19}{10\sqrt{t}}\right),\\
&(u_-(t),v_-(t)) =\\
&\quad\left(-\frac{54}{55}-\frac{670}{27}\sqrt{3}-\frac{2}{3}t\sqrt{t}+28\sqrt{t}-\frac{32}{9\sqrt{t}}, \frac{204}{203}-\frac{107}{8}\sqrt{3}-\frac{2}{3}t+14\sqrt{t}-\frac{15}{8\sqrt{t}}\right).
\end{aligned}
\tag{5.17}
$$

Remarks 5.6. The following spaces are examples of weakly complete Banach lattices:

(i) a reflexive (e.g., a uniformly convex) Banach lattice;

(ii) a uniformly monotone Banach lattice in the sense defined in [1, XV, 14];

(iii) a separable Hilbert space whose order cone is generated by an orthonormal basis;

(iv) $\mathbb{R}^m$ ordered coordinatewise and normed by a p-norm, $1 \leq p \leq \infty$;

(v) l^p, $1 \leq p < \infty$, normed by p-norm and ordered componentwise;

(vi) $L^p(\Omega)$, $1 \leq p < \infty$, normed by p-norm and ordered a.e. pointwise.

The Sobolev spaces $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise, have properties given for E in the hypothesis (B) (cf. [2, Appendix C4]).

Thus the results of Theorems 3.2, 4.2, and 5.2 and Corollaries 3.3, 4.3, and 5.3 hold when E is one of the spaces listed above.

Problems of the form (3.1), (4.1), and (5.1) include many kinds of special types. For instance, they can be

(i) singular, because $\lim_{t\to a+} p(t) = 0$ is allowed;

(ii) nonlocal, because the functions c, d, and f may depend functionally on u;

(iii) discontinuous, because the dependencies of c, d, and f on u can be discontinuous;

(iv) problems on an infinite interval, because cases $a = -\infty$ and/or $b = \infty$ are included;

(v) finite systems when $E = \mathbb{R}^m$;

(vi) infinite systems when $E = l^p$;

(vii) of random type when $E = L^p(\Omega)$ and Ω is a probability space.

According to the hypotheses of Lemma 2.2, the operator G may be discontinuous and noncompact. Moreover, the boundedness hypotheses assumed for functions f, c, and d don't provide means to construct a priori upper and/or lower solutions for problems (3.1), (4.1), and (5.1). Thus, for instance, Schauder's fixed point theorem, ordinary iteration methods or the method of upper and lower solutions are not, in general, applicable alternatives to Lemma 2.2 used in the proofs of Theorems 3.2, 4.2, and 5.2.

As for generalized versions and applications of Lemma 2.2, as well as applications of algorithms of the type presented in Remark 2.3, see, for example, [3, 4, 5, 6, 7].

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